Methods of Functional Analysis and Topology
Vol. 29 (2023), no. 1–2, pp. 16–29
https://doi.org/10.31392/MFAT-npu26_1–2.2023.02

ELLIPTIC PROBLEM IN AN EXTERIOR DOMAIN DRIVEN BY A SINGULARITY WITH A NONLOCAL NEUMANN CONDITION

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Abstract. We prove existence of a ground state solution to the following problem.

$$\begin{aligned}(-\Delta)^s u + u &= \lambda |u|^{-\gamma-1}u + P(x)|u|^{p-1}u \qquad \text{in } \mathbb{R}^N \setminus \Omega,\\ N_s u(x) &= 0 \qquad \text{in } \Omega\end{aligned}$$

where $N \geq 2$, $\lambda > 0$, $0 < s, \gamma < 1$, $p \in (1, 2_s^* - 1)$ with $2_s^* = \frac{2N}{N-2s}$. Moreover, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $(-\Delta)^s$ denotes the s-fractional Laplacian and finally N_s denotes a nonlocal operator that describes the Neumann boundary condition. We further establish existence of infinitely many bounded solutions to the problem.

Доведено існування розв'язку основного стану наступної задачі:

$$\begin{aligned}(-\Delta)^s u + u &= \lambda |u|^{-\gamma-1}u + P(x)|u|^{p-1}u \qquad \text{в } \mathbb{R}^N \setminus \Omega\\ N_s u(x) &= 0 \qquad \text{в } \Omega\end{aligned}$$

де $N \geq 2$, $\lambda > 0$, $0 < s, \gamma < 1$, $p \in (1, 2_s^* - 1)$ з $2_s^* = \frac{2N}{N-2s}$. Крім того, $\Omega \subset \mathbb{R}^N$ — гладка обмежена область, $(-\Delta)^s$ позначає s-дробовий лапласіан і, нарешті, N_s позначає нелокальний оператор, який описує неймановску граничну умову. Далі встановлюємо існування нескінченної кількості обмежених розв'язків задачі.

1. Introduction

As mentioned in the Abstract, we will take up the following problem to study:

$$\begin{aligned}(-\Delta)^s u + u &= \lambda |u|^{-\gamma-1}u + P(x)|u|^{p-1}u \qquad \text{in } \mathbb{R}^N \setminus \Omega,\\ N_s u(x) &= 0 \qquad \text{in } \Omega.\end{aligned} \tag{P}$$

The function P is a continuous function such that

$$(P_1): \qquad P(x) \geq \tilde{P} > 0 \qquad \text{in } \mathbb{R}^N \setminus \Omega$$

and

$$\lim_{|x|\to\infty} P(x) = \tilde{P}.$$

The "nonlocal normal derivative" N_s was first introduced by Dipierro et al. [10] which is given as follows:

$$N_s u(x) = C_{N,s} \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \qquad x \in \Omega.$$

The authors in [10] proved that as $s \to 1^-$, the classical Neumann boundary condition is recovered in the following sense:

$$\lim_{s\to 1^-} \int_{\mathbb{R}^N \setminus \Omega} v N_s u = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu},$$

where ν is an outward drawn normal to the boundary $\partial\Omega$. Elliptic problems considered in an exterior domain is a rarity in the literature. However, when we traced through the literature pertaining to the exterior domain problem, we found a few seminal works.

2020 *Mathematics Subject Classification.* 35R11, 35J75, 35J60, 46E35.

Keywords. Fractional Laplacian, variable order fractional Sobolev space, Kirchhoff operator, ground state solution, singularity.

One of them is due to Benci and Cerami [4] who considered the problem (**P**) with $s = 1$, $V(x) \equiv 1$, $\lambda = 0$, $Q(x) \equiv 1$ and with a zero Dirichlet boundary condition. The authors in [4] showed that it does not have a ground state solution. Thanks to the article due to Esteban [11] who proved that the same problem with Neumann condition has a ground state solution to the following problem:

$$\begin{aligned} -\Delta u + u = |u|^{p-1}u \qquad &\text{in } \mathbb{R}^N \setminus \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \qquad &\text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

A noteworthy work is due to Cao [5] who studied the existence of positive solution to (1.1) under the assumption that

$$P(x) \geq \tilde{P} - Ce^{-a|x|}|x|^{-m} \quad \text{as} \quad |x| \to \infty$$

together with the condition in (P_1). Here $a = \frac{2(p+1)}{p-1}$, $m > N-1$, and $C > 0$. Continuing in this article [5], Cao further proved the existence of sign changing solution (also know as a *nodal* solution) under the additional assumption that

$$P(x) \geq \tilde{P} + Ce^{-\frac{p|x|}{p+1}}|x|^{-m} \quad \text{as} \quad |x| \to \infty$$

together with the condition in (P_1) with $0 < m < \frac{N-1}{2}$, Alves et al. [1] proved that the results found in [5] also hold true for the p-Laplacian operator and for a larger class of nonlinearity. The problem with $N = 2$ and nonlinearity of critical growth has also been considered by Alves in [2].

Off-late, the fractional Laplacian operator gained a mileage as far as attention is concerned as it naturally arises in many different contexts, viz. optimization, thin obstacle problem, finance, crystal dislocation, conservation laws, limits of quantum mechanics, material science and water waves to name a few. Interested readers may also refer to the works [7, 8, 15] purely for mathematical interest. We drew motivation from the work due to of Alves [3] to study the problem (**P**). To the best of our knowledge, there is no article in the literature that addresses the problem (**P**) driven by a singularity in an exterior domain and a Neumann boundary condition. We will first prove the existence of a nonnegative ground state solution to (**P**). Capitalising on this proof, we will further show that the problem has infinitely many bounded solutions for a finite range of λ. The main result concerning the existence of a ground state solution is as follows.

Theorem 1.1. *Suppose $p \in (1, 2_s^* - 1)$ and (P_1) holds, then* (**P**) *has a positive ground state solution. Further, there exists $\lambda_0 > 0$ such that* (**P**) *has infinitely many bounded solutions whenever $\lambda \in (0, \lambda_0)$.*

2. Preliminaries

The section introduces the readers to some well known function spaces besides considering the following limiting problem.

$$\begin{aligned} (-\Delta)^s u + u = \lambda |u|^{-\gamma-1}u + \tilde{P}|u|^{p-1}u \qquad &\text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N). \end{aligned} \tag{P_∞}$$

The operator $(-\Delta)^s$ is defined as follows.

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy \tag{2.2}$$

where $C_{N,s} = 2^{2s-1}\pi^{-\frac{N}{2}}\frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}$. We will denote $H^s(\mathbb{R}^N)$ to be the fractional Sobolev space equipped with the norm

$$\|u\| = \left(\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N}|u|^2dx\right)^{\frac{1}{2}}.$$

Let $D \subset \mathbb{R}^N$ be a smooth domain. We now define the fractional Sobolev space pertaining to an exterior domain as follows:

$$H_D^s = \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(D^c)^2}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_D |u|^2dx < \infty\right\}$$

where $D^c = \mathbb{R}^N \setminus D$. This space is equipped with the norm

$$\|u\|_s = \left(\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(D^c)^2}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_D |u|^2dx\right)^{\frac{1}{2}}.$$

This space H_D^s is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle_{H_D^s}$ given by

$$\langle u, v\rangle_{H_D^s} = \frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(D^c)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx + \int_D uvdx.$$

We now state a few embedding results pertaining to the space H_D^s which can be found in [9].

Lemma 2.1. (1) *Let $H^s(D)$ be the classical fractional Sobolev space equipped with the norm*

$$\|u\|^2_{H^s(D)} = \frac{1}{2}\int_D\int_D\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_D |u|^2dx.$$

Since $D \times D \subset \mathbb{R}^{2N} \setminus (D^c)^2$, the embedding $H_D^s \hookrightarrow H^s(D)$ is continuous.

(2) *The embedding $H^s(\mathbb{R}^N) \hookrightarrow H_D^s$ is continuous.*

(3) *Since $H^s(D) \hookrightarrow L^p(D)$ is continuous for every $p \in \left[2, \frac{2N}{N-2s}\right]$, by (1) we have*

$$H_D^s \hookrightarrow L^p(D) \text{ for all } p \in \left[2, \frac{2N}{N-2s}\right].$$

(4) *If D is bounded, we have the compact embedding*

$$H_D^s \hookrightarrow L^p(D) \text{ for all } p \in \left[1, \frac{2N}{N-2s}\right).$$

Definition 2.2 (Palais Smale condition [14])**.** Let X be a Banach space and $J : X \to \mathbb{R}$ a C^1 functional. It is said to satisfy the *Palais-Smale* condition (PS) if the following holds: whenever $(u_n) \subset X$ is such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ in X^*, the dual space of X, then (u_n) has a convergent subsequence.

Definition 2.3 (Mountain pass theorem of Ambrosetti and Rabinowitz [14])**.** Let $J : X \to \mathbb{R}$ be a C^1 functional satisfying (PS). Let $u_0, u_2 \in X$, $c \in \mathbb{R}$ and $R > 0$ such that

(1) $\|u_1 - u_0\| > R$,
(2) $J(u_0), J(u_1) < c \leq J(v)$, for all v such that $\|v - u_0\| = R$.

Then J has a critical value $\tilde{c} \geq c$ defined by

$$\tilde{c} = \inf_{\delta\in\mathcal{P}}\max_{t\in[0,1]}\{J(\delta(t))\}$$

where $\mathcal{P}$ is the collection of all continuous paths $\delta : [0,1] \to X$ such that $\delta(0) = u_0$ and $\delta(1) = u_1$.

2.1. **Cut-off functional.** We first define a functional $I : H^s_{\Omega^c} \to \mathbb{R}$ corresponding to the problem in (**P**) as follows:

$$\begin{aligned} I(u) = \frac{1}{2}\Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N\setminus\Omega} |u|^2 dx\Big) \\ - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N\setminus\Omega} |u|^{1-\gamma}dx - \frac{1}{p+1}\int_{\mathbb{R}^N\setminus\Omega} |u|^{p+1}dx. \end{aligned} \tag{2.3}$$

Note that, the functional I is not C^1 over $H^s_{\Omega^c}(\mathbb{R}^N)$ due to the presence of the singular term. Towards this, we will define a cut-off functional to overcome this problem. We now prove the following Lemma which will be used to construct the cut-off functional.

Lemma 2.4. *Let $0 < \gamma < 1$, $\lambda, \mu > 0$. Then the following problem:*

$$\begin{aligned} (-\Delta)^s u + u &= \lambda u^{-\gamma} \quad \textit{in } \mathbb{R}^N \setminus \Omega, \\ u &> 0 \quad \textit{in } \Omega, \\ u &= 0 \quad \textit{in } \Omega \end{aligned} \tag{2.4}$$

has a unique weak solution in H^s_{Ω}. This solution is denoted by $\underline{u}_\lambda$, satisfies $\underline{u}_\lambda \geq \epsilon_\lambda v_0$ a.e. in Ω^c, where $\epsilon_\lambda > 0$ is a constant.

Proof. We follow the proof in [12]. First, we note that an energy functional on $H^s_{\Omega^c}$ formally corresponding to (2.4) can be defined as follows:

$$E(u) = \frac{1}{2}\Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N\setminus\Omega} |u|^2 dx\Big) - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N\setminus\Omega} |u|^{1-\gamma}dx \tag{2.5}$$

for $u \in H^s_{\Omega}$. By the Poincaré inequality, this functional is coercive and continuous on $H^s_{\Omega^c}$. It follows that E possesses a global minimizer $u_0 \in H^s_{\Omega^c}$. Clearly, $u_0 \neq 0$ since $E(0) = 0 > E(\epsilon v_0)$ for sufficiently small ϵ and some $v_0 > 0$ in $\mathbb{R}^N \setminus \Omega$.

Secondly, if u_0 is a global minimizer for E, hence $|u_0|$ is also a global minimizer since $E(|u_0|) \leq E(u_0)$. Clearly enough, the equality holds iff $u_0^- = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. In other words we need to have $u_0 \geq 0$, i.e., $u_0 \in H^s_{\Omega^c}$ where

$$H^s_{\Omega^c}{}^+ = \{u \in H^s_\Omega : u \geq 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

is a positive cone in $H^s_{\Omega^c}$.

Third, we will show that $u_0 \geq \epsilon v_0 > 0$ holds a.e. in $\mathbb{R}^N \setminus \Omega$ for small enough ϵ. Observe that

$$\begin{aligned} E'(tv_0)|_{t=\epsilon} = \epsilon\Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N\setminus\Omega} |u|^2 dx\Big) \\ - \lambda\epsilon^{-\gamma}\int_{\mathbb{R}^N\setminus\Omega} |u|^{1-\gamma}dx < 0 \end{aligned} \tag{2.6}$$

whenever $0 < \epsilon \leq \epsilon_\lambda$ for some sufficiently small ϵ_λ. We now prove that $u_0 \geq \epsilon_\lambda v_0$. Suppose not, i.e, $w = (\epsilon_\lambda v_0 - u_0)^+$ does not vanish identically in $\mathbb{R}^N \setminus \Omega$. Denote

$$(\mathbb{R}^N \setminus \Omega)^+ = \{x \in \mathbb{R}^N \setminus \Omega : w(x) > 0\}$$

to be the positive cone in $H^s_{\Omega^c}$. We consider the function $\zeta(t) = E(u_0 + tw)$ of $t \geq 0$. We note that the function ζ is convex. This can be concluded from the definition of ζ over the convex set $H^s_\Omega{}^+$. Further $\zeta'(t) = \langle E'(u_0 + tw), w\rangle$ is nonnegative and nondecreasing for $t > 0$. Consequently for $0 < t < 1$ we have

$$\begin{aligned}0 \leq \zeta'(1) - \zeta'(t) &= \langle E'(u_0 + w) - E'(u_0 + tw), w\rangle \\ &= \int_{(\mathbb{R}^N\setminus\Omega)^+} E'(u_0 + w)dx - \zeta'(t) \\ &< 0\end{aligned} \tag{2.7}$$

by inequality (2.6) and $\zeta'(t) \geq 0$ with $\zeta'(t)$ being nondecreasing for every $t > 0$, which is a contradiction. Therefore $w = 0$ in $\mathbb{R}^N \setminus \Omega$ and hence $u_0 \geq \epsilon_\lambda v_0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

Finally, we conclude that u_0 is the only critical point of E in ${H^s_{\Omega^c}}^+$ because the functional E is strictly convex on ${H^s_{\Omega^c}}^+$. Define $\mathcal{W} = \{x \in \mathbb{R}^N \setminus \Omega : u(x) = \underline{u}_\mu(x)\}$. Obviously $\mathcal{W}$ is a measurable set. therefore, for any $\eta > 0$ there exists a closed subset $\mathcal{V}$ of $\mathcal{W}$ such that $|\mathcal{W} \setminus \mathcal{V}| < \eta$. Further assume that $|\mathcal{W}| > 0$. A quick manipulation of taking the difference between the weak formulations of the problems (**P**) and (2.4) alongwith testing it with

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in V, \\ 0 < \varphi < 1, & \text{if } x \in W \setminus V, \\ 0, & \text{if } x \in (\mathbb{R}^N \setminus \Omega) \setminus W \end{cases} \tag{2.8}$$

yields an absurd relation. Therefore, $|\mathcal{W}| = 0$ which implies that $\mathcal{W} = \emptyset$. Hence, $u > \underline{u}_\mu$ in $\mathbb{R}^N \setminus \Omega$. □

We now define the following cut-off function which will be used to create the required cut-off functional:

$$\bar{f}(x,t) = \begin{cases} \lambda|t|^{-\gamma-1}t + |t|^{p-1}t, & \text{if } |t| > \underline{u}_\lambda, \\ \lambda\underline{u}_\lambda^{-\gamma} + \underline{u}_\lambda^p, & \text{if } |t| \leq \underline{u}_\lambda, \end{cases}$$

where $\underline{u}_\lambda$ is a solution to (2.4). Define

$$\bar{I}(u) = \frac{1}{2}\Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N\setminus\Omega} |u|^2dx\Big) - \int_{\mathbb{R}^N\setminus\Omega} \bar{F}(x,u)dx, \tag{2.9}$$

where $\bar{F}(x,t) = \int_0^t \bar{f}(x,s)ds$. This functional $\bar{I}$ thus defined is in $C^1(H^s_{\Omega^c};\mathbb{R})$ and it is standard to show that

$$\begin{aligned}\langle \bar{I}'(u), v\rangle = \frac{1}{2}\Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N\setminus\Omega} uvdx\Big) \\ - \int_{\mathbb{R}^N\setminus\Omega} \bar{f}(x,u)vdx\end{aligned} \tag{2.10}$$

for all $v \in H^s_{\Omega^c}$.

Similarly, we treat the functional corresponding to the problem defined in (P_∞). The functional is defined as follows:

$$\begin{aligned}I_\infty(u) = \frac{1}{2}\Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N} |u|^2dx\Big) \\ - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} |u|^{1-\gamma}dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}dx.\end{aligned} \tag{2.11}$$

A similar modification as done to the functional I yields us the following:

$$\bar{I}_\infty(u) = \frac{1}{2}\Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N} |u|^2dx\Big) - \int_{\mathbb{R}^N} \bar{G}(x,u)dx. \tag{2.12}$$

Again, it is easy to see that $\bar{I}_\infty \in C^1(H^s(\mathbb{R}^N);\mathbb{R})$ and

$$\langle \bar{I}'_\infty(u), v\rangle = \frac{1}{2}\Big(\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N} uvdx\Big) - \int_{\mathbb{R}^N} \bar{g}(x,u)vdx \quad (2.13)$$

for all $v \in H^s(\mathbb{R}^N)$. Here

$$\bar{g}(x,t) = \begin{cases} \lambda|t|^{-\gamma-1}t + |t|^{p-1}t, & \text{if } |t| > \underline{u}_\lambda^\infty, \\ \lambda(\underline{u}_\lambda^\infty)^{-\gamma} + (\underline{u}_\lambda^\infty)^p, & \text{if } |t| \le \underline{u}_\lambda^\infty, \end{cases}$$

where $\bar{G}(x,t) = \int_0^t \bar{g}(x,s)ds$. Further, $\underline{u}_\lambda^\infty$ is a solution to the following problem:

$$\begin{aligned} (-\Delta)^s u + u &= \lambda u^{-\gamma} \quad \text{in } \mathbb{R}^N, \qquad (P_\infty) \\ u &> 0 \quad \text{in } \mathbb{R}^N. \end{aligned} \quad (2.14)$$

Existence of a unique solution to (2.14) can be proved by following *verbatim* the proof of Lemma 2.4.

Remark 2.5. Instead of studying the problem (2.2), we will study the following problem:

$$\begin{aligned} (-\Delta)^s u + u = \bar{f}(x,u) \quad \text{in } \mathbb{R}^N \setminus \Omega, \\ N_s u(x) = 0 \quad \text{in } \Omega. \end{aligned} \quad (\mathbf{P}')$$

This is because a solution to (**P**') is also a solution to (**P**).

3. Proof of the main theorem

This section is devoted to the proof of the main Theorem 1.1. We begin by stating a *Lions type* lemma that will play a crucial role in the proof of the main theorem.

Lemma 3.1 (Refer [3]). *Let $D \subset \mathbb{R}^N$ be an exterior domain with smooth bounded boundary and let $(u_n) \subset H_D^s$ be a bounded sequence such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{U(y,T)} |u_n|^2 dx = 0 \quad (3.15)$$

for some $T > 0$ and $U(y,T) = B(y,T) \cap D$ with $U(y,T) \neq \phi$. Then

$$\lim_{n\to\infty} \int_D |u_n|^p dx \text{ for all } p \in (2, 2_s^*). \quad (3.16)$$

The next lemma proves that the functional $\bar{I}$ satisfies the Mountain pass geometry.

Lemma 3.2. *The functional $\bar{I}$ obeys the Mountain pass geometry for $\lambda \in (0,\lambda_0)$ with $\lambda_0 < \infty$.*

Proof. Since $p \in \left(1, \frac{N+2s}{N-2s}\right)$ and P is bounded, by Sobolev embedding we obtain

$$\bar{I}(u) \ge \frac{1}{2}\|u\|^2_{H^s_{\Omega^c}} - \frac{C_1\|P\|_\infty}{p+1}\|u\|^{p+1}_{H^s_{\Omega^c}} - \frac{\lambda C_2}{1-\gamma}\|u\|^{1-\gamma}_{H^s_{\Omega^c}}$$

where $C_1, C_2 > 0$. Now for a small $\lambda > 0$, say λ_0, we have that $\frac{1}{2}\|u\|^2_{H^s_{\Omega^c}} - \frac{\lambda C_2}{1-\gamma}\|u\|^{1-\gamma}_{H^s_{\Omega^c}} > 0$. Note that this positivity holds for any $\lambda \in (0,\lambda_0)$. For a sufficiently small $\|u\|_{H^s_{\Omega^c}} = r$, we further have $a(r) = \frac{1}{2}r^2 - \frac{C_1\|P\|_\infty}{p+1}r^{p+1} - \frac{\lambda C_2}{1-\gamma}r^{1-\gamma} > 0$. Therefore, to sum it up we have a pair (λ, r) such that

$$\bar{I}(u) \ge a(r) > 0$$

for any $\lambda \in (0, \lambda_0)$ and for every u such that $\|u\|_{H^s_{\Omega^c}} = r$. On the other hand, taking $u \in H^s_{\Omega^c} \setminus \{0\}$ and $t \geq 0$ we have

$$\bar{I}(tu) = \frac{t^2}{2}\|u\|^2_{H^s_{\Omega^c}} - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_{\Omega^c}|u|^{1-\gamma}dx - \frac{t^{p+1}}{p+1}\int_{\Omega^c}P(x)|u|^{p+1}.$$

Since $p+1 > 2 > 1-\gamma$, we have $\bar{I}(tu) \to -\infty$ as $t \to \infty$. This verifies the second condition of the Mountain pass theorem. □

Remark 3.3. Suppose u is a solution to (**P**), then the following holds.

(1) Since $I(u) = I(|u|)$, it implies that $u^- = 0$ a.e. in Ω^c.
(2) Furthermore, $u > 0$ a.e. in Ω^c owing to the presence of the singular term.

Thus without loss of generality, we assume that the solution is positive.

We have the following result.

Lemma 3.4. *For any fixed $\lambda \in (0, \lambda_0)$, a solution $u > 0$ of (**P**) such that $u \geq \underline{u}_\lambda$ a.e. in $\mathbb{R}^N \setminus \Omega$ exists.*

Proof. Fix $\lambda \in (0, \lambda_0)$ and let $u \in H^s_{\Omega^c}$ be a positive solution to (**P**) and $\underline{u}_\lambda > 0$ be a solution to (2.4). We prove that $u \geq \underline{u}_\lambda$ a.e. in $\mathbb{R}^N \setminus \Omega$. Let $\tilde{\Omega} = \{x \in \mathbb{R}^N \setminus \Omega : u(x) < \underline{u}_\lambda(x)\}$. From the equation obeyed by u, $\underline{u}_\lambda$, we have

$$\langle(-\Delta)^s\underline{u}_\lambda - (-\Delta)^s u, \underline{u}_\lambda - u\rangle_{\underline{\Omega}} + \int_{\underline{\Omega}}|\underline{u}_\lambda - u|^2dx \leq \lambda\int_{\underline{\Omega}}(\underline{u}_\lambda^{-\gamma} - u^{-\gamma})(\underline{u}_\lambda - u)dx \leq 0. \tag{3.17}$$

Further, by the Simon's inequality we have

$$\langle(-\Delta)^s\underline{u}_\lambda - (-\Delta)^s u, \underline{u}_\lambda - u\rangle_{\underline{\Omega}} \geq 0. \tag{3.18}$$

Thus from (3.17) and (3.18), we conclude that $u \geq \underline{u}_\lambda$ a.e. in Ω^c. □

Lemma 3.2 allows us to apply the Mountain pass theorem without the Palais-Smale condition (Definition 2.2) to find a sequence $(u_n) \subset H^s_{\Omega^c}$ such that

$$\bar{I}(u_n) \to c_1 \text{ and } \bar{I}'(u_n) \to 0 \tag{3.19}$$

where

$$c_1 = \inf_{u \in H^s_{\Omega^c}\setminus\{0\}} \sup_{t\geq 0}\bar{I}(tu). \tag{3.20}$$

Moreover, we further have

$$c_1 = \inf_{u\in\mathcal{N}}\bar{I}(u) \tag{3.21}$$

where

$$\mathcal{N} = \{u \in H^s_{\Omega^c} \setminus \{0\} : \langle\bar{I}'(u), u\rangle = 0\} \tag{3.22}$$

is called a Nehari manifold. Henceforth, we say that $u \in H^s_{\Omega^c}$ is a ground state solution to (**P**') when

$$\bar{I}(u) = c_1 \text{ and } \bar{I}'(u) = 0. \tag{3.23}$$

We further recall that by a *ground state* solution we mean a function $\tilde{u}$.

Lemma 3.5. *Suppose (P_1) holds, then*

$$0 < c_1 < c_\infty$$

whenever $\lambda \in (0, \lambda_0)$.

Proof. Let $\overline{u}$ be a nontrivial ground state solution of (P_∞) and define $u_n(x) = \overline{u}(x - \alpha_n)$ where $\alpha_n = (n, 0, \cdots, 0) \in \mathbb{R}^N$. From (3.19),

$$c_1 = \max_{t\geq 0}\{\bar{I}(tu)\}. \tag{3.24}$$

For every $t > 0$ consider the function

$$f(t) = \frac{t^2}{2}\|u_n\|^2_{H^s_{\Omega^c}} - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_{\Omega^c}|u_n|^{1-\gamma}dx - \frac{t^{p+1}}{p+1}\int_{\Omega^c}P(x)|u_n|^{p+1}dx. \tag{3.25}$$

It is clear that $f(0) = 0$, $f(t) > 0$ for t small enough and $f(t) < 0$ for t large enough. Therefore, there exists a unique $\gamma_n \in (0, \infty)$ such that

$$f(\gamma_n) = \bar{I}(\gamma_n u_n) = \max_{t\geq 0}\{\bar{I}(tu_n)\}. \tag{3.26}$$

Therefore, $f'(\gamma_n) = 0$ which amounts to saying that

$$\begin{aligned}\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus\Omega^2}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\Omega^c}|u_n|^2dx \\ = \lambda\gamma_n^{-1-\gamma}\int_{\Omega^c}|u_n|^{1-\gamma}dx + \gamma_n^{p-1}\int_{\Omega^c}P(x)|u_n|^{p+1}dx.\end{aligned} \tag{3.27}$$

From the definition of c_1 given in (3.21) we obtain

$$\begin{aligned} c_1 \leq & \bar{I}(\gamma_n u_n) \\ = & \bar{I}_\infty(u) - \frac{\gamma_n^2}{2}\Big(\frac{1}{2}\iint_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx + \int_\Omega |u_n|^2dx\Big) \\ & + \frac{\gamma_n^{p+1}}{p+1}\int_{\mathbb{R}^N\setminus\Omega}(\tilde{P}-P(x))|u_n|^{p+1}dx + \frac{\gamma_n^{p+1}}{p+1}\int_\Omega \tilde{P}|u_n|^{p+1}dx + \frac{\lambda\gamma_n^{1-\gamma}}{1-\gamma}\int_\Omega |u_n|^{1-\gamma}dx \\ = & \bar{I}_\infty(\gamma_n u_n) - \frac{a_n\gamma_n^2}{2} + \frac{\gamma_n^{p+1}}{p+1}\int_\Omega \tilde{P}|u_n|^{p+1}dx + \frac{\gamma_n^{p+1}}{p+1}\int_{\mathbb{R}^N\setminus\Omega}(\tilde{P}-P(x))|u_n|^{p+1}dx \\ & + \frac{\lambda\gamma_n^{1-\gamma}}{1-\gamma}\int_\Omega |u_n|^{1-\gamma}dx \end{aligned} \tag{3.28}$$

where

$$a_n = \frac{1}{2}\iint_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx + \int_\Omega |u_n|^2dx.$$

We found from (3.27) that (γ_n) is bounded. For if not, then there exists a subsequence of (γ_n), still denoted by (γ_n), such that $\gamma_n \to \infty$. Moreover, as $n \to \infty$, we have

$$\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus\Omega^2}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\Omega^c}|u_n|^2dx \to \|\bar{u}\|^2. \tag{3.29}$$

Employing this in (3.27) leads to an absurdity that $\|\bar{u}\|^2 = \infty$. Therefore, (γ_n) is bounded.Thus, up to a subsequence we have $\gamma_n \to \gamma_0$. We now claim that $\gamma_0 = 1$.

On making a change of variables $\tilde{x} = x - \alpha_n$, $\tilde{y} = y - \alpha_n$, we see that

$$\begin{aligned}\|u_n\|^2_{H^s_{\Omega^c}} &= \frac{1}{2}\iint_{\mathbb{R}^{2N}} \chi_{\mathbb{R}^{2N}\setminus\Omega^2}(x,y)\frac{|\bar{u}(x-\alpha_n)-\bar{u}(y-\alpha_n)|^2}{|x-y|^{N+2s}}dydx \\ &\quad + \int_{\mathbb{R}^N}\chi_{\mathbb{R}^N\setminus\Omega}|\bar{u}(x-\alpha_n)|^2dx \\ &= \frac{1}{2}\iint_{\mathbb{R}^{2N}} \chi_{\mathbb{R}^{2N}\setminus\Omega^2}(x+\alpha_n,y+\alpha_n)\frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}}dydx \\ &\quad + \int_{\mathbb{R}^N}\chi_{\mathbb{R}^N\setminus\Omega}(x+\alpha_n)|\bar{u}(x)|^2dx.\end{aligned} \tag{3.30}$$

Since $|\alpha_n| \to \infty$,

$$\chi_{\mathbb{R}^{2N}\setminus\Omega^2}(x+\alpha_n,y+\alpha_n)\frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}} \to \frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}} \text{ a.e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N$$

and

$$\chi_{\mathbb{R}^N\setminus\Omega}(x+\alpha_n)|\bar{u}(x)|^2 \to |\bar{u}(x)|^2 \text{ a.e. } x\in\mathbb{R}^N.$$

Furthermore,

$$\left|\chi_{\mathbb{R}^{2N}\setminus\Omega^2}(x+\alpha_n,y+\alpha_n)\frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}}\right| \leq \frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{N+2s}} \in L^1(\mathbb{R}^N\times\mathbb{R}^N)$$

and

$$|\chi_{\mathbb{R}^N\setminus\Omega}(x+\alpha_n)|\bar{u}(x)|^2| \leq \bar{u}(x)|^2 \in L^1(\mathbb{R}^N).$$

Thus, by the Lebesgue's dominated convergence theorem we have

$$\|u_n\|^2_{H^s_{\Omega^c}} \to \|\bar{u}\| \text{ as } n\to\infty \tag{3.31}$$

and thus

$$\gamma_n^{1+\gamma}\|u_n\|^2_{H^s_{\Omega^c}} \to \gamma_0^{1+\gamma}\|\bar{u}\| \text{ as } n\to\infty. \tag{3.32}$$

By condition (P_1) we have

$$\gamma_n^{p+\gamma}\chi_{\mathbb{R}^N\setminus\Omega}(x+\alpha_n)P(x+\alpha_n)|\bar{u}(x)|^{p+1} \to \gamma_0^{p+\gamma}|\tilde{u}(x)|^{p+1} \text{ a.e. } x\in\mathbb{R}^N$$

and by the boundedness of P and (γ_n) we obtain

$$|\gamma_n^{p+\gamma}\chi_{\mathbb{R}^N\setminus\Omega}(x+\alpha_n)P(x+\alpha_n)|\bar{u}(x)|^{p+1}| \leq \gamma_0^{p+\gamma}M|\tilde{u}(x)|^{p+1} \text{ a.e. } x\in\mathbb{R}^N \in L^1(\mathbb{R}^N). \tag{3.33}$$

Thus by the Lebesgue's dominated convergence we have

$$\gamma_n^{p+\gamma}\int_{\mathbb{R}^N\setminus\Omega}P(x)|u_n|^{p+1}dx \to \gamma_0^{p+\gamma}\int_{\mathbb{R}^N}\tilde{P}|\bar{u}(x)|^{p+1}dx. \tag{3.34}$$

Similarly, on the other hand,

$$\lambda\chi_{\mathbb{R}^N\setminus\Omega}(x+\alpha_n)|\bar{u}(x)|^{1-\gamma} \to \lambda|\bar{u}(x)|^{1-\gamma} \text{ a.e. } x\in\mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N}\lambda\chi_{\mathbb{R}^N\setminus\Omega}(x+\alpha_n)|\bar{u}(x)|^{1-\gamma} \leq \int_{\mathbb{R}^N}\lambda|\bar{u}(x)|^{1-\gamma} \text{ a.e. } x\in\mathbb{R}^N.$$

Needless to say, by the Lebesgue's dominated convergence theorem again we have

$$\int_{\mathbb{R}^N\setminus\Omega}|u_n|^{1-\gamma}dx \to \int_{\mathbb{R}^N}|\bar{u}(x)|^{1-\gamma}dx. \tag{3.35}$$

Since, $\bar{u}$ is a solution to (P_∞) the limits (3.31), (3.34) and (3.35) give $\gamma_0 = 1$. Further, by (3.28) we also have

$$c_1 \leq \bar{I}_\infty(\bar{u}) - \frac{t_n\gamma_n^2}{2} + s_n = c_\infty - \frac{t_n\gamma_n^2}{2} + s_n. \tag{3.36}$$

Here,

$$b_n = \frac{\gamma_n^{p+1}}{p+1}\Big(\int_\Omega \tilde{P}|u_n|^{p+1}dx + \int_{\mathbb{R}^N\setminus\Omega}(\tilde{P}-P(x))|u_n|^{p+1}dx\Big) + \frac{\lambda\gamma_n^{1-\gamma}}{1-\gamma}\int_\Omega |u_n|^{1-\gamma}dx.$$

We will show that $a_n \to 0$ and $b_n \to \frac{\gamma_0^{p+1}}{p+1}\int_{\mathbb{R}^N}(\tilde{P}-P(x))|\bar{u}|^{p+1}dx < 0$ as $n \to \infty$ which is sufficient to show that $c_1 < c_\infty$.

In order to verify this, we first note that $\bar{u} \in H^s(\mathbb{R}^N)$ and this gives $a_n \to 0$ as $n \to \infty$. Using a similar argument, we get $b_n \to \frac{\gamma_0^{p+1}}{p+1}\int_{\mathbb{R}^N}(\tilde{P}-P(x))|\bar{u}|^{p+1}dx < 0$. Therefore, from (3.36) we have $c_1 < c_\infty$. □

Proof of Theorem 1.1. From (3.19) there exists a sequence $(u_n) \subset H^s_{\Omega^c}$ such that

$$\bar{I}(u_n) \to c_1 \text{ and } \bar{I}'(u_n) \to 0 \text{ as } n \to \infty.$$

Now since (u_n) is bounded, there exists a subsequence still denoted by (u_n), such that $u_n \rightharpoonup u$ for some $u \in H^s_{\Omega^c}$ and $\bar{I}'(u) = 0$. The condition that $\bar{I}'(u_n) \to 0$ implies that each u_n cannot be zero over a non-zero subset of Ω^c. For if it is, then it leads to a contradiction that $\bar{I}(u_n)$ is finite. We claim that $u \neq 0$. Let us assume on the contrary that $u = 0$. Since $c_1 > 0$, Lemma 3.1 guarantees the existence of $\rho, \beta > 0$ and $(z_n) \subset \Omega^c$ with $|z_n| \to \infty$ as $n \to \infty$ such that

$$\int_{B_r(z_n)\cap\Omega^c}|u_n|^2dx \geq \beta \qquad \forall n \in \mathbb{N}.$$

Then for each fixed $T > 0$, there exists $n_0 = n_0(T) \in \mathbb{N}$ such that

$$B(0,T) \subset \mathbb{R}^N \setminus (\Omega - z_n) \qquad \forall n \geq n_0.$$

Let $w_n(x) = u_n(x+z_n)$ for $x \in \Omega$ and $w_n(x) = u_n(x)$ for $x \in \mathbb{R}^N \setminus \Omega$. This defines w_n in the entire $\mathbb{R}^N$. Then we have some subsequence of (w_n), still denoted by (w_n), which is bounded in $H^s(B(0,T))$ for all $T > 0$. This is because (u_n) is bounded in $H^s_{\Omega^c}$ and therefore there exists a positive constant C such that

$$\begin{aligned} C &\geq \iint_{\mathbb{R}^{2N}\setminus\Omega^2}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N\setminus\Omega}|u_n|^2dx \\ &= \iint_{\mathbb{R}^{2N}\setminus(\Omega-z_n)^2}\frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}dydx + \int_{\mathbb{R}^N\setminus(\Omega-z_n)}|w_n|^2dx \\ &\geq \iint_{B(0,T)\times B(0,T)}\frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}dydx + \int_{B(0,T)}|w_n|^2dx = \|w_n\|^2_{H^s(B(0,T))}. \end{aligned} \tag{3.37}$$

This suggests that there exists a subsequence of (w_n), still denoted by (w_n), and a $v \in H^s_{\text{loc}}(\mathbb{R}^N)$ such that $w_n \rightharpoonup w$ in $H^s(B(0,T))$ as $n \to \infty$.

Further, by the lower semicontinuity of the norm we have

$$\|w\|_{H^s(B(0,T))} \leq \liminf_{n\to\infty}\|w_n\|_{H^s(B(0,T))} \leq C$$

for every $T > 0$. It follows from this that $w \in H^s(\mathbb{R}^N)$.

Let $\varphi \in H^s(\mathbb{R}^N)$ be a test function with bounded support. Since, $\bar{I}'(u_n) = o_n(1)$,

$$\langle \bar{I}'(u_n), \varphi(\cdot - z_n)\rangle = o_n(1). \tag{3.38}$$

Hence,

$$\begin{aligned} &\frac{1}{2}\iint_{\mathbb{R}^N\setminus(\Omega-z_n)^2}\frac{(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx+\int_{\mathbb{R}^N\setminus(\Omega-z_n)}w_n\varphi dx \\ &=\int_{\mathbb{R}^N\setminus(\Omega-z_n)}P(x+z_n)|w_n|^{p-1}w_n\varphi dx+\lambda\int_{\mathbb{R}^N\setminus(\Omega-z_n)}|w_n|^{-\gamma-1}w_n\varphi dx. \end{aligned} \tag{3.39}$$

By the weak convergence of w_n to w in $H^s(B(0,T))$, we realize that

$$\begin{aligned} &\frac{1}{2}\iint_{\mathbb{R}^{2N}\setminus(\Omega-z_n)^2}\frac{(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx+\int_{\mathbb{R}^N\setminus(\Omega-z_n)}w_n\varphi dx \\ &\to\frac{1}{2}\iint_{\mathbb{R}^{2N}}\frac{(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx+\int_{\mathbb{R}^N}w\varphi dx \text{ as } n\to\infty. \end{aligned} \tag{3.40}$$

Now suppose that $|z_n|\to\infty$. Then we have

$$P(x+z_n)\to\tilde{P} \text{ a.e. } x\in\mathbb{R}^N \quad \text{as } n\to\infty$$

and therefore

$$P(x+z_n)|w_n|^{p-1}w_n\varphi\to\tilde{P}|w|^{p-1}w\varphi \quad \text{a.e. } x\in\mathbb{R}^N \quad \text{as } n\to\infty.$$

Further,

$$|w_n|^{-\gamma-1}w_n\varphi\to|w|^{-\gamma-1}w\varphi \quad \text{a.e. } x\in\mathbb{R}^N \quad \text{as } n\to\infty.$$

These limits in combination with the boundedness of (w_n) in $L^{p+1}(\mathbb{R}^N\setminus\Omega)$ permits us to apply [13, 4.6], to obtain

$$\int_{B(0,T)}P(x+z_n)|w_n|^{p-1}w_n\varphi dx\to\int_{B(0,T)}\tilde{P}|w|^{p-1}w\varphi dx. \tag{3.41}$$

Note that since $w_n\rightharpoonup w$ as $n\to\infty$, we have by the compact embedding given in Lemma 2.1 (4) that $w_n\to w$ in $L^{p+1}(\mathbb{R}^N)$. Further, since φ is bounded and has bounded support,

$$|w_n|^{p-\gamma-1}w_n\varphi\to|w|^{-\gamma-1}w\varphi dx$$

and therefore

$$\int_{B(0,T)}|w_n|^{p-\gamma-1}w_n\varphi dx\to\int_{B(0,T)}|w|^{-\gamma-1}w\varphi dx. \tag{3.42}$$

Note that this also implies that w can never be zero over a subset (of $\mathbb{R}^N$) of non-zero measure (refer Appendix). Thus, (3.39)—(3.42) give

$$\langle\bar{I}'_\infty(w),\varphi\rangle=0.$$

Now by density we extend our test function space to $H^s(\mathbb{R}^N)$ and the last equality gives that w is a nontrivial solution of (P_∞). On computing the following

$$\begin{aligned} c_\infty&\leq\bar{I}_\infty(w)-\frac{1}{2}\langle\bar{I}'_\infty(w),w\rangle \\ &=\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N}\tilde{P}|w|^{p+1}dx+\lambda\Big(\frac{1}{2}-\frac{1}{1-\gamma}\Big)\int_{\mathbb{R}^N}|w|^{1-\gamma}dx \\ &\leq\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N}\tilde{P}|w|^{p+1}dx \\ &\leq\liminf_{n\to\infty}\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N\setminus(\Omega-z_n)}P(x+z_n)|w_n|^{p+1}dx \\ &=\liminf_{n\to\infty}\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N\setminus\Omega}P(x)|u_n|^{p+1}dx \\ &=\liminf_{n\to\infty}(\bar{I}(u_n)-\frac{1}{2}\langle\bar{I}'(u_n),u_n\rangle)=c_1 \end{aligned} \tag{3.43}$$

we get a violation of the Lemma 3.1. Thus $u \neq 0$. As seen earlier, we have that $\bar{I}(|u|) = \bar{I}(u)$ which yields $u \geq 0$ a.e. in Ω^c. However, by the Appendix we conclude that $u > 0$ a.e. in Ω^c. This proves the existence of a positive ground state solution to (**P**). □

3.1. **Existence of infinitely many solutions.** To begin with, we refer to the symmetric mountain pass theorem which is given in Colasuonno-Pucci [6, Theorem 2.2].

(PS) condition is satisfied by $\bar{I}$: Suppose (u_n) is a sequence such that $\bar{I}(u_n) \to c$ and $\bar{I}'(u_n) \to 0$ as $n \to \infty$. This implies that (u_n) is bounded in $H^s_{\Omega^c}$. Therefore there exists a subsequence, still denoted by (u_n), such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H^s_{\Omega^c}, \\ &u_n \to u \text{ in } L^r(\Omega^c), \quad 1 \leq r < 2^*_s. \end{aligned} \tag{3.44}$$

Consider the following.

$$\begin{aligned} o(1) &= \langle \bar{I}'(u_n), u_n - u \rangle \\ &= \langle u_n, u_n - u \rangle - \int_{\Omega^c} P(x)|u_n|^{p-1}u_n(u_n - u)dx - \lambda \int_{\Omega} |u_n|^{-\gamma-1}u_n(u_n - u)dx \\ &= \|u_n\|^2 - \|u\|^2 + o(1) \end{aligned} \tag{3.45}$$

where we have also used the Appendix. Therefore $u_n \to u$ in $H^s_{\Omega^c}$.

It is well known that if X is a Banach space then we have that

$$X = \bigoplus_{i \geq 1} X_i$$

where $X_i = \text{span}\{e_j\}_{j \geq i}$. Define

$$Y_m = \bigoplus_{1 \leq j \leq m} X_j, \qquad Z_m = \bigoplus_{j \geq m} X_j.$$

Clearly, Y_m is a finite dimensional subspace of X, for each m. Let $X = H^s_{\Omega^c}$. Due to the equivalence of any two norms in Y_m, we have

$$\begin{aligned} \bar{I}(u) &= \frac{1}{2}\|u\|^2_{H^s_{\Omega^c}} - \frac{\lambda}{1-\gamma}\int_{H^s_{\Omega^c}} |u|^{1-\gamma}dx - \frac{1}{p+1}\int_{H^s_{\Omega^c}} P(x)|u|^{p+1}dx \\ &\leq \frac{1}{2}\|u\|^2_{H^s_{\Omega^c}} - \frac{\lambda C_3}{1-\gamma}\|u\|^{1-\gamma}_{H^s_{\Omega^c}} - \frac{\tilde{P}C_4}{p+1}\|u\|^{p+1}_{H^s_{\Omega^c}} \leq 0. \end{aligned} \tag{3.46}$$

Thus for any finite dimensional subspace $\bar{X} \subset H^s_{\Omega^c}$, there exists a sufficiently large $r_0 = r(\bar{X})$ for which we have $\bar{I}(u) \leq 0$ whenever $\|u\| \geq r_0$. Hence, by the symmetric Mountain pass theorem there exists an unbounded sequence of critical values of $\bar{I}$ characterized by a minimax argument. From Remark 2.5 it follows that the problem in (**P'**) possesses infinitely many solutions. Therefore, the problem (**P**) also has infinitely many solutions.

3.2. **Boundedness of any solution to (P).** The idea used here is a usual one that appears in most literatures and hence we will only mention that an improvement of integrability is possible up to L^∞ assuming integrability of a certain order, say p. The boundedness will follow from a *bootstrap* argument. Without loss of generality we consider the set $\Omega' = \{x \in \Omega^c : u(x) > 1\}$ and thus from positivity of a fixed solution, say u, we have $u = u^+ > 0$ a.e. in Ω. Let $u \in L^\beta(\Omega^c)$ for $\beta > 1$. On testing with u^β we obtain the following:

$$\begin{aligned}
&\frac{1}{2}\langle u, u^{\beta}\rangle + \int_{\Omega^c} u^{\beta+1}dx \\
&= \Big(\lambda\int_{\Omega'}|u|^{\beta-\gamma}dx + \int_{\Omega'}|u|^{p-1+\beta}dx\Big)\frac{(\beta+1)^2}{4\beta} \\
&\leq \Big(\lambda\int_{\Omega'}|u|^{\beta}(1+|u|^{p-1})dx\Big)\frac{(\beta+1)^2}{4\beta};\ \text{since in } \Omega' \text{ we have } u>1 \\
&\leq \Big(\lambda\int_{\Omega'}|u|^{\beta}|u|^{p}dx\Big)\frac{(\beta+1)^2}{4\beta} \\
&\leq \lambda\beta C''\|u\|_{\alpha^*}^{p}\|u^{\beta}\|_{t};\ \text{by Hölder's inequality.}
\end{aligned} \tag{3.47}$$

Here $t=\frac{\alpha^*}{\alpha^*-p}$ for some $\alpha^*>1$, $t^*=\frac{tN}{N-ts}<2^*_{s^-}$. We further have

$$\begin{aligned}
C'\|u^{\frac{\beta}{2}}\|_{\alpha^*}^2 &\leq C'\|u^{\frac{\beta+1}{2}}\|_{\alpha^*}^2 \\
&\leq \int_{\Omega'}\frac{\Big|u(x)^{\frac{(\beta+1)}{2}}-u(y)^{\frac{(\beta+1)}{2}}\Big|^2}{|x-y|^{N+2s}}dxdy.
\end{aligned} \tag{3.48}$$

So, from (3.47) and (3.48), we get the following:

$$C'\|u^{\frac{\beta}{2}}\|_{\alpha^*}^2 \leq \lambda\beta C''\|u\|_{\alpha^*}^{p}\|u^{p}\|_{t}. \tag{3.49}$$

For the fixed $\alpha^*>1$ so chosen, we set $\eta=\frac{\alpha^*}{2t}>1$ for a suitable choice of t and $\tau=t\beta$ to get

$$\|u\|_{\eta\tau} \leq (\beta C)^{t/\tau}\|u\|_{\tau} \tag{3.50}$$

where $C=\lambda C''\|u\|_{\alpha^*}^{p}$ is a fixed quantity for a fixed solution u. Let us now iterate with $\tau_0=t$, $\tau_{n+1}=\eta\tau_n=\eta^{n+1}t$. After n iterations, the inequality (3.50) yields

$$\|u\|_{\tau_{n+1}} \leq C^{\sum_{i=0}^{n}\frac{t}{\tau_i}}\prod_{i=0}^{n}\Big(\frac{\tau_i}{t}\Big)^{\frac{t}{\tau_i}}\|u\|_{t}. \tag{3.51}$$

By using the fact that $\eta>1$ and the iterative scheme, i.e., $\tau_0=t$, $\tau_{n+1}=\eta\tau_n=\eta^{n+1}t$, we get

$$\sum_{i=0}^{\infty}\frac{t}{\tau_i}=\sum_{i=0}^{\infty}\frac{1}{\eta^i}=\frac{\eta}{\eta-1}$$

and

$$\prod_{i=0}^{\infty}\Big(\frac{\tau_i}{t}\Big)^{\frac{t}{\tau_i}}=\eta^{\frac{\eta^2}{(\eta-1)^2}}.$$

Therefore, passing to the limit in (3.51) as $n\to\infty$, we obtain

$$\|u\|_{\infty} \leq C^{\frac{\eta}{\eta-1}}\eta^{\frac{\eta^2}{(\eta-1)^2}}\|u\|_{t}. \tag{3.52}$$

Thus $u\in L^{\infty}(\Omega^c)$.

4. Appendix

We now claim that

$$\lim_{n\to+\infty}\int_{\Omega^c}|u_n|^{-\gamma-1}u_n v dx = \int_{\Omega^c}|u|^{-\gamma-1}uvdx<\infty \tag{4.53}$$

for v with bounded support.

Proof of the claim: The proof follows verbatim the one in [16].

5. Acknowledgement

The author DC thanks his teachers Prof. T. Amaranath, Prof. B.S. Padmavathi, Prof. G.P. Raja Sekhar, Prof. R. Radha for teaching him the art of research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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Received