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SECOND DEGREE SEMICLASSICAL LINEAR FUNCTIONALS OF CLASS ONE. THE QUASI-ANTISYMMETRIC CASE

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Abstract. An orthogonal sequence with respect to a regular linear functional w is said to be semiclassical if there exist a monic polynomial Φ and a polynomial Ψ with $\deg(\Psi) \geq 1$, such that $(\Phi w)' + \Psi w = 0$. Recently, all semiclassical monic orthogonal polynomial sequences of class one satisfying a three term recurrence relation with $\beta_0 = -\alpha_0$, $\beta_{n+1} = \alpha_n - \alpha_{n+1}$ and $\gamma_{n+1} = -\alpha_n^2$ with $\alpha_n \neq 0$, $n \geq 0$, have been determined [17].

In this paper, we point sequences of the above family such that their corresponding Stieltjes function $S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}$ satisfies a quadratic equation $B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0$, where B, C, D are polynomials.

Ортогональна послідовність відносно регулярного лінійного функціонала w називається напівкласичною, якщо існує моном Φ і поліном Ψ, $\deg(\Psi) \geq 1$, такі, що $(\Phi w)' + \Psi w = 0$. Останнім часом всі напівкласичні монічні ортогональні поліноміальні послідовності першого класу, що задовольняють тричленному рекурентному відношенню, коли $\beta_0 = -\alpha_0$, $\beta_{n+1} = \alpha_n - \alpha_{n+1}$ і $\gamma_{n+1} = -\alpha_n^2$ з $\alpha_n \neq 0$, $n \geq 0$, були визначені [17].

В статті вказуються послідовності вищевказаної сім'ї такі, що їх відповідна функція Стілтьєса $S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}$ задовольняє квадратичному рівнянню $B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0$, де B, C, D – поліноми.

1. Introduction and preliminary results

A 1995 paper of Maroni [10] provides an introduction to second degree linear functionals. These linear functionals are characterized by the fact that their formal Stieltjes function $S(w)$ satisfies a quadratic equation $B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0$. They have been studied in [7, 20] and [16] in the framework of orthogonality on several intervals. Later in [11] and [10], an algebraic approach to such second degree linear functionals as an extension of the Tchebychev linear functionals was given. The second degree linear functional set is a part of the semiclassical one [9, 10]. Among the second degree semiclassical sequences of orthogonal polynomials only those which are of class $s = 0$ and those of class $s = 1$, which are symmetric and quasi-symmetric, are completely described in the literature [1, 2, 3, 19]. See also [4, 5, 8, 18].

In this contribution we are dealing with second degree linear functionals which are semiclassical of class $s = 1$ and such that their corresponding sequence of monic orthogonal polynomials $\{W_n\}_{n\geq 0}$ verifies the recurrence relation

$$W_{n+2}(x) = \Big(x - (\alpha_n - \alpha_{n+1})\Big)W_{n+1}(x) + \alpha_n^2 W_n(x), \quad n \geq 0\,,$$
$$W_1(x) = x + \alpha_0, \qquad W_0(x) = 1,$$

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with $\alpha_n \neq 0\,,\ n \geq 0$. This family has been a subject of some works. For instance, Maroni [12, 15] characterized such sequences by a particular quadratic decomposition and by a perturbation of a symmetric linear functional.

The structure of the manuscript is as follows. The first section is devoted to preliminary results and notations used in the sequel. In the second section, we focus our attention on second degree linear functionals. More precisely, all second degree semiclassical linear functionals of class $s = 1$ such that their corresponding MOPS verify the above mentioned recurrence relation, are determined. Finally, the polynomial coefficients of the second degree equation fulfilled by the corresponding formal Stieltjes function are deduced.

In the sequel, we will recall some basic definitions and results. The field of complex numbers is denoted by $\mathbb{C}$. The vector space of polynomials with coefficients in $\mathbb{C}$ is denoted by $\mathcal{P}$ and its dual space is denoted by $\mathcal{P}'$. We denoted by $\langle w, f\rangle$ the value of $w \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(w)_n = \langle w, x^n\rangle$, $n \geq 0$, the moments of w. For any linear functional (form) w, any polynomial h, let $Dw = w^{'}$, hw and δ_0 be the linear functionals defined by

$$\langle w', f\rangle := -\langle w, f'\rangle\,, \qquad \langle hw, f\rangle := \langle w, hf\rangle\,, \quad \langle \delta_0, f\rangle = f(0), \quad f \in \mathcal{P}.$$

We recall the definition of right-multiplication of a linear functional w by a polynomial $h(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$:

$$(wh)(x) := \Big\langle w, \frac{xh(x) - \xi h(\xi)}{x - \xi}\Big\rangle = \sum_{m=0}^{n}\Big(\sum_{\nu=m}^{n} a_\nu (w)_{\nu-m}\Big)x^m. \tag{1.1}$$

By duality, we obtain the Cauchy's product of two linear functionals:

$$\langle wv, f\rangle := \langle w, vf\rangle, \qquad w, v \in \mathcal{P}', \quad f \in \mathcal{P}.$$

We define [14] the form $(x-c)^{-1}w\,,\ c \in \mathbb{C}$, through

$$\big\langle (x-c)^{-1}w, f\big\rangle := \langle w, \theta_c f\rangle\,,$$

with

$$\big(\theta_c f\big)(x) = \frac{f(x) - f(c)}{x - c}, \qquad c \in \mathbb{C}, \quad f \in \mathcal{P}. \tag{1.2}$$

From the definition, one gets

$$(w\theta_0 f)(x) = \langle w, \frac{f(x) - f(\xi)}{x - \xi}\rangle, \qquad w \in \mathcal{P}^{'}, \quad f \in \mathcal{P}. \tag{1.3}$$

We introduce an operator $\sigma : \mathcal{P} \to \mathcal{P}$ defined by $(\sigma f)(x) = f(x^2)$ for all $f \in \mathcal{P}$. By transposition, we define σw as

$$\langle \sigma w, f\rangle = \langle w, \sigma f\rangle, \qquad w \in \mathcal{P}^{'}, \quad f \in \mathcal{P}. \tag{1.4}$$

We will also use the so-called formal Stieltjes function associated with $w \in \mathcal{P}^{'}$ and defined by

$$S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}. \tag{1.5}$$

The following results are fundamental [13, 14].

Lemma 1.1. *For any $f \in \mathcal{P}$ and $w \in \mathcal{P}^{'}$,*

$$x^{-1}(xw) = w - (w)_0\delta_0, \tag{1.6}$$

$$(fw)^{'} = fw^{'} + f^{'}w, \tag{1.7}$$

$$f(x)(\sigma w) = \sigma\big(f(x^2)w\big), \tag{1.8}$$

$$\sigma w^{'} = 2\big(\sigma(xw)\big)^{'}, \tag{1.9}$$

$$\big((x^{-1}w)f\big)(z) = \big(\theta_0(wf)\big)(z) = \big(w(\theta_0 f)\big)(z), \tag{1.10}$$

$$\big(w(\sigma f)\big)(z) = \big((\sigma w)f\big)(z^2) + z\big(\sigma(\xi w)(\theta_0 f)\big)(z^2), \tag{1.11}$$

$$\big(\theta_0(\sigma f)\big)(z) = z\big(\sigma(\theta_0 f)\big)(z). \tag{1.12}$$

Let us recall that a linear functional w is said to be regular (quasi-definite) if there exists a sequence $\{W_n\}_{n\geq 0}$ of polynomials with $\deg W_n = n$, $n \geq 0$, such that

$$\langle w, W_n W_m\rangle = r_n\delta_{n,m}, \qquad r_n \neq 0, \quad n \geq 0.$$

We can always assume that each W_n is monic, i.e. $W_n(x) = x^n +$ (lower degree terms). Then the sequence $\{W_n\}_{n\geq 0}$ is said to be orthogonal with respect to w (monic orthogonal polynomial sequence (MOPS) in short).

It is a very well-known fact that the sequence $\{W_n\}_{n\geq 0}$ satisfies a three-term recurrence relation (see, for instance, the monograph by Chihara [6]),

$$\begin{aligned} W_{n+2}(x) &= (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x), \quad n \geq 0, \\ W_1(x) &= x - \beta_0, \qquad W_0(x) = 1, \end{aligned} \tag{1.13}$$

with $\big(\beta_n, \gamma_{n+1}\big) \in \mathbb{C} \times (\mathbb{C} - \{0\})$, $n \geq 0$. By convention we set $\gamma_0 = (w)_0$.

A linear functional w is said to be normalized if $(w)_0 = 1$. In this paper, we suppose that any linear functional is normalized.

We recall that a linear functional w is called symmetric if $(w)_{2n+1} = 0, n \geq 0$. The conditions $(w)_{2n+1} = 0, n \geq 0$, are equivalent to the fact that the corresponding MOPS $\{W_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (1.13) with $\beta_n = 0, n \geq 0$ [6].

Now, let us recall some features of the second degree semiclassical character [10].

Definition 1.2. A linear functional w is said to be a second degree linear functional if it is regular and there exist two polynomials B and C such that

$$B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0, \tag{1.14}$$

where D depends on B, C and w, and

$$D(z) = (w\theta_0 C)(z) - (w^2\theta_0^2 B)(z). \tag{1.15}$$

The regularity of w means that we must have $B \neq 0$, $C^2 - 4BD \neq 0$ and $D \neq 0$.

The following expressions are equivalent to (1.14) [10]:

$$B(x)w^2 = xC(x)w, \qquad \langle w^2, \theta_0 B\rangle = \langle w, C\rangle. \tag{1.16}$$

In the sequel, we shall assume B to be monic.

Let us recall that a linear functional w is called semiclassical if it is regular and there exist two polynomials Φ and Ψ, where $\Phi(x)$ is monic and $\deg(\Psi) \geq 1$, such that

$$(\Phi w)^{'} + \Psi w = 0. \tag{1.17}$$

The class of a semiclassical linear functional w is $s = \max\big(\deg(\Phi) - 2, \deg(\Psi) - 1\big)$ if and only if the following condition is satisfied:

$$\prod_{c\in\mathcal{Z}_\Phi}\Big(|\Phi'(c) + \Psi(c)| + |\langle w, \theta_c^2\Phi + \theta_c\Psi\rangle|\Big) \neq 0,$$

where $\mathscr{Z}_\Phi$ is the set of zeros of Φ [15].

If $s = 0$, w is called a classical linear functional.

As a result, if w is a semiclassical linear functional of class s satisfying (1.17), then the shifted linear functional $\widehat{w} = \big(h_{a^{-1}} \circ \tau_{-b}\big)w$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$, is also semiclassical having the same class as that of w and satisfying the equation

$$\big(\widehat{\Phi}\widehat{w}\big)' + \widehat{\Psi}\widehat{w} = 0, \tag{1.18}$$

with

$$\widehat{\Phi}(x) = a^{-t}\Phi(ax+b),\ \widehat{\Psi}(x) = a^{1-t}\Psi(ax+b), \qquad t = \deg(\Phi),$$

where, for each polynomial f,

$$\langle \tau_b w, f\rangle := \langle w, f(x+b)\rangle, \quad \langle h_a w, f\rangle := \langle w, f(ax)\rangle.$$

A second degree linear functional w is a semiclassical linear functional and satisfies (1.17), with [10]

$$k\Phi(x) = B(x)\big(C^2(x) - 4B(x)D(x)\big),$$
$$k\Psi(x) = -\frac{3}{2}B(x)\big(C^2(x) - 4B(x)D(x)\big)', \qquad k \neq 0,$$

where k is the normalization factor.

A second degree character is kept by shifting. Indeed, if w is a second degree linear functional satisfying (1.16), then $\widehat{w}$ is also a second degree linear functional [10]. It satisfies

$$\widehat{B}(x)\widehat{w}^2 = x\widehat{C}(x)\widehat{w}, \quad \langle \widehat{w}^2, \theta_0\widehat{B}\rangle = \langle \widehat{w}, \widehat{C}\rangle,$$

with

$$\widehat{B}(x) = a^{-r}B(ax+b), \quad \widehat{C}(x) = a^{1-r}C(ax+b), \quad r = \deg(B).$$

We finish this section by recalling an important result.

Theorem 1.3. [3] *Among the classical linear functionals, only the Jacobi linear functionals $\mathcal{J}(p-\frac{1}{2}, q-\frac{1}{2})$ are second degree linear functionals, provided $p+q \geq 0$, $p, q \in \mathbb{Z}$.*

2. Second degree quasi-antisymmetric semiclassical linear functionals of class one

From now on, let w be a semiclassical linear functional of class $s = 1$ satisfying (21) and its corresponding MOPS $\{S_n\}_{n\geq 0}$ fulfills

$$\begin{gathered} W_{n+2}(x) = \Big(x - (\alpha_n - \alpha_{n+1})\Big)W_{n+1}(x) + \alpha_n^2 W_n(x), \quad n \geq 0, \\ W_1(x) = x + \alpha_0, \qquad W_0(x) = 1, \end{gathered} \tag{2.19}$$

with $\alpha_n \neq 0$, $n \geq 0$.

Then, its associated linear functional w is said to be quasi-antisymmetric (i.e $(w)_{2n+2} = 0, n \geq 0$). Equivalently, $x\sigma w = 0$. For more information about these linear functionals see [12, 15].

Let us begin with an example ϑ of second degree quasi-antisymmetric semiclassical linear functional of class one. This example is given in [10]. The linear functional ϑ satisfies (1.14) with

$$B(z) = z, \quad C(z) = 2(iz+1), \quad D(z) = 2i, \tag{2.20}$$

and (1.17) with

$$\Phi(x) = x^3 - x, \quad \Psi(x) = -3x^2.$$

Theorem 2.1. *The quasi-antisymmetric linear functional w is a second degree linear functional if and only if the form $(w)_1 v = \sigma(xw)$ is a second degree linear functional.*

For the proof, we need the following lemmas.

Lemma 2.2. *If w is a quasi-antisymmetric linear functional, then we have*

$$x\sigma w^2 = \big(\sigma(xw)\big)^2, \tag{2.21}$$
$$\sigma(xw^2) = 2\sigma(xw). \tag{2.22}$$

Proof. By using equations (1.1) and (1.4), we obtain

$$\begin{aligned}\langle x\sigma w^2, x^n\rangle &= \langle w^2, x^{2n+2}\rangle \\ &= \langle w, \sum_{k=0}^{2n+2} x^k (w)_{2n+2-k}\rangle \\ &= \langle w, \sum_{k=0}^{n} x^{2k+1} (w)_{2n+1-2k}\rangle \\ &= \langle \sigma(xw), \sum_{k=0}^{n} x^k \big(\sigma(xw)\big)_{n-k}\rangle \\ &= \langle \big(\sigma(xw)\big)^2, x^n\rangle.\end{aligned}$$

Hence (2.21) follows.

Now, taking into account the relations (1.1) and (1.4), we get

$$\begin{aligned}\langle \sigma(xw^2), x^n\rangle &= \langle w^2, x^{2n+1}\rangle \\ &= \langle w, \sum_{k=0}^{2n+1} x^k (w)_{2n+1-k}\rangle \\ &= \langle w, x^{2n+1} + (w)_{2n+1} + \sum_{k=1}^{n} x^{2k} (w)_{2n+1-2k}\rangle \\ &= 2\langle \sigma(xw), x^n\rangle + \langle \sigma w, \sum_{k=1}^{n} x^k (w)_{2n+1-2k}\rangle \\ &= 2\langle \sigma(xw), x^n\rangle + \langle x\sigma w, \sum_{k=1}^{n} x^{k-1} (w)_{2n+1-2k}\rangle.\end{aligned}$$

Here we have the result (2.22). □

Lemma 2.3. *If w is a quasi-antisymmetric linear functional and $f \in \mathcal{P}$, then we have*

$$\big((\sigma w)f\big)(z) = f(z), \tag{2.23}$$
$$\big(w\theta_0(\sigma f)\big)(z) = \big((\sigma\xi w)(\theta_0 f)\big)(z^2) + z(\theta_0 f)(z^2), \tag{2.24}$$
$$\big(w^2\theta_0(\sigma f)\big)(z) = z\big((\sigma\xi w)^2(\theta_0^2 f)\big)(z^2) + 2\big((\sigma\xi w)(\theta_0 f)\big)(z^2) + z(\theta_0 f)(z^2), \tag{2.25}$$
$$\big(w^2\theta_0^2(\sigma f)\big)(z) = \big((\sigma\xi w)^2(\theta_0^2 f)\big)(z^2) + 2\big((\sigma\xi w)(\theta_0^2 f)\big)(z^2) + (\theta_0 f)(z^2). \tag{2.26}$$

Proof. The proof of (2.23) is evident from (1.1). Then, (2.24) is a consequence of them and (1.10)—(1.11). The property (2.25) is obtained from (1.6), (1.10)—(1.12) and (2.21)—(2.22). Finally, (2.26) is evident from (1.10) and (2.25). □

Proof of Theorem 2.1. Let us write the polynomials B, C and D according to their even and odd parts,

$$\begin{cases} B(x) = B^e(x^2) + xB^o(x^2), \\ C(x) = C^e(x^2) + xC^o(x^2), \\ D(x) = D^e(x^2) + xD^o(x^2). \end{cases} \tag{2.27}$$

Now, from (1.5) and the fact that w is a quasi-antisymmetric linear functional, we have

$$(w)_1 zS(v)(z^2) = zS(w)(z) + 1. \tag{2.28}$$

Then, from (2.28), (1.14) reduces to

$$B(z)\Big((w)_1 zS(v)(z^2) - 1\Big)^2 + zC(z)\Big((w)_1 zS(v)(z^2) - 1\Big) + z^2 D(z) = 0.$$

Equivalently,

$$(w)_1^2 z^2 B(z) S^2(v)(z^2) + (w)_1\big(z^2C(z) - 2zB(z)\big)S(v)(z^2) + z^2D(z) + B(z) - zC(z) = 0.$$

Taking into account (2.27) and the even component in left hand side, we get

$$B_1(z)S^2(v)(z) + C_1(z)S(v)(z) + D_1(z) = 0,$$

with

$$\begin{cases} B_1(z) = (w)_1^2 zB^e(z), \\ C_1(z) = (w)_1 z\big(C^e(z) - 2B^o(z)\big), \\ D_1(z) = z\big(D^e(z) - C^o(z)\big) + B^e(z). \end{cases} \tag{2.29}$$

Now, taking into account (1.11) and (2.23)–(2.27), we get from (1.15), after some calculations, that

$$D^e(z) = (w)_1\Big(\big(v\theta_0 C^e\big)\big)(z) - 2\big(v\theta_0 B^o\big)(z)\Big) - (w)_1^2\big(v^2\theta_0^2 B^e\big)(z) + C^o(z) - \big(\theta_0 B^e\big)(z). \tag{2.30}$$

Or, for each polynomial f, we have

$$(wf)(z) = z\big(w\theta_0 f\big)(z) + \langle w, f\rangle. \tag{2.31}$$

Then, from (2.30) and (2.31), we get from (2.29) that

$$D_1(z) = (v\theta_0 C_1(z) - (v^2\theta_0^2 B_1)(z) + (w)_1\langle v, 2B^o - C^e\rangle + (w)_1^2\langle v^2, \theta_0 B^e\rangle + B^e(0).$$

But, from (1.16) and (2.27), we can deduce

$$\langle \sigma w^2, B^e\rangle + \langle(\sigma\xi w^2), B^o\rangle = \langle(\sigma\xi w), C^e\rangle.$$

Thus, from (1.6) and (2.21)–(2.22), we get

$$(w)_1^2\langle v^2, \theta_0 B^e\rangle + 2(w)_1\langle v, B^o\rangle + B^e(0) = (w)_1\langle v, C^e\rangle.$$

Therefore, $D_1(z) = (v\theta_0 C_1(z) - (v^2\theta_0^2 B_1)(z)$ and also v is a second degree linear functional.

Conversely, we assume that v is a second degree linear functional. Then there exist two polynomials B_1 and C_1 such that

$$B_1(z)S^2(v)(z) + C_1(z)S(v)(z) + D_1(z) = 0, \tag{2.32}$$

with

$$D_1(z) = \big(v\theta_0 C_1\big)(z) - \big(v^2\theta_0^2 B_1\big)(z).$$

Making the change of variable $z \to z^2$ in (2.32) and substituting (2.28) into the obtained equation, we get (1.14) with

$$\begin{cases} B(z) = \frac{z^2}{(w)_1^2} B_1(z^2), \\ C(z) = \frac{z^2}{(w)_1} C_1(z^2) + 2\frac{z}{(w)_1^2} B_1(z^2), \\ D(z) = \frac{1}{(w)_1^2} B_1(z^2) + \frac{z}{(w)_1} C_1(z^2) + z^2 D_1(z^2). \end{cases} \tag{2.33}$$

Taking into account (2.33) and (1.2), we get

$$(w\theta_0 C)(z) = \frac{1}{(w)_1}\big(w(\xi C_1(\xi^2))\big)(z) + \frac{2}{(w)_1^2}\big(wB_1(\xi^2)\big)(z).$$

With a use of equation (1.1), the last equation becomes

$$\begin{aligned}(w\theta_0 C)(z) &= \frac{1}{(w)_1^2}\Big\langle w, \frac{(w)_1 z^2 C_1(z^2) + 2zB_1(z^2) - (w)_1\xi^2 C_1(\xi^2) - 2\xi B_1(\xi^2)}{z-\xi}\Big\rangle \\ &= \frac{1}{(w)_1^2}\Big\langle w, (z+\xi)\frac{(w)_1 z^2 C_1(z^2) + 2zB_1(z^2) - (w)_1\xi^2 C_1(\xi^2) - 2\xi B_1(\xi^2)}{z^2-\xi^2}\Big\rangle \\ &= \frac{1}{(w)_1^2}\Big\langle w, (w)_1 z C_1(z^2) + 2B_1(z^2) + (w)_1\xi C_1(\xi^2) + (w)_1 z\xi^2\big(\theta_{z^2}C_1\big)(\xi^2) \\ &\quad + 2\xi^2\big(\theta_{z^2}B_1\big)(\xi^2)\Big\rangle \\ &\quad + \frac{1}{(w)_1^2}\Big\langle w, (w)_1 z^2\xi\frac{C_1(z^2)-C_1(\xi^2)}{z^2-\xi^2} + 2z\xi\frac{B_1(z^2)-B_1(\xi^2)}{z^2-\xi^2}\Big\rangle.\end{aligned}$$

Or,

$$\begin{aligned}&\Big\langle w, (w)_1 z\xi^2\big(\theta_{z^2}C_1\big)(\xi^2) + 2\xi^2\big(\theta_{z^2}B_1\big)(\xi^2)\Big\rangle \\ &\qquad = (w)_1\Big\langle v, (w)_1 z\big(\theta_{z^2}C_1\big)(\xi^2) + 2\big(\theta_{z^2}B_1\big)(\xi^2)\Big\rangle = 0,\end{aligned}$$

because w is a quasi-antisymmetric linear functional. Then,

$$\begin{aligned}(w\theta_0 C)(z) &= \frac{1}{(w)_1^2}\Big(\langle w, (w)_1\xi C_1(\xi^2) + (w)_1 z^2\xi\frac{C_1(z^2)-C_1(\xi^2)}{z^2-\xi^2} + 2z\xi\frac{B_1(z^2)-B_1(\xi^2)}{z^2-\xi^2}\rangle \\ &\quad + (w)_1 z C_1(z^2) + 2B_1(z^2)\Big) \\ &= \frac{1}{(w)_1}\Big(\langle v, (w)_1 C_1(\xi) + (w)_1 z^2\frac{C_1(z^2)-C_1(\xi)}{z^2-\xi} + 2z\frac{B_1(z^2)-B_1(\xi)}{z^2-\xi}\rangle\Big) \\ &\quad + \frac{1}{(w)_1^2}\Big((w)_1 z C_1(z^2) + 2B_1(z^2)\Big),\end{aligned}$$

by virtue of (1.3). Therefore,

$$\begin{aligned}(w\theta_0 C)(z) &= \frac{1}{(w)_1}\Big(\langle v, (w)_1 C_1(\xi)\rangle + (w)_1 z^2\Big(v\theta_0 C_1(\xi)\Big)(z^2) + 2z\Big(v\theta_0 B_1(\xi)\Big)(z^2)\Big) \\ &\quad + \frac{1}{(w)_1^2}\Big((w)_1 z C_1(z^2) + 2B_1(z^2)\Big). \qquad (2.34)\end{aligned}$$

Using (2.33) and (1.2), we can deduce that

$$\big(w^2\theta_0^2 B\big)(z) = \frac{1}{(w)_1^2}\big(w^2 B_1(\xi^2)\big)(z).$$

Taking into account (2.21)—(2.22) and using the same process as we did to obtain (2.34), we get

$$\begin{aligned}\big(w^2\theta_0^2 B\big)(z) &= \frac{1}{(w)_1^2}B_1(z^2) \\ &\quad + \frac{1}{(w)_1}\Big(\langle v^2, (\theta_0 B_1)(\xi)\rangle + 2z\Big(v\theta_0 B_1(\xi)\Big)(z^2) + z^2\Big(v\theta_0^2 B_1(\xi)\Big)(z^2)\Big). \qquad (2.35)\end{aligned}$$

Therefore, on account of (1.16) and (2.33)—(2.35), we conclude that the polynomials B, C and D given by (2.33) verify (1.15). Here w is also a second degree linear functional. $\square$

Theorem 2.4. [17] *For a semiclassical linear functional w of class $s = 1$ fulfilling (1.17) such that the corresponding MOPS satisfies (2.19), we get the following.*

a) *If $\Phi(x) = x$, we have*

$$(xu)' + 2x^2u = 0,$$

with

$$\alpha_{2n} = -\lambda\sqrt{\pi}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}, \quad \alpha_{2n+1} = \frac{\Gamma(n+\frac{3}{2})}{\lambda\Gamma(n+1)}, \qquad n \geq 0.$$

b) *If $\Phi(x) = x^3 - x$, we have*

$$\begin{cases} \big((x^3-x)u\big)' - 2(\alpha+1)x^2u = 0, \\ |\alpha| + |(2\alpha+1)\lambda+1| \neq 0, \end{cases} \tag{2.36}$$

with, for $n \geq 0$,

$$\alpha_{2n} = \begin{cases} -\frac{2\lambda\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})\Gamma(n+1)\Gamma(n+\alpha+1)}{(4n+2\alpha+1)\Gamma(\alpha+1)\Gamma(n+\alpha+\frac{1}{2})\Gamma(n+\frac{1}{2})}, & \text{if } \alpha \neq -\frac{1}{2}, \\ \lambda, & \text{if } \alpha = -\frac{1}{2}, \end{cases}$$

and

$$\alpha_{2n+1} = \begin{cases} -\frac{2\Gamma(\alpha+1)\Gamma(n+\alpha+\frac{3}{2})\Gamma(n+\frac{3}{2})}{\lambda\sqrt{\pi}(4n+2\alpha+3)\Gamma(\alpha+\frac{3}{2})\Gamma(n+1)\Gamma(n+\alpha+1)}, & \text{if } \alpha \neq -\frac{1}{2}, \\ \frac{1}{4\lambda}, & \text{if } \alpha = -\frac{1}{2}. \end{cases} \tag{2.37}$$

In the sequel, we denote by $\mathcal{L}(\alpha)$ the linear functional w that satisfies (2.36)—(2.37). Therefore, we have $\vartheta = \mathcal{L}(\frac{1}{2})$.

Theorem 2.5. *Among the quasi-antisymmetric semiclassical linear functionals of class $s = 1$ only the linear functionals $\mathcal{L}(p - \frac{1}{2})$ are second degree linear functionals, provided $p \in \mathbb{N}$.*

For the proof, we need the following lemma.

Lemma 2.6. [2] *Let w be a second degree semiclassical linear functional satisfying (1.17). The class of w is $s = \deg(\Phi) - 2 = \deg(\Psi) - 1$.*

Proof of Theorem 2.5. According to Theorem 2.4, we distinguish two canonical cases for Φ.

– *First case:* $\Phi(x) = x$.
According to Lemma 2.6, this case is excluded.

– *Second case:* $\Phi(x) = x^3 - x$.
Multiplying equation (2.36) by x and using (1.7), we obtain

$$\big(x^2(x^2-1)\mathcal{L}(\alpha)\big)' + \big(-(2\alpha+3)x^2 + x\big)\mathcal{L}(\alpha) = 0.$$

Applying the operator σ to the previous equation and using (1.8)—(1.9), we get

$$\Big(x(x-1)\big(\sigma x\mathcal{L}(\alpha)\big)\Big)' + \frac{1}{2}\Big(-(2\alpha+3)x+1\Big)\big(\sigma x\mathcal{L}(\alpha)\big) = 0. \tag{2.38}$$

Let us make a suitable shift for $\sigma x\mathcal{L}(\alpha)$,

$$\widehat{\sigma x\mathcal{L}(\alpha)} = \big(h_2 \circ \tau_{-\frac{1}{2}}\big)\sigma x\mathcal{L}(\alpha).$$

Using (2.38), we see that $\widehat{\sigma x\mathcal{L}(\alpha)}$ satisfies (1.18) with

$$\widehat{\Phi}(x) = x^2 - 1, \quad \widehat{\Psi}(x) = \frac{1}{2}\big(-(2\alpha+3)x + 2\alpha + 1\big).$$

Therefore, we have

$$\big(h_2 \circ \tau_{-\frac{1}{2}}\big)\sigma x\mathcal{L}(\alpha) = \mathcal{J}(\alpha, -\frac{1}{2}),$$

where $\mathcal{J}(a,b)$ is the classical Jacobi linear functional that satisfies

$$\Big((x^2-1)\mathcal{J}(a,b)\Big)' + \Big(-(a+b+2)x + a - b\Big)\mathcal{J}(a,b) = 0.$$

According to Theorem 1.3, Theorem 2.1 and the fact that the shifted linear functional of a second degree linear functional is also a second degree linear functional, we obtain that $\mathcal{L}(\alpha)$ is a second degree semiclassical linear functional of class $s=1$ if and only if $\alpha = p - \frac{1}{2}$, $p \in \mathbb{N}$. □

Let us now give the polynomial coefficients B, C and D of (1.14) corresponding to these linear functionals. For this, we need the following lemmas.

Lemma 2.7. *We have*

$$(x^2-1)\mathcal{L}(\alpha) = -\mathcal{L}(\alpha+1). \tag{2.39}$$

Proof. The linear functional $\mathcal{L}(\alpha)$ satisfies (2.36). Multiplying by x^2-1, we obtain

$$\Big((x^3-x)\big((x^2-1)\mathcal{L}(\alpha)\big)\Big)' - 2(\alpha+2)\big((x^2-1)\mathcal{L}(\alpha)\big) = 0.$$

Hence (2.39) follows. □

Lemma 2.8. [3] *Let w and u be two regular linear functionals satisfying the following relation:*

$$M(x)w = N(x)u,$$

where $M(x)$ and $N(x)$ are two polynomials.

If w is a second degree linear functional verifying (1.14), then u is also a second degree linear functional and fulfils

$$\tilde{B}(z)S^2(u)(z) + \tilde{C}(z)S(u)(z) + \tilde{D}(z) = 0,$$

with

$$\begin{cases} \tilde{B}(z) = B(z)N^2(z), \\ \tilde{C}(z) = N(z)\{2B(z)[\big(u\theta_0 N\big)(z) - \big(w\theta_0 M\big)(z)] + M(z)C(z)\}, \\ \tilde{D}(z) = B(z)[\big(u\theta_0 N\big)(z) - \big(w\theta_0 M\big)(z)]^2 + M(z)C(z)[\big(u\theta_0 N\big)(z) \\ \qquad\qquad - \big(w\theta_0 M\big)(z)] + M^2(z)D(z). \end{cases}$$

Using Lemma 2.7 and Lemma 2.8 and the fact that $\vartheta = \mathcal{L}(\frac{1}{2})$ satisfies (1.14) with (2.20), the elements B, C and D in (1.14) are given.

Proposition 2.9. *Let us consider $w = \mathcal{L}(p - \frac{1}{2})$ where $p \in \mathbb{N}$. Then, we have*

$$(x^2-1)w = (-1)^{p+1}(x^2-1)^p\vartheta, \qquad p \geq 0,$$

$$\begin{cases} B(z) = z(z^2-1)^2, \\ C(z) = (z^2-1)\{2z\mathcal{X}(z) - 2(-1)^p(iz+1)(z^2-1)^p\}, \\ D(z) = z\mathcal{X}^2(z) - 2(-1)^p(iz+1)(z^2-1)^p\mathcal{X}(z) + 2i(z^2-1)^{2p}, \end{cases}$$

where

$$\mathcal{X}(z) = z + \lambda + (-1)^p\Big(\vartheta\theta_0\big(\xi^2-1)^p\big)\Big)(z).$$

Remark 2.10. 1) From (1.6) and (2.36), we have

$$w = \lambda x^{-1}\mathcal{J}(\alpha,\alpha) + \delta_0. \tag{2.40}$$

The linear functional $\mathcal{J}(\alpha,\alpha)$ has the following integral representation [14]:

$$\langle \mathcal{J}(\alpha,\alpha), f\rangle = \frac{\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}\Gamma(\alpha+1)}\int_{-1}^{1}(1-x^2)^\alpha f(x)dx, \quad f \in \mathcal{P}, \quad \mathcal{R}e(\alpha) > -1. \tag{2.41}$$

Using (2.40) and (2.41), we obtain

$$\langle w, f\rangle = f(0) + \frac{\lambda\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}\Gamma(\alpha+1)} PV\int_{-1}^{1}\frac{(1-x^2)^{\alpha}}{x}f(x)dx, \quad f\in\mathcal{P}, \quad \mathcal{R}e(\alpha) > -1, \tag{2.42}$$

where PV means Cauchy's principal value of the integral.

Therefore, from Theorem 2.5 and (2.42), we have

$$\langle \mathcal{L}(p-\frac{1}{2}), f\rangle = f(0) + \frac{\lambda\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} PV\int_{-1}^{1}\frac{(1-x^2)^{p}}{x\sqrt{1-x^2}}f(x)dx, \quad f\in\mathcal{P}, \quad p\in\mathbb{N}. \tag{2.43}$$

The case $p = 1$ is ϑ.

2) From (2.43), we get for $n \geq 0$:

$$\begin{aligned}\big(\mathcal{L}(p-\frac{1}{2})\big)_{2n+1} &= \langle \mathcal{L}(p-\frac{1}{2}), x^{2n+1}\rangle \\ &= \frac{\lambda\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}\int_{-1}^{1}(1-x^2)^{p-\frac{1}{2}}x^{2n}dx \\ &= \Big(\frac{\lambda\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}\Big)\Big(\frac{\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{\Gamma(n+p+1)}\Big) \\ &= \frac{\lambda\Gamma(p+1)\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+p+1)}.\end{aligned}$$

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