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ON REEB GRAPHS INDUCED FROM SMOOTH FUNCTIONS ON 3-DIMENSIONAL CLOSED MANIFOLDS WHICH MAY NOT BE ORIENTABLE

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Abstract. The *Reeb space* of a smooth function is a topological and combinatorial object. It is important in understanding the manifold. It is a graph defined as the quotient space of the manifold where the equivalence relation is as follows: two points in the manifold are equivalent if and only if they are in a same connected component of a level set. If the function is a *Morse*(*-Bott*) function for example, then this is the graph (*Reeb graph*) whose vertex set is the set of all points containing some singular points in the corresponding connected component of the level set.

The author previously constructed explicit smooth functions on suitable 3-dimensional closed and orientable manifolds whose Reeb graphs are isomorphic to prescribed graphs and whose preimages are of prescribed types. The present paper concerns a variant in the case where the 3-dimensional manifolds may not be non-orientable.

Простір Реба гладкої функції є топологічним і комбінаторним об'єктом. Він грає важливу роль для розуміння многовиду. Він є графом, який визначається як фактор-простір многовиду, де відношення еквівалентності таке: дві точки многовида еквівалентні тоді і тільки тоді, коли вони знаходяться в одному і тому ж зв'язному компоненті поверхні рівня. Якщо функція є *функцією Морса*(*-Ботта*), тоді це є графом (*графом Реба*), множина вершин якого є множиною всіх точок, що містять певні особливі точки у відповідних зв'язних компонентах множини рівнів.

Раніше автор побудував явні гладкі функції на відповідних 3 - вимірних замкнутих і орієнтованих многовидах, графи Реба яких ізоморфні заданим графам і прообрази яких мають задані типи. У цій статті розглядається варіант в випадку, коли 3-мірні многовиди можуть не бути неорієнтованими.

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1. Introduction

1.1. **Reeb spaces, Reeb graphs and Main Problem.** The *Reeb space* of a smooth map c is defined as follows.

For a smooth map $c : X \to Y$, we can define an equivalence relation $\sim_c$ on X as follows: $x_1 \sim_c x_2$ holds if and only if they are in a same connected component of a preimage $c^{-1}(y)$.

Definition 1. The quotient space $W_c := X/\sim_c$ is the *Reeb space* of c.

Hereafter, $q_c : X \to W_c$ denotes the quotient space. We can define a map $\bar{c}$ uniquely by the relation $c = \bar{c} \circ q_c$. For a manifold and a polyhedron X, its dimension is uniquely defined and let $\dim X$ denote this. For a smooth manifold X, T_pX denotes the tangent space at p. A *singular* point of a smooth map $c : X \to Y$ is a point $p \in X$ where the rank of the differential $dc_p : T_pX \to T_c(p)Y$ is smaller than $\min\{\dim X, \dim Y\}$. The *singular set* of c is the set of all singular points of c. For a smooth map c, the *singular value* is a point $c(p)$ which is a value at some singular point p. A *regular value* of the map is a point which is not a singular value in Y.

The Reeb space of a smooth function f which is not so wild is a graph. Reeb spaces are graphs for smooth functions on compact manifolds with finitely many singular values for example ([18]). In the present paper we only concentrate on such smooth functions essentially. One of pioneering papers on Reeb spaces is [15] for example. They have been fundamental and important topological objects and tools in algebraic topological studies and differential topological ones on differentiable manifolds. They inherit topological information such as homology groups and cohomology rings in several cases. See author's works [7, 8, 9, 6, 4, 5], for example, for related expositions. They essentially concentrate on *fold* maps such that preimages of regular values are disjoint unions of spheres. *Fold* maps are higher dimensional variants of so-called Morse functions and the definition of a fold map is introduced in the next section. We present another important problem, which is our Main Problem. Before presenting this, we introduce several terminologies and explanations on graphs. A *graph* $G := (V, E)$ is an object consisting of the *vertex set* V and the *edge set* E. The edge set is a set consisting of a pair of a subset of V consisting of exactly two elements in V and an integer. It is also a so-called *multigraph* with no *loops*. A *vertex* of the graph is an element of the vertex set. An *edge* of the graph is an element of the edge set. If V and E are finite sets, then G is called a *finite* graph. If the following condition is satisfied, then the graph is said to be *connected*: for any two distinct vertices $v_1, v_2 \in V$, there exists a sequence $\{v_j\}_{j=1}^{l_{v_1,v_2}} \subset V$ of length l_{v_1,v_2} satisfying the following conditions for some integer $l_{v_1,v_2} > 1$:

- $v_j \neq v_{j+1}$ for $1 \leq j \leq l_{v_1,v_2} - 1$.
- For the two-element set $V_j := \{v_j, v_{j+1}\}$, there exists an integer $i(j)$ and $(V_j, i(j)) \in E$ for $1 \leq j \leq l_{v_1,v_2} - 1$.

By regarding each edge as a closed interval and each vertex a point in a natural way, we can regard a graph as a topological space. For a connected graph, the topological space is connected and arcwise connected. For a finite graph, it is regarded as a so-called 1-dimensional finite simplicial complex.

Hereafter, we only consider finite and connected graphs as graphs essentially. We can regard such graphs as 1-dimensional finite simplicial complexes and regard as objects in the PL category, or equivalently, the piecewise smooth category. An *isomorphism* between two such graphs G_1 and G_2 is a (PL or piecewise smooth) homeomorphism from G_1 to G_2 mapping the vertex set of G_1 onto the vertex set of G_2. If there exists an isomorphism from a graph G_1 to G_2, then they are said to be *isomorphic*.

Main Problem. For a finite and connected graph with at least one edge, can we construct a smooth function on a (compact) manifold (satisfying some good conditions) whose Reeb graph is isomorphic to the graph?

Paper [20] is pioneering on this topic. In [3, 12, 13, 18], some related studies, among other, were conducted. The author also obtained results in [10] and [11].

Recently Reeb graphs and Reeb spaces also became important in applications of mathematics such as data analysis and visualizations. This problem will play important roles in such scenes, [19] is a related article.

1.2. **Notions and notation for our Main Theorems.** We introduce notions and notation we need. The k-dimensional Euclidean space $\mathbb{R}^k$ including the line ($\mathbb{R} := \mathbb{R}^1$) and the plane ($\mathbb{R}^2$) are simplest smooth manifolds. They are also Riemannian manifolds endowed with the standard Euclidean metrics. The sphere S^k which is centered at the origin of $\mathbb{R}^{k+1}$ and whose radius is 1 is the k-dimensional unit sphere. The disk D^k which is centered at the origin of $\mathbb{R}^k$ and whose radius is 1 is the k-dimensional unit disk.

A *height* function of the unit disk is a Morse function with exactly one singular point p with $i(p) = 0$ in the interior. In other words, a height function is a smooth function having the form $(x_1, \cdots, x_m) \mapsto \pm\Sigma_{j=1}^{m} {x_j}^2 + c$ for suitable coordinates and a constant value $c \in \mathbb{R}$.

Definition 2. Let $m \geq n \geq 1$ be integers. A smooth map from an m-dimensional manifold with no boundary into an n-dimensional manifold with no boundary is said to be a *fold* map if:

(1) for each singular point p, there exists an integer satisfying $0 \leq i(p) \leq \frac{m-n+1}{2}$;
(2) around each singular point p, the map has the form

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{j=1}^{m-i(p)-n+1} {x_{j+n-1}}^2 - \sum_{j=1}^{i(p)} {x_{j+m-i(p)}}^2)$$

for suitable coordinates.

The case where the manifold of the target is the line $\mathbb{R}$ is for *Morse* functions on manifolds with no boundaries. Together with Morse functions, we mainly consider Morse functions on compact smooth manifolds with non-empty boundaries or non-compact manifolds with no boundaries in the present paper.

Proposition 1. *For a fold map in Definition 2, the following properties are satisfied.*

(1) *The integer $i(p)$ is unique for any singular point p and the set of all singular points of an arbitrary fixed $i(p)$ is a smooth regular submanifold of dimension $n-1$ with no boundary. If the manifold of the domain is closed, then the set of the singular points is compact.*
(2) *Furthermore, the map obtained by restricting the original map to the previous $(n-1)$-dimensional submanifold is a smooth immersion.*
(3) *Around each singular point, the fold map is locally represented as the product map of a Morse function and the identity map on a small open neighborhood of the singular point for suitable coordinates. Here the small open neighborhood is chosen in the singular set.*

In this proposition, we call $i(p)$ the *index* of p.

Definition 3. A fold map is said to be *special generic* if the index of a singular point is always 0.

For fundamental theory on singularity theory and differential topological properties of fold maps, see [2, 16, 17, 18] for example.

Definition 4. A continuous real-valued function g on a graph G is said to be *good* if it is injective on each edge.

1.3. **Main Theorems.** In our arguments, for graphs, we consider a graph G with a good function g and an integer valued function r on the edge set E of G. Note that we do not consider the notion of an isomorphism and the notion of isomorphic graphs for such graphs with functions. The triplet (G, g, r) is called a graph *associated with a good function g and a family r of integer labels to edges.*

Main Theorem 1. *Let $G := (V, E)$ be a connected and finite graph satisfying $E \neq \emptyset$. Let (G, g, r_G) be a graph associated with a good function g and a family r_G of integer labels. Assume also that for each edge $e \in E$, either of the following two conditions holds:*

- $r_G(e) \geq 0$.
- $r_G(e)$ *is even and negative.*

Then there exist a 3-dimensional closed, connected and orientable manifold M and a smooth function $f : M \to \mathbb{R}$ enjoying the following four properties.

(1) *The Reeb graph W_f of f is isomorphic to G and we can take a suitable isomorphism $\phi : W_f \to G$.*
(2) *For each point $\phi(p) \in G$ ($p \in W_f$) in the interior of an arbitrary edge e, the preimage ${q_f}^{-1}(p)$ is a closed, connected, and orientable surface of genus $r_G(e)$ if $r_G(e) \geq 0$ and a non-orientable one of genus $-r_G(e)$ if $r_G(e) < 0$.*
(3) *For a point $p \in M$ mapped by q_f to a vertex $v_p := q_f(p) \in W_f$, we have $f(p) = g \circ \phi(v_p)$.*
(4) *Around each singular point p, locally, the function has either of the following forms.*
   (a) *Assume that at the vertex $q_f(p)$, g does not have a local extremum. Then the local function is a Morse function.*
   (b) *Assume that the vertex $q_f(p)$ is of degree 1, that at the vertex, g has a local extremum and that at the edge e containing the vertex, $r_G(e) = 0$ is satisfied. Then the local function is a height function.*
   (c) *Assume that the vertex $q_f(p)$ is of degree greater than 1 and that g has a local extremum there. Then the local function is the composition of a Morse function with a height function.*
   (d) *Assume that the vertex $q_f(p)$ is of degree 1 and that g has a local extremum there. Assume also that at the edge e containing the vertex, $r_G(e) = -2$ is satisfied. In this case, the local function is obtained in the following way.*
      (i) *Define a smooth function h of a well-known type and enjoying the following properties: $h(x) := 0$ for $x \leq 0$ and $h(x) := \epsilon_h e^{-\frac{1}{x}}$ for $x > 0$ with a real number $\epsilon_h > 0$.*
      (ii) *Consider a special generic map $f_p : S^1 \tilde{\times} S^2 \to \mathbb{R}^2$ on the total space $S^1 \tilde{\times} S^2$ of a non-trivial smooth bundle over a circle whose fiber is diffeomorphic to the 2-dimensional unit sphere enjoying the following three properties.*
         (A) *The restriction to the singular set is a smooth embedding.*
         (B) *The image of the singular set is the disjoint union of two circles centered at $(0, 0)$ and of radii $\frac{1}{2}$ and $\frac{3}{2}$, respectively.*
         (C) *The image of the map is the closure of the domain surrounded by the two embedded circles just before.*
      (iii) *We consider the restriction of the previous map to the preimage of the unit disk $D^2 \subset \mathbb{R}^2$. After that we compose this map with the height function mapping (x_1, x_2) to ${x_1}^2 + {x_2}^2$.*

(iv) *We compose the resulting function with a smooth function mapping a real number* x *to* $\pm h(x-\frac{1}{4})$. h *is as before. Furthermore, here the domain is restricted suitably.*

(v) *We consider the sum of the previous function and a (suitable) real-valued constant function. This is our desired local function.*

Furthermore, for example, for this local function, the preimage of $q_f(p)$ *is a circle.*

(e) *Assume that the vertex* $q_f(p)$ *is of degree* 1 *and that* g *has a local extremum there. Assume also that at the edge* e *containing the vertex,* $r_G(e) \neq 0, -2$ *is satisfied. In this case, the composition of a fold map into the interior of the unit disk with a height function is a desired local function. Furthermore, the fold map is constructed as a map enjoying the following properties.*

(i) *Each connected component of the preimage of each point in the interior of the unit disk of the target is either of the following three.*

(A) *A circle. This case is for connected components of the preimage of each regular value or connected components of the preimage of some singular value having no singular points.*

(B) *The bouquet of two circles. This case is for some connected component of the preimage of some singular value.*

(C) *A* 1*-dimensional polyhedron obtained by an iteration of identifying two points whose small open neighborhoods are homeomorphic to an open interval in distinct connected* 1*-dimensional polyhedra, starting from finitely many circles. This case is for some connected component of the preimage of some singular value.*

(ii) *For the singular set of the fold map, remove suitable finitely many singular points and consider the restriction to the resulting* 1*-dimensional manifold. Then we have an embedding. The finitely many singular points removed before are all in the preimage of* $(0,0)$ *(where the space of the target of the presented fold map is considered).*

As Remark 1 says, it was announced that we could immediately show this. However, we need additional arguments. As another result, we also show the following result. Hereafter, let $\sharp X$ denote the size of a finite set X.

Main Theorem 2. *Let* $G := (V, E)$ *be a connected and finite graph* $E \neq \emptyset$. *Let* (G, g, r_G) *be a graph associated with a good function* g *and a family* r_G *of integer labels. We also assume the following two conditions.*

- *For each vertex* v*, the difference* $D_v := \sharp A_{\mathrm{up},v} - \sharp A_{\mathrm{low},v}$ *of the sizes of the two following finite sets is even.*
  - *The set* $A_{\mathrm{up},v}$ *of all edges satisfying the following conditions.*
    - $e \in A_{\mathrm{up},v}$ *contains* v *as a point.*
    - *The restriction of the function* g *to* $e \in A_{\mathrm{up},v}$ *has the minimum at* v.
    - $r_G(e)$ *is odd and negative for* $e \in A_{\mathrm{up},v}$.
  - *The set* $A_{\mathrm{low},v}$ *of all edges satisfying the following conditions.*
    - $e \in A_{\mathrm{low},v}$ *contains* v *as a point.*
    - *The restriction of the function* g *to* $e \in A_{\mathrm{low},v}$ *has the maximum at* v.
    - $r_G(e)$ *is odd and negative for* $e \in A_{\mathrm{low},v}$.
- *Let* v *be an arbitrary vertex of* G *satisfying* $D_v \neq 0$ *such that* g *does not have a local extremum at* v.
  - *Let* $D_v > 0$. *Let* $B_{\mathrm{low},v}$ *denote the set of all edges satisfying the following conditions.*
    - $e \in B_{\mathrm{low},v}$ *contains* v *as a point.*

- $*$ *The restriction of the function* g *to* $e \in B_{\mathrm{low},v}$ *has the maximum at* v.
- $*$ $r_G(e)$ *is negative for* $e \in B_{\mathrm{low},v}$.

For $e \in B_{\mathrm{low},v}$, *define* ${r_G}'(e)$ *as the greatest even number satisfying* ${r_G}'(e) \leq |r_G(e)|$. *The sum* $\Sigma_{e \in B_{\mathrm{low},v}} {r_G}'(e)$ *satisfies* $D_v \leq \Sigma_{e \in B_{\mathrm{low},v}} {r_G}'(e)$.

– *Let* $D_v < 0$. *Let* $B_{\mathrm{up},v}$ *denote the set of all edges satisfying the following conditions.*
  - $*$ $e \in B_{\mathrm{up},v}$ *contains* v *as a point.*
  - $*$ *The restriction of the function* g *to* $e \in B_{\mathrm{up},v}$ *has the minimum at* v.
  - $*$ $r_G(e)$ *is negative for* $e \in B_{\mathrm{up},v}$.

  For each edge $e \in B_{\mathrm{up},v}$, *define* ${r_G}'(e)$ *as the greatest even number satisfying* ${r_G}'(e) \leq |r_G(e)|$. *The sum* $\Sigma_{e \in B_{\mathrm{up},v}} {r_G}'(e)$ *satisfies* $|D_v| \leq \Sigma_{e \in B_{\mathrm{up},v}} {r_G}'(e)$.

Then there exist a 3*-dimensional closed and connected manifold* M *and a smooth function* $f : M \to \mathbb{R}$ *enjoying the properties (1)–(4) in Main Theorem 1.*

1.4. **Additional comments.** Paper [10] concerns cases where the 3-dimensional manifolds and preimages of regular values are orientable. Main Theorems extend this and this says that the author has a good idea for the present problem (Problem 3). Paper [18] generalizes the main result of [10] partially and can be regarded as a paper motivated by it. This also generalizes Main Theorems partially. More precisely, [18] generalizes the manifold assigned to each edge to a general compact (closed) manifold. For edges containing a vertex, the manifolds satisfy a condition from theory of cobordisms of manifolds there. On the other hand, explicit classes of smooth functions are not considered and used functions are essentially ones obtained by integrating smooth functions which are not-real analytic.

In [10] and the present paper, explicit functions with mild singularities are used.

Moreover, other related studies introduced before are essentially ones for smooth functions on surfaces or Morse functions such that preimages of regular values are disjoint unions of spheres.

We prove Main Theorems in the next section.

2. Proofs of Main Theorems.

A proof of Main Theorem 1. We need to add several arguments to the proof of Theorem 1 of [10].

First we introduce several smooth functions. Let $a < b$ be real numbers and $\{t_j\}_{j=1}^{l}$ be a sequence of real numbers in (a,b) of length $l \geq 0$ such that either of the following holds: $t_{j_1} \leq t_{j_2}$ for any pair $j_1 < j_2$ or $t_{j_1} \geq t_{j_2}$ for any pair $j_1 < j_2$). Let $\tilde{f}_{\mathrm{P},\{t_j\},[a,b]}$ denote a function satisfying the following six.

- ◦ The manifold of the domain is diffeomorphic to the manifold obtained in the following way.
  - – Prepare a manifold represented as a connected sum of l copies of the real projective plane.
  - – Remove the interiors of two copies of the 2-dimensional unit disk smoothly and disjointly embedded in the previous manifold.
- ◦ The Reeb space is (PL) homeomorphic to a closed interval.
- ◦ The preimage of $\{a,b\}$ and the boundary of the manifold of the domain agree.
- ◦ There exist exactly l singular points. We can take the sequence $\{s_j\}_{j=1}^{l}$ of all singular points so that $\tilde{f}_{\mathrm{P},\{t_j\},[a,b]}(s_j) = t_j$ holds.
- ◦ Around each singular point, the function is locally a Morse function.
- ◦ Preimages of regular values are always circles.

Let $\tilde{f}_{\mathrm{K},\mathrm{up},t,[a,b]}$ ($\tilde{f}_{\mathrm{K},\mathrm{low},t,[a,b]}$) denote a function on a 3-dimensional compact and connected manifold satisfying the following:

- The function has exactly one singular point and t is the singular value. The function is a Morse function there.
- The Reeb space is (PL) homeomorphic to a closed interval.
- The preimage of $\{a,b\}$ and the boundary of the manifold of the domain agree.
- The preimage of a (resp. b) is diffeomorphic to the 2-dimensional unit sphere.
- The preimage of b (resp. a) is diffeomorphic to the Klein bottle.

We also need several fundamental arguments on relationship among Morse functions and singular points, and the manifolds, to know the existence of these functions. The theory of so-called *handles* will help us to understand the structures of the presented functions and other smooth functions in the present paper. In the proof of Main Theorem 2, we present the theory of handles in short with a short proof on the construction of another new function $\tilde{f}_{\mathrm{P},0,t,[a,b]}$. For systematic theory on relationship between handles and singular points of Morse functions, see [14] for example.

We come back to the proof. Besides the original proof of Theorem 1 of [10], we need construction of a local function around a vertex $v \in V \subset G$ in the following four cases.

Case 1 We consider the following case.

- v is contained in some edge e satisfying $r_G(e) < 0$.
- At v, g does not have a local extremum.

We consider a small regular neighborhood $N(v) \subset G$. We can regard this as a graph whose edge set consists of closed intervals being subsets of edges of G and containing v. Furthermore, this satisfies the following.

- Mutually distinct edges of the graph $N(v)$ are always closed intervals in mutually distinct edges of G.
- For each edge of G containing v, there exists a unique edge of $N(v)$ being also a closed interval in the edge of G.

By using methods of Michalak ([13]) and the author ([10]), we construct a local smooth function $\tilde{f_{v,0}}$ onto a small closed interval $[g(v)-\epsilon_v, g(v)+\epsilon_v]$ enjoying the following five properties where $\epsilon_v > 0$ is a sufficiently small real number.

- $\tilde{f_{v,0}}$ is a Morse function with exactly one singular value $g(v)$.
- The preimage of $\{g(v)-\epsilon_v, g(v)+\epsilon_v\}$ and the boundary of the manifold of the domain agree.
- The Reeb space $W_{\tilde{f_{v,0}}}$ has the structure of a graph such that the vertex set consists of the following two.
  - All elements in $(\tilde{f_{v,0}})^{-1}(\{g(v)-\epsilon_v, g(v)+\epsilon_v\})$.
  - The unique element in $(\tilde{f_{v,0}})^{-1}(g(v))$.

  There exists a suitable isomorphism $\phi_{v,0}$ from the graph $W_{\tilde{f_{v,0}}}$ onto $N(v)$ mapping $(\tilde{f_{v,0}})^{-1}(g(v)) \subset W_{\tilde{f}_{v,0}}$ to the one-point set $\{v\}$.
- Consider the edge $e' \ni v$ of $N(v)$ contained in the edge e of G. For each point $x \in e' - \{v\}$, $(\phi_{v,0} \circ q_{\tilde{f_{v,0}}})^{-1}(x)$ is a closed, connected and orientable surface of genus $r_G(e)$ if $r_G(e) \geq 0$.
- Consider the edge $e' \ni v$ of $N(v)$ contained in the edge e of G. For each point $x \in e' - \{v\}$, $(\phi_{v,0} \circ q_{\tilde{f_{v,0}}})^{-1}(x)$ is diffeomorphic to the 2-dimensional unit sphere if $r_G(e) < 0$.

We change this function to a desired local smooth function $\tilde{f_v}$. We can choose finitely many trivial smooth bundles over the image $[g(v)-\epsilon_v, g(v)+\epsilon_v]$ whose fibers are diffeomorphic to the unit disk D^2 disjointly and which are apart from the singular set of the function. We apply such arguments throughout the present paper. We can do this is due to the structures of Morse functions with a relative version of an important theorem in [1]. It says that smooth submersions with compact preimages give smooth bundles. We can attach new functions instead. We do this procedure according to the following rules and steps around each edge of $N(v)$.

- Around an edge $e' \ni v$ of $N(v)$ contained in the uniquely defined edge e of G satisfying $r_G(e) < 0$ and $\tilde{\tilde{f}}_{v,0}(e') = [g(v)-\epsilon_v, g(v)]$.
  - We can choose $\frac{|r_G(e)|}{2}$ trivial smooth bundles over the image $[g(v)-\epsilon_v, g(v)+\epsilon_v]$ whose fibers are diffeomorphic to the unit disk D^2 disjointly and which satisfy the following conditions. We remove them.
    * They are apart from the singular set of the function.
    * For the total space $B_{e',j}$ of each of the $\frac{|r_G(e)|}{2}$ trivial smooth bundles before, the image of the restriction of $q_{\tilde{f_{v,0}}}$ to $B_{e',j} \bigcap {\tilde{f_{v,0}}}^{-1}([g(v)-\epsilon_v, g(v)])$ is e'.
  - We prepare a function $\tilde{f}_{\mathrm{K,low},g(v),[g(v)-\epsilon_v,g(v)+\epsilon_v]}$ before. We can choose a trivial smooth bundle over $[g(v)-\epsilon_v, g(v)+\epsilon_v]$ whose fiber is diffeomorphic to the unit disk D^2 and which is apart from the singular set. As before, we can do this due to the structure of the function with the relative version of the important theorem. We remove this and we have a new smooth function.
  - We attach $\frac{|r_G(e)|}{2}$ copies of the previously constructed function here to the function obtained first here. We glue the functions one after another preserving the values at all points of the manifolds of the domains.
- Around an edge $e' \ni v$ of $N(v)$ contained in the uniquely defined edge e of G satisfying $r_G(e) < 0$ and $\tilde{\tilde{f}}_{v,0}(e') = [g(v), g(v)+\epsilon_v]$.
  - We can choose $\frac{|r_G(e)|}{2}$ trivial smooth bundles over the image $[g(v)-\epsilon_v, g(v)+\epsilon_v]$ whose fibers are diffeomorphic to the unit disk D^2 disjointly and which satisfy the following conditions. We remove them.
    - They are apart from the singular set of the function.
    - For the total space $B_{e',j}$ of each of the $\frac{|r_G(e)|}{2}$ trivial smooth bundles before, the image of the restriction of $q_{\tilde{f_{v,0}}}$ to $B_{e',j} \bigcap {\tilde{f_{v,0}}}^{-1}([g(v), g(v)+\epsilon_v])$ is e'.
  - We prepare a function $\tilde{f}_{\mathrm{K,up},g(v),[g(v)-\epsilon_v,g(v)+\epsilon_v]}$ before. We can choose a trivial smooth bundle over $[g(v)-\epsilon_v, g(v)+\epsilon_v]$ whose fiber is diffeomorphic to the unit disk D^2 and which is apart from the singular set. We can do this due to the structure of the function with the relative version of the important theorem as before. We remove this and we have a new smooth function.
  - As the first case here, we attach $\frac{|r_G(e)|}{2}$ copies of the previously constructed function in the present case to the function obtained first in the present case. We glue the functions one after another preserving the values at all points of the manifolds of the domains.

Thus we have a desired local smooth function onto $[g(v)-\epsilon_v, g(v)+\epsilon_v]$.

Hereafter, we omit rigorous notation and expositions on identifications of original abstract graphs and Reeb graphs of local or global smooth functions. We naturally identify them in similar arguments.

Case 2 We consider the following case.

- v is of degree greater than 1.
- v is contained in some edge e satisfying $r_G(e) < 0$.
- At v, the function g has a local extremum.

As in [10], we can construct a local function as Case 1 and we compose this with a suitable smooth embedding into $\mathbb{R}^2$. We can take the smooth embedding before as one enjoying the following properties and we choose this as the embedding.

- The image is a parabola $\{(t, g(v) \pm t^2) \mid t \in [-\epsilon_v, \epsilon_v]\} \subset \mathbb{R}^2$ for a sufficiently small positive number $\epsilon_v > 0$.
- The embedding maps the unique singular value of the local function to $(0, g(v))$.

We compose the resulting map into the $\mathbb{R}^2$ with the canonical projection to the second component. By doing the construction of the smooth function like one in Case 1 suitably first, we have a desired local function. We choose the sign $+$ $(-)$ according to the condition that $g(v)$ is the local minimum (resp. maximum) of g.

Case 3 We consider the following case.

- v is of degree 1.
- v is contained in the edge e satisfying $r_G(e) = -2$.
- At v, the function g has a local extremum.

We can have a special generic map on the total space of a non-trivial smooth bundle over a circle whose fiber is diffeomorphic to the 2-dimensional unit sphere into $\mathbb{R}^2$. We can construct one enjoying the following properties.

- The restriction to the singular set is a smooth embedding.
- The image of the singular set is the disjoint of the following two circles: they are centered at $(0,0)$ and their radii are $\frac{1}{2}$ and $\frac{3}{2}$, respectively.
- The image of the map is the closure of the domain surrounded by the previous two embedded circles.

We consider the restriction of the map to the preimage of the unit disk D^2 in $\mathbb{R}^2$ and compose this with a height function mapping (x_1, x_2) to ${x_1}^2 + {x_2}^2$. We compose this with a smooth function mapping x to $\pm h(x - \frac{1}{4})$ with the domain restricted suitably. Here the sign $+$ $(-)$ is chosen according to the condition that $g(v)$ is the local minimum (resp. maximum) of g. We consider the sum of the resulting function and the constant function whose values are always $g(v)$. We thus have a desired local function.

Case 4 We consider the following case.

- v is of degree 1.
- v is contained in the edge e satisfying $r_G(e) \neq -2$ and $r_G(e) = -2(l_0 + 1)$ for a positive integer $l_0 > 0$.
- At v, the function g has a local extremum.

For this case, see also our proof of Theorem 1 in [10]. Arguments there are somewhat similar. However, functions and maps are different in preimages for examples.

Let $\epsilon_v > 0$ be a small real number. Let P be the surface of the domain of $\tilde{f}_{\mathrm{P},\{t_j\},[-\epsilon_v,\epsilon_v]}$ such that $\{t_j\}_{j=1}^{l_0}$ is of length l_0 and defined in the following way.

- For $l_0 = 1$, let $t_j = t_1 := 0$.
- For $l_0 > 1$, let $t_1 := -\frac{\epsilon_v}{2}$ and $t_{l_0} := \frac{\epsilon_v}{2}$ and $t_j := -\frac{\epsilon_v}{2} + (j-1)\frac{\epsilon_v}{l_0 - 1}$ for $1 < j < l_0$.

We can define $\tilde{f}_{\mathrm{P},\{t_j + \frac{t+1}{2}(t_{l_0-j+1} - t_j)\}_{j=1}^{l_0},[-\epsilon_v,\epsilon_v]}$ for $-1 \leq t \leq 1$ and set

$$F_{v,t} := \tilde{f}_{\mathrm{P},\{t_j + \frac{t+1}{2}(t_{l_0-j+1} - t_j)\},[-\epsilon_v,\epsilon_v]}$$

for $-1 \leq t \leq 1$ suitably to define a smooth deformation $F_v : P \times [-1,1] \to \mathbb{R} \times [-1,1]$ enjoying the following properties.

- $F_{v,t}(x) = F_v(x,t)$ for $(x,t) \in P \times [-1,1]$.
- The singular set consists of all points of the form $(s_{t,j}, t)$ where $s_{t,j}$ is a singular point of the Morse function $F_{v,t}$.
- On the interior of $P \times [-1,1]$, represented as Int $P \times (-1,1)$, the map F_v is a fold map.
- For the fold map before, the restriction to the subset of the singular set obtained by removing all singular points in the preimage of $(0,0)$ is a smooth embedding.

We consider the restriction of this map to the preimage of the following open disk: the open disk is centered at $(0,0)$ and its radius is $\frac{1}{2}$. We compose the restriction with a height function mapping (x_1, x_2) to $\pm({x_1}^2 + {x_2}^2)$. We consider the sum of the resulting function and a constant function whose values are always $g(v)$. Thus we have a desired local function.

For a sufficiently small real number $x_{g(v)} > 0$, the preimage of $g(v) \pm x_{g(v)}$ is diffeomorphic to a closed, connected and non-orientable surface obtained by identifying the two connected components of the boundary of the surface of the domain of $\tilde{f}_{\mathrm{P},\{t_j\},[a,b]}$ with $l := 2l_0$. The genus is $2 + 2l_0$. The sign $+$ $(-)$ is chosen according to the condition that $g(v)$ is the local minimum (resp. maximum) of g.

For this exposition, see also [18]. This is on differential topological theory on structures around preimages of smooth maps whose codimensions are -1. This is theory of so-called *fibers*.

This completes the proof. □

Remark 1. In an earlier version of [10], it was announced that we immediately have a similar answer to Problem 3 of [10]. However, the answer is not so trivial. In fact we need Case 3 in our proof for example.

We prepare two important lemmas.

Lemma 1. *Let $m > 1$ be an integer. Suppose there exist two smooth functions $\tilde{f_1}$ and $\tilde{f_2}$ on m-dimensional smooth, compact, and connected manifolds onto a small closed interval $[a-\epsilon_a, a+\epsilon_a]$ enjoying the following three properties with a and $\epsilon_a > 0$ being real numbers.*

- *$\tilde{f_1}$ and $\tilde{f_2}$ are both Morse functions with exactly one singular value a.*
- *The preimages of $a - \epsilon_a$ are diffeomorphic to closed manifolds $F_{\mathrm{low},1}$ and $F_{\mathrm{low},2}$ respectively. The preimages of $a + \epsilon_a$ are diffeomorphic to closed manifolds $F_{\mathrm{up},1}$ and $F_{\mathrm{up},2}$ respectively.*
- *The preimages ${\tilde{\bar{f_1}}}^{-1}(a)$ and ${\tilde{\bar{f_2}}}^{-1}(a)$ are both one-point sets.*

Then we have a smooth function $\tilde{f_{1,2}}$ on an m-dimensional smooth, compact, and connected manifold onto $[a-\epsilon_a, a+\epsilon_a]$ enjoying the following properties.

(1) *$\tilde{f_{1,2}}$ is a Morse function with exactly one singular value a.*
(2) *The preimage of $a-\epsilon_a$ is diffeomorphic to the disjoint union of $F_{\mathrm{low},1}$ and $F_{\mathrm{low},2}$. The preimage of $a+\epsilon_a$ is diffeomorphic to the disjoint union of $F_{\mathrm{up},1}$ and $F_{\mathrm{up},2}$.*
(3) *The preimage ${\tilde{\bar{f_{1,2}}}}^{-1}(a)$ is a one-point set.*

Proof. Besides $\tilde{f_1}$ and $\tilde{f_2}$, we construct a similar smooth function $\tilde{f_{\mathrm{S}}}$ on an m-dimensional smooth, compact and connected manifold onto the small closed interval $[a-\epsilon_a, a+\epsilon_a]$ enjoying the following three properties. We can do this as in Case 1 of the proof of Main Theorem 1.

- This is a Morse function with exactly one singular value a.

- The preimages of $a-\epsilon_a$ and $a+\epsilon_a$ are diffeomorphic to the disjoint union of two copies of the $(m-1)$-dimensional unit disk.
- The preimage ${\bar{\tilde{f}}_{\mathrm{S}}}^{-1}(a)$ is a one-point set.

For $\tilde{f}_{\mathrm{S}}$, we can choose two trivial smooth bundles over the image $[a-\epsilon_a, a+\epsilon_a]$ whose fibers are diffeomorphic to the unit disk D^{m-1} disjointly and which are apart from the singular set of the function. We can do as in Case 1 of the proof of Main Theorem 1. We can also choose them so that the following two conditions hold.

- The total spaces are mapped to the unions of two edges by the quotient map to the Reeb space.
- The sets of the two edges are mutually disjoint for these two bundles.

We remove the interiors of the total spaces of the trivial bundles. We have a new function. Note that the resulting Reeb space is regarded as a graph with exactly four edges.

For $\tilde{f}_j$ $(j=1,2)$, we can choose a trivial smooth bundle over the image $[a-\epsilon_a, a+\epsilon_a]$ whose fiber is diffeomorphic to the unit disk D^{m-1} and which is apart from the singular set of the function similarly.

We remove the interiors as before. We have another two functions. We glue these three resulting functions together to obtain a desired smooth function preserving the value at each point as in Case 1 of the proof of Main Theorem 1. This completes the proof. □

Lemma 2. *Let $m>1$ be an integer. Suppose two smooth functions $\tilde{f}_1$ and $\tilde{f}_2$ on m-dimensional smooth compact and connected manifolds onto a small closed interval $[a-\epsilon_a, a+\epsilon_a]$ enjoying the following three properties exist where a and $\epsilon_a>0$ are real numbers.*

- *$\tilde{f}_1$ and $\tilde{f}_2$ are both Morse functions with exactly one singular value a.*
- *The preimages of $a-\epsilon_a$ are diffeomorphic to closed manifolds $F_{\mathrm{low},1}$ and $F_{\mathrm{low},2}$ respectively. Let $F_{\mathrm{low},1,0} \subset F_{\mathrm{low},1}$ and $F_{\mathrm{low},2,0} \subset F_{\mathrm{low},2}$ be connected components of the manifolds. The preimages of $a+\epsilon_a$ are diffeomorphic to closed manifolds $F_{\mathrm{up},1}$ and $F_{\mathrm{up},2}$ respectively.*
- *The preimages ${\bar{\tilde{f}}_1}^{-1}(a)$ and ${\bar{\tilde{f}}_2}^{-1}(a)$ are both one-point sets in the Reeb spaces.*

Then we have a smooth function $\widetilde{f_{1,2}}$ on an m-dimensional smooth, compact and connected manifold onto $[a-\epsilon_a, a+\epsilon_a]$ enjoying the following properties.

1. *$\widetilde{f_{1,2}}$ is a Morse function with exactly one singular value a.*
2. *The preimage of $a-\epsilon_a$ is diffeomorphic to the disjoint union of a manifold diffeomorphic to $F_{\mathrm{low},1}-F_{\mathrm{low},1,0}$, a manifold diffeomorphic to $F_{\mathrm{low},2}-F_{\mathrm{low},2,0}$, and a manifold represented as a connected sum of the two manifolds $F_{\mathrm{low},1,0}$ and $F_{\mathrm{low},2,0}$: the connected sum is considered in the smooth category. The preimage of $a+\epsilon$ is diffeomorphic to the disjoint union of $F_{\mathrm{up},1}$ and $F_{\mathrm{up},2}$.*
3. *The preimage ${\bar{\widetilde{f_{1,2}}}}^{-1}(a)$ is a one-point set.*

Proof. Besides $\tilde{f}_1$ and $\tilde{f}_2$, we construct a similar smooth function $\widetilde{f_{\mathrm{S,up}}}$ on an m-dimensional compact, connected and smooth manifold onto the small closed interval $[a-\epsilon_a, a+\epsilon_a]$ enjoying the following three. We can do this as in Case 1 of the proof of Main Theorem 1.

- This is a Morse function with exactly one singular value a.
- The preimages of $a-\epsilon$ and $a+\epsilon$ are diffeomorphic to the $(m-1)$-dimensional unit sphere and the disjoint union of two copies of the $(m-1)$-dimensional unit sphere respectively.
- The preimage ${\bar{\widetilde{f_{\mathrm{S,up}}}}}^{-1}(a)$ is a one-point set.

For $\widetilde{f_{\mathrm{S,up}}}$, we can choose two trivial smooth bundles over the image $[a-\epsilon_a, a+\epsilon_a]$ whose fibers are diffeomorphic to the unit disk D^{m-1} disjointly and which are apart from the singular set of the function. We remove the interiors. We can do this as in Case 1 of the proof of Main Theorem 1. We can also choose the trivial bundles so that the following two properties hold.

- The two total spaces are mapped to the unions of two edges by the quotient map to the Reeb space.
- The pairs of the two edges before are distinct for these two bundles.

We have one new function. Note that the resulting Reeb space is regarded as a graph with exactly three edges.

We identify $F_{\mathrm{low},1,0} \subset F_{\mathrm{low},1}$ and $F_{\mathrm{low},2,0} \subset F_{\mathrm{low},2}$ with the preimages of $a - \epsilon$ for functions $\tilde{f_1}$ and $\tilde{f_2}$ in a suitable way. We can choose a trivial smooth bundle over the image $[a-\epsilon_a, a+\epsilon_a]$ enjoying the following properties for $j = 1, 2$. We remove the interiors. We can also do this as in Case 1 of the proof of Main Theorem 1 for $j = 1, 2$.

- The fiber is diffeomorphic to the unit disk D^{m-1}.
- The fiber of this bundle is smoothly embedded in $F_{\mathrm{low},j,0} \subset F_{\mathrm{low},j}$.
- The total space of the bundle is apart from the singular set of each of these two functions.

We have another two functions.

Thus we have three functions. We can glue the resulting functions together to obtain a desired smooth function preserving the value at each point as in Case 1 of the proof of Main Theorem 1. This completes the proof. □

A proof of Main Theorem 2. We first define several smooth functions.

Let $a < t < b$ be real numbers and let $l > 0$. Let $\tilde{f}_{\mathrm{P},0,t,[a,b]}$ denote a smooth function on a 3-dimensional compact and connected manifold enjoying the following properties.

- The Reeb space is homeomorphic to a closed interval.
- The preimage of $\{a, b\}$ and the boundary of the manifold of the domain agree.
- The preimages of a and b are both diffeomorphic to the real projective plane.
- There exist exactly two singular points. t is the unique singular value.
- Around each singular point, the function is a Morse function.

We present the structure of such a function, since this needs a non-trivial argument. We consider the product of a copy P_S of the real projective plane and a closed interval $[0,1]$. We attach a so-called 1-*handle*. It is diffeomorphic to $D^2 \times [0,1] \supset D^2 \times \{0,1\}$. We attach this to a smoothly and disjointly embedded two copies of the unit disk D^2 in $P_S \times \{0\}$. We also attach a so-called 2-*handle*. It is diffeomorphic to $D^2 \times [0,1] \supset \partial D^2 \times [0,1]$. We can find a smoothly embedded surface which is diffeomorphic to $\partial D^2 \times [0,1]$ and apart from the two embedded copies of the disk D^2 before, in $P_S \times \{0\}$. We attach the 2-handle to the surface. Furthermore, we can do this in such a way that the following properties are satisfied.

- After removing the interior of the surface diffeomorphic to $\partial D^2 \times [0,1]$ from $P_S \times \{0\}$, remaining two connected components are diffeomorphic to the 2-dimensional unit disk and the Möbius band respectively.
- The two embedded copies of the disk D^2 before are in different connected components. This is a non-trivial argument on this construction and this produces a desired function.

This exposition is based on well-known fundamental correspondence between singular points of Morse functions and so-called (k-)*handles* where k is a non-negative integer smaller than or equal to the dimension of the manifold of the domain. This exposition is also used in the proof of our Main Theorems and propositions in [13]. (An earlier version of) [10] adopts expositions using terminologies and notions of this theory. For this theory, consult [14] as presented in the proof of Main Theorem 1.

Let $a < t < b$ be real numbers. Let $\tilde{f}_{\mathrm{N,up},l,t,[a,b]}$ ($\tilde{f}_{\mathrm{N,low},l,t,[a,b]}$) denote a function on a 3-dimensional compact and connected manifold enjoying the following properties. This function can be also understood by a fundamental argument on 1-handles and we omit the exposition.

- The Reeb space is homeomorphic to a finite and connected graph.
- The preimage of $\{a, b\}$ and the boundary of the manifold of the domain agree.
- The preimage of a (resp. b) is diffeomorphic to the disjoint union of $l + 1$ copies of the real projective plane.
- The preimage of b (resp. a) is diffeomorphic to a closed, connected, and non-orientable surface of genus $1 + l$.
- There exist exactly l singular points. t is the unique singular value.
- Around each singular point, the function is locally a Morse function.

Proving that we can construct a desired local function around a vertex v in the following cases and our proof of Main Theorem 1 will be completed.

Case 1 v is contained in some edge $e \in A_{\mathrm{low},v}$ or $e \in A_{\mathrm{up},v}$. In addition at v, the function g does not have a local extremum.

Case 2 v is contained in some edge $e \in A_{\mathrm{low},v}$ or $e \in A_{\mathrm{up},v}$. In addition at v, the function g has a local extremum.

We construct our desired local functions.

Part 1 Case 1.

We present the construction of Case 1. First take a sufficiently small real number $\epsilon_v > 0$. Hereafter, let E_v denote the set of all edges containing v. Let $E_{\mathrm{low},v}$ denote the set of all edges containing v as the point at which the restrictions of the function g to the edges have the maxima. Let $E_{\mathrm{up},v}$ denote the set of all edges containing v as the point at which the restrictions of the function g to the edges have the maxima.

Part 1-1 The case where $D_v = 0$ ($\sharp A_{\mathrm{up},v} = \sharp A_{\mathrm{low},v}$).

First prepare $\sharp A_{\mathrm{up},v} = \sharp A_{\mathrm{low},v}$ copies of the function $\tilde{f}_{\mathrm{P},0,g(v),[g(v)-\epsilon_v,g(v)+\epsilon_v]}$ before.

Part 1-1-1 The case $E_{\mathrm{low},v} - A_{\mathrm{low},v}$ and $E_{\mathrm{up},v} - A_{\mathrm{up},v}$ are empty.
We can apply Lemma 1 one after another and apply Case 1 of the proof of Main Theorem 1 to complete the construction. The latter argument increases the genus of a closed, connected and non-orientable surface in the preimage of a point in the interior of an edge by 2. Note that here we regard the Reeb spaces naturally as graphs.

Part 1-1-2 The case where $E_{\mathrm{low},v} - A_{\mathrm{low},v}$ and $E_{\mathrm{up},v} - A_{\mathrm{up},v}$ are both non-empty.
We first construct a function as the previous case such that the Reeb space is homeomorphic to a graph with exactly $2\sharp A_{\mathrm{up},v}$ vertices of degree 1, 1 vertex of degree $2\sharp A_{\mathrm{up},v}$ and $2\sharp A_{\mathrm{up},v}$ edges. Only the preimage of the vertex of degree $2\sharp A_{\mathrm{up},v}$ has some singular points. The genus of each connected component of the preimage of each point in the interior of each edge respects r_G as the previous case.

$E_{\mathrm{low},v} - A_{\mathrm{low},v}$ and $E_{\mathrm{up},v} - A_{\mathrm{up},v}$ can be regarded to play roles played by $E_{\mathrm{low},v}$ and $E_{\mathrm{up},v}$ in the case of Main Theorem 1. We restrict the original r_G in a natural way and apply the proof of Main Theorem 1 to obtain another function. We can then apply Lemma 1 to these obtained two functions to complete the costruction.

Part 1-1-3 The case where either $E_{\mathrm{low},v} - A_{\mathrm{low},v}$ or $E_{\mathrm{up},v} - A_{\mathrm{up},v}$ is non-empty.

We first construct a function as we do in Part 1-1-1 and Part 1-1-2. The Reeb space is homeomorphic to a graph with exactly $2\sharp A_{\mathrm{up},v}$ vertices of degree 1, exactly 1 vertex of degree $2\sharp A_{\mathrm{up},v}$ and exactly $2\sharp A_{\mathrm{up},v}$ edges. Only the preimage of the vertex of degree $2\sharp A_{\mathrm{up},v}$ has some singular points. The genus of each connected component of the preimage of each point in the interior of each edge respects r_G as we do in Part 1-1-1 and Part 1-1-2.

We choose the empty set between $E_{\mathrm{low},v} - A_{\mathrm{low},v}$ and $E_{\mathrm{up},v} - A_{\mathrm{up},v}$. We replace the chosen set by a one-element set. We have two sets: one of them does not change and the other is the one-element set. Note that we can argue similarly in both cases here.

The resulting two sets can be regarded play roles played by $E_{\mathrm{low},v}$ and $E_{\mathrm{up},v}$ in the case of Main Theorem 1. We restrict the original r_G in the natural way and define the value at the element of the one-element set before as 0. After that we apply the proof of Main Theorem 1 as before. We can then apply Lemma 2 in a suitable way to complete the construction.

This completes the construction of a desired function in Part 1-1.

Part 1-2 The case where $D_v \neq 0$ ($\sharp A_{\mathrm{up},v} \neq \sharp A_{\mathrm{low},v}$).

Without loss of generality, we may assume $\sharp A_{\mathrm{up},v} > \sharp A_{\mathrm{low},v}$.

By the assumption on D_v and ${r_G}'(e)$ for $e \in B_{\mathrm{low},v}$, we can choose some edges of $B_{\mathrm{low},v}$, define the subset $C_{\mathrm{low},v} \subset B_{\mathrm{low},v}$, and choose an even integer ${r_G}''(e) > 0$ for $e \in C_{\mathrm{low},v}$ under the following rules.

- $\Sigma_{e \in C_{\mathrm{low},v}} {r_G}''(e) = D_v$.
- ${r_G}''(e) \leq {r_G}'(e)$.

We prepare a copy of the function $\tilde{f}_{\mathrm{N},\mathrm{low},{r_G}''(e)-1,g(v),[g(v)-\epsilon_v,g(v)+\epsilon_v]}$ before for $e \in C_{\mathrm{low},v}$.

To construct a smooth function we do the following procedure.

- We remove D_v edges in $A_{\mathrm{up},v}$ from $E_{\mathrm{up},v}$, without changing $E_{\mathrm{low},v}$. ${A_{\mathrm{up},v}}'$ denotes the subset of $A_{\mathrm{up},v}$ obtained after the D_v edges being removed.
- We restrict r_G to the set of edges obtained after the D_v edges being removed. We change the values of r_G on ${A_{\mathrm{up},v}}' \sqcup A_{\mathrm{low},v}$ to -1 and those on $((B_{\mathrm{up},v} - (A_{\mathrm{up},v} - {A_{\mathrm{up},v}}')) \sqcup B_{\mathrm{low},v}) - ({A_{\mathrm{up},v}}' \sqcup A_{\mathrm{low},v})$ to 0 and do not change the values elsewhere. Let $\tilde{r_G}$ denote the resulting function on the resulting set of these edges. We can also have a new good function $\tilde{g}$ canonically.

Respecting this new graph associated with the new good function $\tilde{g}$ and the family $\tilde{r_G}$ of integer labels, we construct a desired function as in Part 1-1. Let $\tilde{f_{D_v,0}}$ denote the resulting function.

We apply Lemma 2 one after another. At the j-th step, we do the following two where we do this for $1 \leq j \leq \sharp C_{\mathrm{low},v}$.

- Define the preimage of $g(v)-\epsilon_v$ for the copy of the function $\tilde{f}_{\mathrm{N},\mathrm{low},{r_G}''(e_j)-1,g(v),[g(v)-\epsilon_v,g(v)+\epsilon_v]}$ before for the j-th edge $e_j \in C_{\mathrm{low},v}$. This is for the connected component identified with $F_{\mathrm{low},1,0}$ in the situation of Lemma 2.
- Consider the intersection of the preimage of $g(v) - \epsilon_v$ and the preimage of e_j for the quotient map onto the Reeb space of the function $\tilde{f_{D_v,j-1}}$. This can be regarded as the connected component identified with $F_{\mathrm{low},2,0}$ in the situation of Lemma 2. We regard the intersection before as the connected component identified with $F_{\mathrm{low},2,0}$ in Lemma 2.
- We apply Lemma 2 and define $\tilde{f_{D_v,j}}$ as the resulting function.

We have a function $\tilde{f_{D_v,\sharp C_{\mathrm{low},v}}}$. Last we apply arguments in Case 1 of the proof of Main Theorem 1. For any integer $k_e \geq 0$, this can decrease the Euler number of the preimage for each edge e in the resulting Reeb space, regarded as a graph in a natural way, by $2k_e$.

By virtue of definitions of notions, conditions and properties related to $C_{\mathrm{low},v}$, r_G, $\tilde{r_G}$, ${r_G}'$ and ${r_G}''$, this completes the proof.

More precisely, for $e \in C_{\mathrm{low},v} \bigcap A_{\mathrm{low},v}$, we have

$$-1 - {r_G}''(e) \geq -1 - {r_G}'(e) = r_G(e)$$

and for $e \in C_{\mathrm{low},v} \bigcap (B_{\mathrm{low},v} - A_{\mathrm{low},v})$, we have

$$0 - {r_G}''(e) \geq 0 - {r_G}'(e) = r_G(e)$$

as inequalities. This implies that applying the arguments in Case 1 of the proof of Main Theorem 1 to edges (if we need) completes the proof. We can show similarly if $\sharp A_{\mathrm{up},v} < \sharp A_{\mathrm{low},v}$. We use $\tilde{f}_{\mathrm{N},\mathrm{up},{r_G}''(e)-1,g(v),[g(v)-\epsilon_v,g(v)+\epsilon_v]}$ instead for example.

This completes the construction of a desired function of Part 1-2.

Part 2 Case 2.

We present the construction of Case 2. We apply technique in Case 2 of the proof of Main Theorem 1. We first construct a local function as in Part 1. We can and must construct the map regarding "$\sharp A_{\mathrm{up},v} = \sharp A_{\mathrm{low},v}$" there by the conditions on r_G. We can apply the presented technique to complete the construction.

Last, we glue the local functions as in the proof of Main Theorem 1.

This completes the proof. □

We end the present paper by the following remark.

Remark 2. In obtaining results similar to Main Theorems, for example, the first constraint that D_v is even in Main Theorem 2 is a necessary condition. This is due to constraints from the cobordism theory of closed manifolds. Consult [18].

There remain several cases satisfying the condition on this theory. For example, consider the following case for a vertex v.

(1) E_v consists of exactly 3 edges.
(2) $E_{\mathrm{up},v}$ consists of exactly 2 edges. The values of r_G there are odd and negative.
(3) At the unique edge of $E_{\mathrm{low},v}$, the value of r_G is non-negative.

By fundamental discussions on handles we can see that we cannot construct Morse functions. In the situation of [18], we have a positive result. On the other hand, we do not use explicit functions such as Morse(-Bott) functions there.

Remark 3. In Main Theorems, functions we have obtained are Morse functions around singular points where the functions do not have local extrema. Related to Remark 2, we do not know whether we can weaken the conditions on the family r_G of integer labels or preimages of regular values to obtain a Morse function.

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