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SOLVABILITY OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DERIVATIVES

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Abstract. Existence of solutions of a functional differential equation where the sate depends on its derivative will be studied. Uniqueness of the solution will be analyzed and continuous dependence of the unique solution will be proved. Some examples will be given.

Буде вивчено існування розв'язків функціонально-диференціального рівняння, стан якого залежить від його похідної. Буде проаналізована єдиність розв'язку і доведено неперервну залежність єдиного розв'язку. Наведені деякі приклади.

1. Introduction

Differential and integral equations with deviating arguments usually involve the deviation of the argument that depends only on the time itself, see [2, 3]. However, another case, in which the deviating arguments depend on both the state variable x and the time t, is of importance in theory and practice. Several papers have appeared recently that are devoted to such kind of differential equations, see for example [4, 5, 6, 7, 11, 15, 16, 19, 20] and the references cited therein. This kind of equations play an important role in nonlinear analysis, and have many applications especially in the class of problems that have past memories for example in hereditary phenomena, see [1, 13, 14, 17, 18]. Eder [6] studied existence of a unique solution for the differential equation

$$x'(t) = x(x(t)), \qquad t \in B \subset \mathbb{R},$$

where $x(0) = x_0$.

Féckan in [11] studied existence of solutions for the differential equations

$$x'(t) = f(x(x(t))),$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$, and $x(0) = 0$.

Buicá [5] studied existence and continuous dependence of solutions on x_0 for the problem

$$\begin{aligned} x'(t) &= f(t, x(x(t))), \quad t \in [a, b], \\ x(t_0) &= x_0, \end{aligned}$$

where $t_0, x_0 \in [a, b]$ and $f \in C([a, b], [a, b])$.

EL-Sayed and Ebead [9] relaxed the assumptions of Buicá and generalized their results. They studied the equation

$$x(t) \ = \ g(t, \int_0^t f\big(s, x(x(s))\big) \, ds), \quad t \in [0, T],$$

where f satisfies the Carathèodory condition.

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El-Sayed and Ebead [8, 10] proved existence, uniqueness, and continuous dependence of the solution of the initial value problem of the delay-refereed differential equation

$$\begin{aligned} x'(t) &= f(t, x(g(t, x(t)))), \quad a.e. \;\; t \in (0, T], \\ x(0) &= x_0. \end{aligned}$$

Our aim in this work is to study the initial value problem for the functional differential equation where the state depends on its derivative,

$$\frac{dx}{dt} = f\Big(t, x(\phi_1(x(t))), x(\phi_2(\frac{dx}{dt}))\Big), \quad t \in (0, 1], \tag{1.1}$$

with the initial condition

$$x(0) = x_0. \tag{1.2}$$

Existence of solutions $x \in C[0, 1]$ will be studied under suitable assumptions on the function f. Uniqueness of the solution will be proved. Continuous dependence of the unique solution on the derivative and the initial data x_0 will be analyzed. Some special cases and examples will be given. As an application of our results the initial value problem

$$\frac{dx}{dt} = f\Big(t, x(\frac{dx}{dt})\Big), \quad t \in (0, 1],$$

with the initial condition

$$x(0) = x_0.$$

will be considered.

2. Existence of solution

Consider the initial value problem (1.1)—(1.2) with the assumptions:

(i) $f : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is continuous and satisfies the Lipschitz condition

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le k_1|x_1 - x_2| + k_2|y_1 - y_2|.$$

(ii) $\phi_i : \; [0, 1] \to [0, 1]$ is continuous such that,

$$|\phi_i(t) - \phi_i(s)| \le |t - s|,$$

and $\phi_i(0) = 0, \quad i = 1, 2.$

(iii) $\sup_{t\in[0,1]} |f(t, 0, 0)| = M.$

(vi) There exists a real positive root r_2 for the equation

$$k_2 r_2^2 + (k_1 - 1)r_2 + ((k_1 + k_2)x_0 + M) = 0.$$

(v) $r + x_0 < 1.$

Theorem 2.1. *Let the assumptions* (i)—(v) *be satisfied. Then the initial value problem* (1.1)—(1.2) *has at least one solution.*

Proof. Let

$$\frac{dx}{dt} = y,$$

thus

$$x(t) = x_0 + \int_0^t y(s)\; ds$$

and

$$y(t) = f\Big(t, x(\phi_1(x(t))), x(\phi_2(y(t)))\Big). \tag{2.1}$$

Now let the operator F be defined by

$$Fy(t) = f\Big(t, x(\phi_1(x(t))), x(\phi_2(y(t)))\Big),$$

Let $Q_r = \{y \in C[0,1] : \|y\| \leq r\}$,

First, we prove that F is uniformly bounded

$$\begin{aligned}
|Fy(t)| &= |f(t, x(\phi_1(x(t))), x(\phi_2(y(t))))| \\
&\leq |f(t, x(\phi_1(x(t))), x(\phi_2(y(t)))) - f(t,0,0)| + |f(t,0,0)| \\
&\leq k_1|x(\phi_1(x(t)))| + k_2|x(\phi_2(y(t)))| + |f(t,0)|.
\end{aligned}$$

But

$$\begin{aligned}
x(t) &= x_0 + \int_0^t y(s)ds, \\
x(\phi_2(y(t))) &= x_0 + \int_0^{\phi_2(y(t))} y(s)ds \\
&\leq x_0 + \int_0^{\phi_2(y(t))} r_2 \ ds \\
&\leq x_0 + r_2 \ \phi_2(y(t)) \\
&\leq x_0 + r_2 \ y(t) \\
&\leq x_0 + r_2^2.
\end{aligned}$$

Similarly,

$$x(\phi_1(x(t))) \leq x_0 + r.$$

Hence

$$|Fy(t)| \quad \leq \quad k_1(x_0 + r) + k_2(x_0 + r^2) + M.$$

Thus $\|Fy\| \leq r$, which prove that $F : Q_r \to Q_r$ and the class of functions $\{Fy(t)\}$ is uniformly bounded in Q_r.

Now, we will show that $\{Fy\}$ is equi-continuous. Let $t_1, t_2 \in (0,1]$ be such that $|t_2 - t_1| < \delta$. Then

$$\begin{aligned}
&|Fy(t_2) - Fy(t_1)| \\
&\quad = |f(t_2, x(\phi_1(x(t_2))), x(\phi_2(y(t_2)))) - f(t_1, x(\phi_1(x(t_1))), x(\phi_2(y(t_1))))| \\
&\quad = |f(t_2, x(\phi_1(x(t_2))), x(\phi_2(y(t_2)))) - f(t_2, x(\phi_1(x(t_1))), x(\phi_2(y(t_1)))) \\
&\qquad + f(t_2, x(\phi_1(x(t_2))), x(\phi_2(y(t_1)))) - f(t_1, x(\phi_1(x(t_1))), x(\phi_2(y(t_1))))| \\
&\quad \leq k_1|x(\phi_1(x(t_2))) - x(\phi_1(x(t_1))))| + k_2|x(\phi_2(y(t_2))) - x(\phi_2(y(t_1)))| + \theta(\delta),
\end{aligned}$$

where

$$\theta(\delta) = \sup_{u,v \in Q_r} \{|f(t_2, u, v) - f(t_1, u, v)|, \quad \forall \ u, v \in Q_r, \quad |t_1 - t_2| < \delta\}.$$

But

$$\begin{aligned}
\big|x(\phi_1(x(t_2))) - x(\phi_1(x(t_1))))| &\leq \Big|\int_0^{\phi_1(x(t_2))} y(s)ds - \int_0^{\phi_1(x(t_1))} y(s)ds\Big| \\
&\leq \Big|\int_{\phi_1(x(t_1))}^{\phi_1(x(t_2))} y(s)ds\Big| \\
&\leq r \ |\phi_1(x(t_2)) - \phi_1(x(t_1))| \\
&\leq r \ |x(t_2) - x(t_1)| \\
&\leq r^2 \ |t_2 - t_1|.
\end{aligned}$$

And

$$\begin{aligned}|x(\phi_2(y(t_2))) - x(\phi_2(y(t_1)))| &= \Big|\int_0^{\phi_2(y(t_2))} y(s)ds - \int_0^{\phi_2(y(t_1))} y(s)ds\Big| \\ &= \Big|\int_{\phi_2(y(t_1))}^{\phi_2(y(t_2))} y(s)ds\Big|.\end{aligned}$$

Using continuity of the function ϕ_2 and the Lebesgue theorem [12],

$$\int_{\phi_2(y(t_1))}^{\phi_2(y(t_2))} y(s)ds \to 0 \quad \text{as} \quad |t_2 - t_1| \to 0.$$

This means that the class of functions $\{Fy\}$ is equi-continuous in Q_r.

Finally, we prove that F is continuous. Let $\{y_n\} \in Q_r$, $y_n \to y$. Then

$$\begin{aligned}|x(\phi_2(y_n(t))) - x(\phi_2(y(t)))| &= \Big|\int_0^{\phi_2(y_n(t))} y(s)ds - \int_0^{\phi_2(y(t))} y(s)ds\Big| \\ &= \Big|\int_{\phi_2(y(t))}^{\phi_2(y_n(t))} y(s)ds\Big| \\ &\leq r\ |\phi_2(y(t)) - \phi_2(y_n(t))| \\ &\leq r\ |y_n(t) - y(t)| \leq \epsilon.\end{aligned}$$

Now, from continuity of the functions f and ϕ_2, we have

$$\begin{aligned}\lim_{n\to\infty} Fy_n(t) &= \lim_{n\to\infty} f(t, x(\phi_1(x(t))), x(\phi_2(y_n(t)))), \\ &= f(t, x(\phi_1(x(t))), x(\phi_2(y(t)))). \end{aligned} \tag{2.2}$$

This mean that the operator F is continuous. Hence by the Schauder fixed point theorem [12], there exists at least one solution $y \in C[0,1]$. Consequently, there exist at least one solution $x \in C[0,1]$ and also this solution is given by $x(t) = x_0 + \int_0^t y(s)ds$. ☐

As an application of our work, consider the following initial value problem

$$\frac{dx}{dt} = f\Big(t, x(\frac{dx}{dt})\Big), \quad t \in (0,1], \tag{2.3}$$

with the initial condition

$$x(0) = x_0, \tag{2.4}$$

under the next assumptions:

(1) $f : [0,1] \times \mathbb{R} \to [0,1]$ is continuous and satisfies the Lipschitz condition,

$$|f(t,x) - f(t,u)| \leq b|x - u|.$$

(2) There exists a real positive root r_2 for the equation

$$br_2^2 - r_2 + (bx_0 + M_2) = 0,$$

where $M_2 = \sup_{t\in[0,1]} |f(t,0)|$.

Theorem 2.2. *Let the assumptions* (1) *and* (2) *be satisfied. Then the functional differential equation* (2.3)—(2.4) *has at least one solution* $x \in C[0,1]$.

Proof. The proof can be carried out similarly to the one in Theorem 2.1. ☐

2.1. **Uniqueness of the solution.**

Theorem 2.3. *Let the assumptions (i)—(iv) be satisfied. If* $\Big(k_1\ (1+r)+\ 2k_2\ r\Big) \leq 1$, *then the solution of the initial value problem* (1.1)—(1.2) *is unique.*

Proof. From Theorem (2.1) it follows that the solution of the initial value problem (1.1)—(1.2) exists. Let y^* be another solutions of (2.1). Then

$$\begin{aligned}|y(t)-y^*(t)| &= |f(t,x(\phi_1(x(t))),x(\phi_2(y(t))))-f(t,x^*(\phi_1(x^*(t))),x^*(\phi_2(y^*(t))))|\\ &\leq k_1|x(\phi_1(x(t)))-x^*(\phi_1(x^*(t)))|+k_2|x(\phi_2(y(t)))-x^*(\phi_2(y^*(t)))|.\end{aligned}$$

But

$$\begin{aligned}|x(\phi_1(x(t)))-x^*(\phi_1(x^*(t)))| &= \Big|\int_0^{\phi_1(x(t))} y(s)ds-\int_0^{\phi_1(x^*(t))} y^*(s)ds\Big|\\ &\leq \Big|\int_0^{\phi_1(x(t))} y(s)ds-\int_0^{\phi_1(x(t))} y^*(s)ds\Big|\\ &\quad+\Big|\int_0^{\phi_1(x(t))} y^*(s)ds-\int_0^{\phi_1(x^*(t))} y^*(s)ds\Big|\\ &= \Big|\int_0^{\phi_1(x(t))} \big(y(s)-y^*(s)\big)ds\Big|\\ &\quad+\Big|\int_{\phi_1(x(t))}^{\phi_1(x^*(t))} y^*(s)ds\Big|\\ &\leq \|y-y^*\|\phi_1(x(t))+r\ |\phi_1(x(t))-\phi_1(x^*(t))|\\ &\leq \|y-y^*\|+r\ |x(t)-x^*(t)|\\ &\leq \|y-y^*\|+r\ \Big|\int_0^t y(s)ds-\int_0^t y^*(s)ds\Big|\\ &\leq \|y-y^*\|+r\ \|y-y^*\| = (1+r)\ \|y-y^*\|.\end{aligned}$$

Similarly,

$$|x(\phi_2(y(t)))-x^*(\phi_2(y^*(t)))| \leq 2r\ \|y-y^*\|.$$

Hence

$$\|y(t)-y^*(t)\| \leq \big(k_1\ (1+r)+\ 2k_2\ r\big)\|y-y^*\|.$$

Since $\big(k_1\ (1+r)+\ 2k_2\ r\big)<1$, we have $y(t)=y^*(t)$ and the solution of the initial value problem (1.1)—(1.2) is unique. □

Corollary 2.1. *Let the assumptions* $(1)---(2)$ *be satisfied. If* $2br_2 \leq 1$, *then the solution of the initial value problem* (2.3)—(2.4) *is unique.*

2.2. **Continuous dependence.**

2.2.1. *Continuous dependence on the derivative* y.

Definition 2.1. The solution x of the initial value problem (1.1)-(1.2) depends continuously on the derivative y, if

$$\forall\epsilon>0\quad\exists\quad\delta(\epsilon)\quad \text{s.t.}\quad \|y-y^*\|<\delta \Rightarrow \|x-x^*\|<\epsilon,$$

where x^* is the solution of the initial value problem

$$\frac{dx^*}{dt} = f(t,x^*(\phi_1(x^*(t))),x^*(y^*(t))),\quad t\in(0,1], \tag{2.5}$$

$$x^*(0)=x_0. \tag{2.6}$$

Theorem 2.4. *Let the assumptions of Theorem (2.3) be satisfied. Then the solution x of the initial value problem (1.1)—(1.2) depends continuously on the function y.*

Proof. Let x, x^* be two solutions of the initial value problems (1.1)—(1.2) and (2.5)—(2.6) respectively. Then

$$\begin{aligned}|x(t)-x^*(t)| &= \left|x_0+\int_0^t y(s)ds - x_0 - \int_0^t y^*(s)ds\right| \\ &\le \int_0^t |y(s)-y^*(s)|ds \\ &\le \|y-y^*\| \le \delta.\end{aligned}$$

Hence

$$\|x-x^*\| \le \delta = \epsilon.$$

This mean that the solution x of the initial value problem (1.1)—(1.2) depends continuously on the derivative y. The proof is completed. □

2.2.2. *Continuous dependence on the initial data x_0.*

Definition 2.2. The solution y of the functional equation (2.1) depends continuously on the initial data x_0 if

$$\forall \epsilon > 0 \quad \exists \quad \delta(\epsilon) \quad \text{s.t.} \quad |x_0 - x_0^*| < \delta \Rightarrow \|y-y^*\| < \epsilon,$$

where y^* is the solution of the functional equation

$$y^*(t) = f(t, x^*(\phi_1(x^*(t))), x^*(\phi_2(y^*(t)))), \quad t \in (0,1], \tag{2.7}$$

$$x^*(t) = x_0^* + \int_0^t y^*(s)ds.$$

Theorem 2.5. *Let the assumptions of Theorem (2.3) be satisfied. Then the solution y of the functional equation (2.1) depends continuously on the initial data x_0.*

Proof. Let y, y^* be two solutions of the functional equations (2.1) and (2.7) respectively, then

$$\begin{aligned}|y(t)-y^*(t)| &= |f(t,x(\phi_1(x(t))),x(\phi_2(y(t)))) - f(t,x^*(\phi_1(x^*(t))),x^*(\phi_2(y^*(t))))| \\ &\le k_1|x(\phi_1(x(t))) - x^*(\phi_1(x^*(t)))| + k_2|x(\phi_2(y(t))) - x^*(\phi_2(y^*(t)))|.\end{aligned}$$

But

$$\begin{aligned}&|x(\phi_1(x(t))) - x^*(\phi_1(x^*(t)))| \\ &\quad = \left|x_0 + \int_0^{\phi_1(x(t))} y(s)ds - x_0^* - \int_0^{\phi_1(x^*(t))} y^*(s)ds\right| \\ &\quad \le |x_0-x_0^*| + \left|\int_0^{\phi_1(x(t))} y(s)ds - \int_0^{\phi_1(x(t))} y^*(s)ds\right| \\ &\qquad + \left|\int_0^{\phi_1(x(t))} y^*(s)ds - \int_0^{\phi_1(x^*(t))} y^*(s)ds\right| \\ &\quad \le |x_0-x_0^*| + \|y-y^*\|\phi_1(x(t)) + r\ |\phi_1(x(t)) - \phi_1(x^*(t))| \\ &\quad \le |x_0-x_0^*| + \|y-y^*\| + r\ |x(t)-x^*(t)| \\ &\quad \le |x_0-x_0^*| + \|y-y^*\| + r\ \left|x_0 + \int_0^t y(s)ds - x_0^* - \int_0^t y^*(s)ds\right| \\ &\quad \le |x_0-x_0^*| + r\ |x_0-x_0^*| + \|y-y^*\| + r\ \|y-y^*\| \\ &\quad = (1+r)\ \delta + (1+r)\ \|y-y^*\|.\end{aligned}$$

Also,

$$|x(\phi_2(y(t))) - x^*(\phi_2(y^*(t)))| \leq \delta + 2r \ \|y - y^*\|.$$

Thus

$$|y(t) - y^*(t)| \leq (k_1(1+r) + k_2) \ \delta + \big(k_1 \ (1+r) + \ 2k_2 \ r\big)\|y - y^*\|.$$

Hence

$$\|y - y^*| \leq \frac{(k_1(1+r) + k_2) \ \delta}{1 - \big(k_1 \ (1+r) + \ 2k_2 \ r\big)}.$$

Since $\big(k_1 \ (1+r) + \ 2k_2 \ r\big) < 1$, the solution y of the functional equation (2.1) depends continuously on the initial data x_0. □

Definition 2.3. The solution x of the initial value problem (1.1)—(1.2) depends continuously on the initial data x_0 if

$$\forall \epsilon > 0 \quad \exists \quad \delta(\epsilon) \quad \text{s.t} \quad |x_0 - x_0^*| < \delta \Rightarrow \|x - x^*\| < \epsilon,$$

where x^* is the solution of the initial value problem

$$\frac{dx^*}{dt} = f(t, x^*(\phi_1(x^*(t))), x^*(y^*(t))), \quad t \in (0,1], \tag{2.8}$$

$$x^*(0) = x_0^*. \tag{2.9}$$

Theorem 2.6. *Let the assumptions of Theorem (2.3) be satisfied. Then the solution x of the initial value problem (1.1)—(1.2) depends continuously on the initial data x_0.*

Proof. Let x, x^* be two solutions of the initial value problems (1.1)—(1.2) and (2.8)—(2.9), respectively. Then

$$\begin{aligned}
|x(t) - x^*(t)| &= \Big|x_0 + \int_0^t y(s)ds - x_0^* - \int_0^t y^*(s)ds\Big| \\
&\leq |x_0 - x_0^*| + \int_0^t |y(s) - y^*(s)|ds \\
&\leq \delta + \|y - y^*\| \\
&\leq \delta + \frac{(k_1(1+r) + k_2) \ \delta}{1 - \big(k_1 \ (1+r) + \ 2k_2 \ r\big)}.
\end{aligned}$$

Hence

$$\|x - x^*\| \leq \delta + \frac{(k_1(1+r) + k_2) \ \delta}{1 - \big(k_1 \ (1+r) + \ 2k_2 \ r\big)} = \epsilon.$$

This mean that the solution x of the initial value problem (1.1)—(1.2) depends continuously on the initial data x_0. The proof is completed. □

3. Examples

Example 3.1. Consider the initial value problem

$$\frac{dx}{dt} = \frac{1+2t}{12} + \frac{1}{4} \ x(\frac{dx}{dt}), \quad t \in [0,1], \tag{3.1}$$

$$x(0) = \frac{1}{2}. \tag{3.2}$$

Here we have: $f : [0,1] \times [0,1] \to [0,1]$ defined by

$$\frac{dx}{dt} = f\big(t, x(\frac{dx}{dt})\big) = \frac{1+2t}{12} + \frac{1}{4} \ x\big(\frac{dx}{dt}\big),$$

thus

$$f(t,x) \leq \frac{1+2t}{12} + \frac{1}{4} \ x,$$

and $b = \frac{1}{4}$, $M_2 = \frac{1}{4}$, $x_0 = \frac{1}{2}$, and $r_2 \simeq 0.2$, thus $r_2 + x_0 < 1$. Clearly, all the assumptions of Theorem 2.2 are satisfied, hence the solution of the initial value problem (3.1)—(3.2) exists.

Moreover, $2br_2 < 1$, thus according to Corollary 2.1, the solution of (3.1)—(3.2) is unique.

Example 3.2. Consider the initial value problem

$$\frac{dx}{dt} = \frac{e^{-t}}{t+8} + \frac{1}{4}\, x\big(x^2(t)\big) + \frac{1}{8}\, x\big(\beta\, \frac{dx}{dt}\big), \tag{3.3}$$

$$x(0) = \frac{1}{8}. \tag{3.4}$$

where $t \in [0,1]$, $\beta \in (0,1]$. Here we have

$$\phi_1(x) = x^2 \quad \text{and} \quad \phi_2(x) = \beta\, x,$$

$f : [0,1] \times [0,1] \times [0,1] \to [0,1]$ is defined by

$$\frac{dx}{dt} = f\Big(t, x(\phi_1(x(t))), x(\phi_2(\frac{dx}{dt}))\Big) = \frac{e^{-t}}{t+8} + \frac{1}{4}\, x\big(x^2(t)\big) + \frac{1}{8}\, x\big(\beta\, \frac{dx}{dt}\big),$$

thus

$$f(t,x,y) \leq \frac{e^{-t}}{t+8} + \frac{1}{4}\, x + \frac{1}{8}\, y,$$

and $k_1 = \frac{1}{4}$, $k_2 = \frac{1}{8}$, $M = \frac{1}{8}$, $x_0 = \frac{1}{8}$, and $r \simeq 0.25$, thus $r + x_0 < 1$. It is clear that all the assumptions of Theorem 2.1 are satisfied, so the solution of the initial value problem (3.3)-(3.4) exists. Moreover,

$$\Big(k_1\ (1+r) +\ 2k_2\ r\Big) \simeq 0.1 < 1,$$

thus according to Theorem 2.3, the solution of (3.3)—(3.4) is unique.

4. Conclusion

In this paper, we have proved existence, uniqueness, and continuous dependence of the solution of an initial value problem where the state depends on its derivative under suitable assumptions. Moreover, some special cases and examples were considered.

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