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NEW OPTICAL SOLUTIONS FOR THE WU-ZHANG SYSTEM WITH TIME FRACTIONAL CONFORMABLE DERIVATIVE

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Abstract. In this paper, the sine-Gordon expansion method is implemented to obtain new explicit solutions for the nonlinear Wu-Zhang system with a time-fractional conformable derivative. The solutions constructed are plotted with the Maple software and expressed by three types of functions: hyperbolic function solution, exponential function solution and trigonometric function solution. The nonlinear fractional partial differential equation is converted into an ordinary differential equation of integer order. This method is used to solve a fractional Wu-Zhang system. These solutions might be important and highly useful in various scientific fields. It is shown that this method is very efficient for constructing exact solutions of nonlinear fractional partial differential equations.

Реалізовано метод розширення синус-Гордона для отримання нових явних розв'язків для нелінійної системи Ву-Жанга із дробове-конформною похідною за часом. Отримані розв'язки будуються за допомогою програмного забезпечення Maple і виражаються трьома типами функцій: гіперболічними функціями, показниковими функціями та тригонометричними функціями. Нелінійне диференціальне рівняння з дробовими похідними перетворюється в звичайне диференціальне рівняння з цілим порядком. Цей метод використовується для розв'язку системи У-Чжан з дробовими похідними. Рішення можуть бути важливими і дуже корисними у різних галузях науки. Показано, що це метод є дуже ефективним для побудови точних розв'язків нелінійних рівнянь з дробовими похідними.

1. Introduction

Fractional calculus has attracted great interest and it has been considered as a powerful tool to model many physical phenomena in various scientific areas such as physics, fluid mechanics, chemistry, biology and mathematical physics. The same importance and interest are given to fractional partial differential equations, due to their applications in various branches of nonlinear sciences including mechanics, electrodynamics, elasticity and other applications. Consequently, many authors tried to solve these equations through several efficient techniques, such as homotopy perturbation Sumudu transform technique [1, 2, 3, 4], $\tan(\phi(\xi)/2)-$ expansion method [13], Riccati equation expansion technique [16], Lie symmetry method [6, 7], Adomian decomposition technique [5], homotopy perturbation technique [14], generalized trigonometry functions [15], Jacobian elliptic function technique [17] and extended Jacobian elliptic function technique [18].

In the current paper, we use the effective sine-Gordon expansion method to construct a new exact solution for the Wu-Zhang system [25, 26, 27, 28] with a time-fractional conformable derivative.

On the other hand, the following sinh-Gordon equation

$$\frac{\partial^2 u}{\partial x \partial t} = \alpha \sinh u, \tag{1.1}$$

arises in several scientific fields, where α is a constant.

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Using the wave transformation

$$u(x,t) = U(\xi), \qquad \xi = \mu\,(x + y - \lambda t)\,,$$

equation (1.1) becomes an ordinary differential equation as follows:

$$\frac{\partial^2 U}{\partial \xi^2} = -\frac{\alpha}{\mu^2\lambda}\sinh U, \tag{1.2}$$

where λ and μ are respectively the wave speed and wave number.

Now, multiplying both sides of (1.2) by the first order derivative of U, we get

$$\frac{\partial U}{\partial \xi} \times \frac{\partial^2 U}{\partial \xi^2} = -\frac{\alpha}{\mu^2\lambda}\sinh U \times \frac{\partial U}{\partial \xi},$$

which implies

$$\frac{\partial U}{\partial \xi} \times \frac{\partial^2 U}{\partial \xi^2} = -\frac{2\alpha}{\mu^2\lambda}\sinh\left(\frac{1}{2}U\right) \times \cosh\left(\frac{1}{2}U\right) \times \frac{\partial U}{\partial \xi}, \tag{1.3}$$

by integrating (1.3), we obtain

$$\left(\frac{d}{d\xi}\frac{1}{2}U\right)^2 = -\frac{\alpha}{\mu^2\lambda}\sinh^2\left(\frac{1}{2}U\right) + c, \tag{1.4}$$

where c is a constant.

Taking into consideration

$$c = 0, \quad \alpha = -\mu^2\lambda, \quad \frac{1}{2}U = w,$$

equation (1.4) becomes

$$\frac{dw(\xi)}{d\xi} = \sinh w(\xi).$$

The Jacobi elliptic function solutions are obtained by converting equation (1.2) into

$$\frac{d^2 w}{d\xi^2} = \frac{1}{2}\sinh 2w, \tag{1.5}$$

with the assumptions $U = 2w$ and $\alpha = -\mu^2\lambda$. Equation (1.5) takes the form:

$$\left(\frac{dw}{d\xi}\right)^2 = \sinh^2 w + c, \tag{1.6}$$

which can be used in the implemented method, where c is a constant of integration. Therefore, the solutions of (1.6) are as follows:

$$\sinh\left[w(\xi)\right] = cs(\xi; m), \tag{1.7}$$

$$\cosh\left[w(\xi)\right] = ns(\xi; m), \tag{1.8}$$

where m is a Jacobian elliptic functions module.

$$cs(\xi; m) = \frac{cn(\xi; m)}{sn(\xi; m)}, \quad ns(\xi; m) = \frac{1}{sn(\xi; m)},$$

with the properties

$$\frac{dcs(\xi; m)}{d\xi} = -ns(\xi; m)ds(\xi; m), \quad \frac{dns(\xi; m)}{d\xi} = -cs(\xi; m)ds(\xi; m).$$

Inserting (1.7) and (1.8) into (1.6) shows that the constant c must satisfy

$$c = 1 - m^2. \tag{1.9}$$

The rest of this article is arranged as follows. In Section 2, we provide some fundamental properties of the fractional conformable derivative. Section 3, is devoted to the main steps of the sinh-Gordan expansion method. In Section 4, we apply this technique to construct new explicit solutions of the fractional Wu-Zhang system with conformable derivative. Finally, some concluding remarks are given in Section 5.

2. Conformable derivative properties

. There are various definitions of fractional derivative [19, 20, 21, 22, 29]. In the last years, the new definition called fractional conformable derivative is proposed by Khalil and all. [23]. In this section, we give its properties.

Definition 2.1. The conformable derivative of order α for a function $f:[0,\infty)\to R$ is defined as

$$T_\alpha(f)(t)=\lim_{\epsilon\to 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon},$$

where $t>0,\alpha\in(0,1)$.

Now, we provide some properties of this novel fractional derivative:

- $T_\alpha(\gamma f+\beta g)=\gamma T_\alpha(f)+\beta T_\alpha(g)$ for all real constant γ and β,
- $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$,
- $T_\alpha(t^p)=\alpha t^{\alpha-p}$ for all α,
- $T_\alpha\left(\frac{g}{f}\right)=\frac{fT_\alpha(g)-gT_\alpha(f)}{f^2}$,
- $T_\alpha(C)=0$, where C is a constant.

Moreover, the differentiability of f implies that $T_\alpha(f)=t^{1-\alpha}\frac{df}{dt}(t)$.

Theorem 2.2. *Suppose that $f:[0,\infty)$ is differentiable and conformable differerentiable of order α and the function g is also differentiable. Then*

$$T_\alpha(fog)=t^{1-\alpha}g^{'}(t)f^{'}(g(t)).$$

3. Description of the method

The sinh-Gordon equation expansion technique [24], is highly efficient for finding new explicit solutions of engineering and physical fractional problems appearing in various scientific areas. This method is based on equation (1.5) or equation (1.6) and it will be described as follows.

Consider the following equation with the fractional time-conformable derivative:

$$\mathbf{N}\left(u,T_t^\alpha u,T_x^\alpha u,T_y^\alpha u,\ldots\right)=0. \tag{3.10}$$

Using the transformation

$$u(x,y,t)=U(\xi),\quad \xi=\mu\left(\frac{x^\alpha}{\alpha}+\frac{y^\alpha}{\alpha}-\lambda\frac{t^\alpha}{\alpha}\right),$$

equation (3.10) is transformed into an ordinary differential equation,

$$\mathbf{Q}\left(U,U^{'},\mu U^{'},-\lambda U^{'},U^{''},\mu^2U^{''},\ldots\right)=0. \tag{3.11}$$

Now, we suppose that a solution for (3.11) takes the following form:

$$U(w(\xi))=A_0+\sum_{i=1}^{n}\cosh^{i-1}w\left[A_i\sinh w+B_i\cosh w\right], \tag{3.12}$$

where $w=w(\xi)$ satisfies (1.5) or (1.6) and (1.9), $A_i(i=0,1,2,\ldots,n)$, $B_i(i=0,1,2,\ldots,n)$, are constants to be found later.

We apply the balance principle by taking nonlinear terms and the higher derivative in equation (3.11) to find the value of the integer n.

Now, put the coefficients of $\sinh^i w\cosh^j w$ that have the same power to be zero, to obtain a system of equations including unknowns $\mu,\lambda,A_i(i=0,1,2,\ldots,n),B_j(j=0,1,2,\ldots,n)$.

Finally, we solve the obtained system by using the Maple software, then we substitute $A_0, A_1, B_1, \ldots, A_n,\, B_n, \mu, \lambda$ in (3.12).

Remark 3.1. . If $m \to 1$,

$$cs(\xi, m) \to csch(\xi), \qquad ns(\xi, m) \to \coth(\xi),$$

If $m \to 0$,

$$cs(\xi, m) \to cot(\xi), \qquad ns(\xi, m) \to csc(\xi).$$

4. Implementation of the method

Consider the nonlinear fractional Wu-Zhang system

$$\begin{cases} T_t^{\alpha} u = -uu_x - v_x, \\ T_t^{\alpha} v = -vu_x - uv_x - \frac{1}{3}u_{xxx}, \end{cases} \tag{4.13}$$

where $\alpha \in (0,1)$, $u = u(x,t)$ is the surface velocity of water and $v = v(x,t)$ is the water elevation. The following wave transformation

$$\begin{cases} u(x,t) = U(\xi), \\ v(x,t) = V(\xi), \\ \xi = x - \lambda \frac{t^{\alpha}}{\alpha}, \end{cases}$$

where λ is a constant, reduces system (4.13) to the following system of ODEs:

$$\begin{cases} T_t^{\alpha} u = -\lambda U^{'}, \\ uu_x = UU^{'}, \\ V_x = V^{'}, \\ T_t^{\alpha} v = -\lambda V^{'}, \\ vu_x = VU^{'}, \\ uv_x = UV^{'}. \\ \frac{1}{3}u_{xxx} = \frac{1}{3}U^{'''}. \end{cases}$$

Then, we obtain the new system as follows:

$$\begin{cases} \lambda U^{'} = UU^{'} + V^{'}, \\ \lambda V^{'} = VU^{'} + UV^{'} + \frac{1}{3}U^{'''}. \end{cases} \tag{4.14}$$

By taking the integration constant to be zero, we integrate the first equation in system (4.14) and obtain

$$V = \lambda U - \frac{U^2}{2}. \tag{4.15}$$

Inserting equation (4.15) into the second equation of system (4.14), we have the following nonlinear differential equation

$$2U^{''} - 3U^3 + 9\lambda U^2 - 6\lambda^2 u = 0. \tag{4.16}$$

Now, balancing the terms $U^{''}$ and U^3, we get $n = 1$. Therefore, the solutions of equation (4.16) take the following form:

$$U(\xi) = A_0 + A_1 \sinh\left(w\left(\xi\right)\right) + B_1 \cosh\left(w\left(\xi\right)\right). \tag{4.17}$$

Substituting (4.17) into (4.16), we get the following family of equations for λ, A_0, A_1 and B_1:

$$\begin{cases} eq1 = -9\,{A_1}^2 B_1 - 3\,{B_1}^3 + 4\,B_1, \\ eq2 = -3\,{A_1}^3 - 9\,A_1\,{B_1}^2 + 4\,A_1, \\ eq3 = -9\,A_0\,{A_1}^2 - 9\,A_0\,{B_1}^2 + 9\,{A_1}^2\lambda + 9\,{B_1}^2\lambda, \\ eq4 = -18\,A_0\,A_1\,B_1 + 18\,A_1\,B_1\,\lambda, \\ eq5 = -9\,{A_0}^2 B_1 + 18\,A_0\,B_1\,\lambda + 9\,{A_1}^2 B_1 - 6\,B_1\,\lambda^2 + 2\,B_1\,c - 4\,B_1, \\ eq6 = -9\,{A_0}^2 A_1 + 18\,A_0\,A_1\,\lambda + 3\,{A_1}^3 - 6\,A_1\,\lambda^2 + 2\,A_1\,c - 2\,A_1, \\ eq7 = -3\,{A_0}^3 + 9\,\lambda\,{A_0}^2 + 9\,A_0\,{A_1}^2 - 6\,\lambda^2 A_0 - 9\,{A_1}^2\lambda. \end{cases}$$

Solving the family of the above equations, we obtain

Case I:

$$\begin{cases} A_0 = \frac{1}{3}\sqrt{6\,m^2+6}, & \lambda = \frac{1}{3}\sqrt{6\,m^2+6}, \\ B_1 = \frac{2}{3}\sqrt{3}, & A_1 = 0. \end{cases} \tag{4.18}$$

By using (4.17) and (4.18), we get

$$U_1(\xi) = -\frac{1}{3}\sqrt{6\,m^2+6} + \frac{2\sqrt{3}}{3}\text{ns}\,(\xi, m)\,, \tag{4.19}$$

and

$$V_1(\xi) = -\frac{2}{3}\,m^2 - \frac{2}{3} + \frac{2}{9}\sqrt{6\,m^2+6}\sqrt{3}ns\,(\xi, m) - \frac{1}{2}\left(-\frac{1}{3}\sqrt{6\,m^2+6} + \frac{2}{3}\sqrt{3}ns\,(\xi, m)\right)^2,$$

where $\xi = x - \lambda\frac{t^\alpha}{\alpha}$.

Case II:

$$\begin{cases} A_0 = \frac{1}{3}\sqrt{6\,m^2-12}, & \lambda = \frac{1}{3}\sqrt{6\,m^2-12}, \\ A_1 = \frac{2}{3}\sqrt{3}, & B_1 = 0. \end{cases} \tag{4.20}$$

From (4.17) and (4.20), we have

$$U_2(\xi) = \frac{1}{3}\sqrt{6\,m^2-12} + \frac{2}{3}\sqrt{3}\text{cs}\,(\xi, m)\,, \tag{4.21}$$

and

$$V_2(\xi) = -\frac{2}{3}\,m^2 + \frac{4}{3} - \frac{2}{9}\sqrt{6\,m^2-12}\sqrt{3}\text{cs}\,(\xi, m) - \frac{1}{2}\left(\frac{1}{3}\sqrt{6\,m^2-12} + \frac{2}{3}\sqrt{3}\text{cs}\,(\xi, m)\right)^2,$$

where $\xi = x - \lambda\frac{t^\alpha}{\alpha}$.

Case III:

$$\begin{cases} A_0 = \frac{1}{3}\sqrt{6\,m^2-3}, & \lambda = \frac{1}{3}\sqrt{6\,m^2-3}, \\ A_1 = \frac{1}{3}\sqrt{3}, & B_1 = \frac{1}{3}\sqrt{3}. \end{cases} \tag{4.22}$$

Using (4.17) and (4.22), we obtain

$$U_3(\xi) = \frac{1}{3}\sqrt{6\,m^2-3} + \frac{1}{3}\sqrt{3}\text{cs}\,(\xi, m) + \frac{1}{3}\sqrt{3}\text{ns}\,(\xi, m)\,, \tag{4.23}$$

and

$$\begin{aligned} V_3(\xi) = {}& \frac{2}{3}\,m^2 - \frac{1}{3} + \frac{1}{9}\sqrt{6\,m^2-3}\sqrt{3}\text{cs}\,(\xi, m) + \frac{1}{9}\sqrt{6\,m^2-3}\sqrt{3}\text{ns}\;(\xi, m) \\ & - \frac{1}{2}\left(\frac{1}{3}\sqrt{6\,m^2-3} + \frac{1}{3}\sqrt{3}\text{cs}\,(\xi, m) + \frac{2}{3}\sqrt{3}\text{ns}\,(\xi, m)\right)^2, \end{aligned}$$

where $\xi = x - \lambda\frac{t^\alpha}{\alpha}$.

If $m = 0$, from (4.19), (4.21) and (4.23), we get new solitary wave solutions of (4.14),

$$\begin{aligned}
U_4(\xi) &= -\frac{1}{3}\sqrt{6} + 2/3\sqrt{3}\csc(\xi), \\
V_4(\xi) &= -\frac{2}{3} + \frac{2}{9}\sqrt{6}\sqrt{3}\csc(\xi) - \frac{1}{2}\left(-\frac{1}{3}\sqrt{6} + \frac{2}{3}\sqrt{3}\csc(\xi)\right)^2, \\
U_5(\xi) &= \frac{2}{3}i\sqrt{3} + \frac{2}{3}\sqrt{3}\cot(\xi), \\
V_5(\xi) &= \frac{4}{3} - \frac{4}{3}i\cot(\xi) - \frac{1}{2}\left(\frac{2}{3}i\sqrt{3} + \frac{2}{3}\sqrt{3}\cot(\xi)\right)^2, \\
U_6(\xi) &= \frac{1}{3}i\sqrt{3} + \frac{1}{3}\sqrt{3}\cot(\xi) + \frac{1}{3}\sqrt{3}\csc(\xi), \\
V_6(\xi) &= -\frac{1}{3} + \frac{1}{3}i\cot(\xi) + \frac{1}{3}i\csc(\xi) - \frac{1}{3}\left(\frac{1}{3}i\sqrt{3} + \frac{1}{3}\sqrt{3}\cot(\xi) + \frac{1}{3}\sqrt{3}\csc(\xi)\right)^2,
\end{aligned} \tag{4.24}$$

where $\xi = x - \lambda\frac{t^\alpha}{\alpha}$.

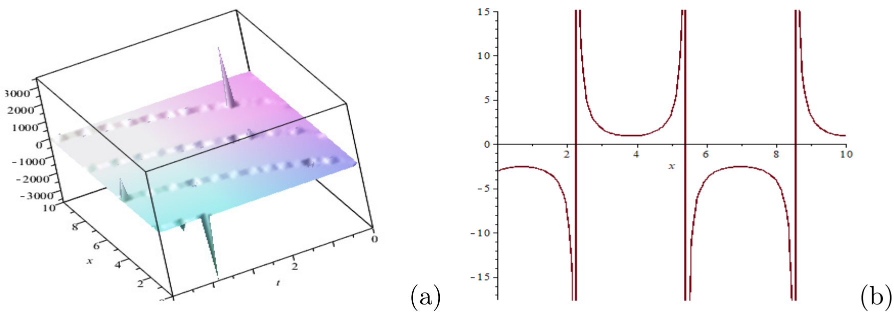

FIGURE 1. Profiles of solutions: (a) 3D solution of $u_4(x,t)$, (b) 2D solution of $u_4(x,y)$,

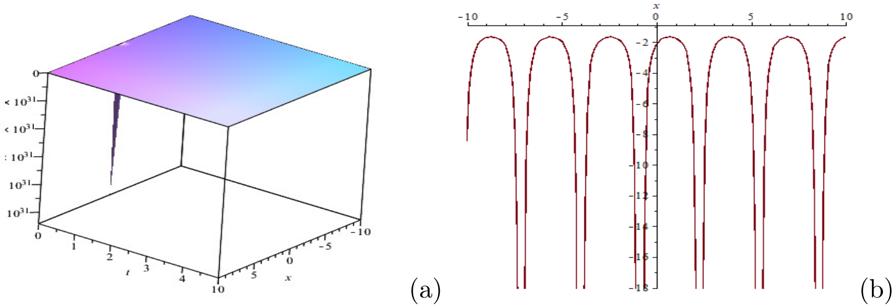

FIGURE 2. Profiles of solutions: (a) 3D solution of $v_4(x,y)$, (b) 2D solution of $v_4(x,y)$,

If $m = 1$, the following solutions of (4.14) are generated from (4.19), (4.21) and (4.24):

$$U_7(\xi) = -\frac{2}{3}\sqrt{3} + \frac{2}{3}\sqrt{3}\coth(\xi),$$

$$V_7(\xi) = -\frac{4}{3} + \frac{4}{3}\coth(\xi) - \frac{1}{2}\left(-\frac{2}{3}\sqrt{3} + \frac{2}{3}\sqrt{3}\coth(\xi)\right)^2,$$

$$U_8(\xi) = \frac{1}{3}i\sqrt{6} + \frac{2}{3}\sqrt{3}csch(\xi),$$

$$V_8(\xi) = -\frac{2}{3} + \frac{2}{9}i\sqrt{6}\sqrt{3}csch(\xi) - \frac{1}{2}\left(-\frac{1}{3}i\sqrt{6} + \frac{2}{3}\sqrt{3}csch(\xi)\right)^2,$$

$$U_9(\xi) = \frac{1}{3}\sqrt{3} + \frac{1}{3}\sqrt{3}csch(\xi) + \frac{1}{3}\sqrt{3}\coth(\xi),$$

$$V_9(\xi) = \frac{1}{3} + \frac{1}{3}csch(\xi) + \frac{1}{3}\coth(\xi) - \frac{1}{2}\left(\frac{1}{3}\sqrt{3} + \frac{1}{3}\sqrt{3}csch(\xi) + \frac{1}{3}\sqrt{3}\coth(\xi)\right)^2,$$

where $\xi = x - \lambda\frac{t^\alpha}{\alpha}$.

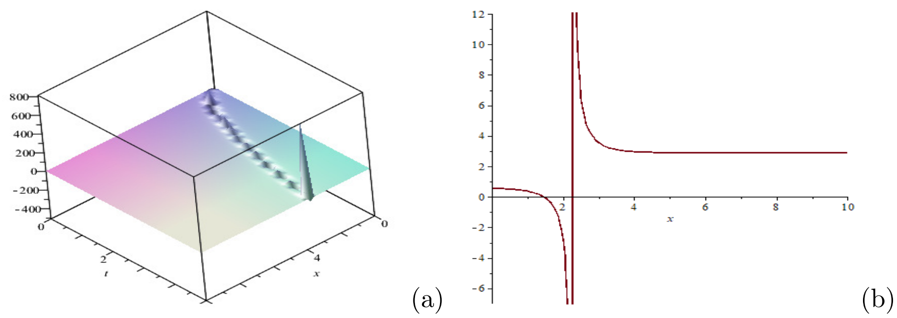

FIGURE 3. Profiles of solutions: (a) 3D solution of $u_7(x,y)$, (b) 2D solution of $u_7(x,y)$,

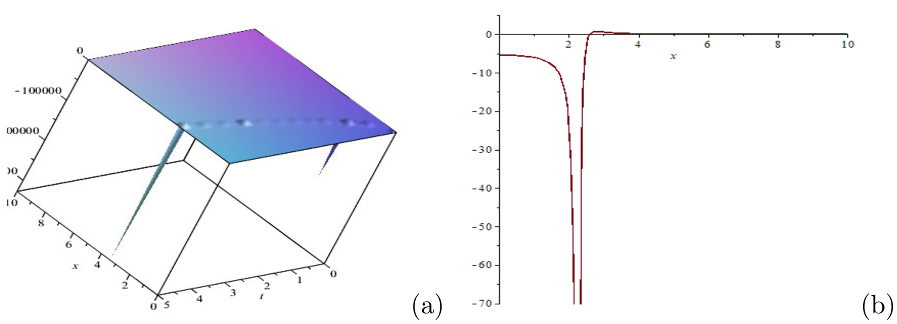

FIGURE 4. Profiles of solutions: (a) 3D solution of $v_7(x,y)$, (b) 2D solution of $v_7(x,y)$,

5. Conclusion

In this work, we have found new explicit solutions to the nonlinear fractional Wu-Zhang system with a time conformable derivative by using the sinh-Gordan equation method. A construction of various kinds of exact solutions for this system such as hyperbolic, exponential and trigonometric solutions has been carried out. These solutions might be very useful in various branches of science. According to the shown results, we can conclude that this method is highly effective, simple to use and can be applied to solve many other nonlinear fractional systems in different domains of science. The method can be also extended to higher-dimensional nonlinear fractional differential equations involving new fractional derivatives.

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