Methods of Functional Analysis and Topology
Vol. 27 (2021), no. 2, pp. 130–141
https://doi.org/10.31392/MFAT-npu26_2.2021.02

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

UNIFORM AND MEAN ERGODIC THEOREMS FOR C_0-SEMIGROUPS

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Abstract. Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on a complex Banach space $\mathcal{X}$. In this paper, we study the uniform ergodicity for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ via the discrete ergodicity of a bounded linear operator $T(t_0)$, for some $t_0 > 0$. We show that for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\lim\limits_{t\to\infty} \frac{\|T(t)\|}{t} = 0$, the Cesàro averages $\frac{1}{t}\int_0^t T(s)ds$ of $\{T(t)\}_{t\geq 0}$ converges uniformly as $t \to \infty$ if and only if the Cesàro means $\frac{1}{n}\sum_{k=0}^{n-1} T^k(t_0)$ of an operator $T(t_0)$, for $t_0 > 0$, converges uniformly as $n \to \infty$. Furthermore, we investigate the strong convergence of the Cesàro averages of $\{T(t)\}_{t\geq 0}$, so that we give some sufficient conditions implying that $\{T(t)\}_{t\geq 0}$ is mean ergodic.

Нехай $\{T(t)\}_{t\geq 0}$ – C_0-півгрупа обмежених лінійних операторів у комплексному банаховому просторі $\mathcal{X}$. Вивчається її рівномірна ергодичність шляхом зведення до дискретної ергодичності обмеженого лінійного оператора $T(t_0)$, для деякого $t_0 > 0$. Показано, що для C_0-півгрупи $\{T(t)\}_{t\geq 0}$, такої, що $\lim\limits_{t\to\infty} \frac{\|T(t)\|}{t} = 0$, середні Чезаро $\frac{1}{t}\int_0^t T(s)ds$ рівномірно збігаються при $t \to \infty$ тоді й тільки тоді, коли середні Чезаро $\frac{1}{n}\sum_{k=0}^{n-1} T^k(t_0)$ оператора $T(t_0)$, де $t_0 > 0$, рівномірно збігаються при $n \to \infty$. Крім того, досліджується сильна збіжність середніх Чезаро від $\{T(t)\}_{t\geq 0}$; даються достатні умови, за яких $\{T(t)\}_{t\geq 0}$ ергодична в середньому.

1. Introduction

Throughout this paper, $\mathcal{B}(\mathcal{X})$ denotes the Banach algebra of all bounded linear operators on a Banach space $\mathcal{X}$ into itself, with the unit element I. If $T \in \mathcal{B}(\mathcal{X})$, then we denote by $N(T)$, $R(T)$, $\sigma(T)$, $\rho(T)$, and $\mathcal{R}(\lambda, T)$ the kernel, the range, the spectrum, the resolvent set and the resolvent function of T, respectively. The notation $\mathcal{X} = Y \oplus Z$ means $\mathcal{X}$ is a topological direct sum of linear subspaces Y and Z; where Y and Z are closed.

Let $T \in \mathcal{B}(\mathcal{X})$, the Cesàro means of T are defined by

$$\mathcal{M}_n(T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k \ , \quad \text{where} \quad n \in \mathbb{N}^*. \tag{1.1}$$

If $\mathcal{M}_n(T)$ converges in the norm (resp. strong) operator topology, then T is called uniformly (resp. mean) ergodic. The interest in such operators has its origins in statistical mechanics and probability theory. In such a setting, one also considers continuous processes Φ_t, with t specifying time, such that $\Phi_t\big(\Phi_s(u)\big) = \Phi_{t+s}(u)$, for all points u in a phase space and all times s and t. The abstract setting consists of a semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on $\mathcal{X}$, and one investigates the long term behavior of $\{T(t)\}_{t\geq 0}$ via its Cesàro averages

$$\mathcal{C}(t) = \frac{1}{t}\int_0^t T(s)\ ds, \quad \text{for} \quad t > 0. \tag{1.2}$$

2020 *Mathematics Subject Classification.* 47A35, 47D03, 47A10.
Keywords. C_0-semigroup, Cesàro averages, Mean ergodic operator, Uniform ergodic theorems, Ergodic decomposition.

If $\lim_{t\to\infty} \mathcal{C}(t)$ exists in the norm (resp. strong) operator topology, then $\{T(t)\}_{t\geq 0}$ is called uniformly (resp. mean) ergodic. Fix $t_0 > 0$ and $n \in \mathbb{N}$, we have $\mathcal{C}(nt_0) = \mathcal{M}_n\big(T(t_0)\big)\mathcal{C}(t_0)$, hence we see the connection between the Cesàro averages $\mathcal{C}(nt_0)$ of the semigroup $\{T(t)\}_{t\geq 0}$ and the discrete Cesàro averages $\mathcal{M}_n\big(T(t_0)\big)$ of the individual operator $T(t_0)$. For this, we are interested to examine simultaneously the uniform ergodicity of family semigroup $\{T(t)\}_{t\geq 0}$ and the uniform ergodicity of individual operator $T(t_0)$, for some $t_0 > 0$. It is worth mentioning that the strong convergence of the Cesàro averages $\mathcal{C}(t)$, as $t \to \infty$, does not imply the strong convergence of the discrete Cesàro averages $\mathcal{M}_n\big(T(t_0)\big)$, as $n \to \infty$. For more details see [8, Note p.83].

The ergodic theory was launched by Von Neumann in the 1930s [23], who proved that for every unitary operator T in a complex Hilbert space $\mathcal{H}$, the limit of Cesàro averages $\mathcal{M}_n(T)x$ exists for all $x \in \mathcal{H}$, and the limit P is a projection of $\mathcal{H}$ onto the kernel $N(I-T)$ along $(I-T)\mathcal{H}$. A simple proof of Von Neumann's mean ergodic theorem, due to F. Riesz, appeared in 1937, and was followed by more general results (see e.g. [13, 17]). Kakutani and Yosida in 1938 [25], obtained characterizations of the convergence of the Cesàro averages $\mathcal{M}_n(T)x$ for x in a Banach space $\mathcal{X}$: A power bounded operator $T \in \mathcal{B}(\mathcal{X})$ is mean ergodic if and only if $\mathcal{X}$ has the following decomposition (called the ergodic decomposition)

$$\mathcal{X} = \{x \in \mathcal{X} :\ Tx = x\} \oplus \overline{(I-T)\mathcal{X}}. \tag{1.3}$$

Generally, the right-hand side of (1.3) is precisely the subspace of $x \in \mathcal{X}$ (called the mean ergodic subspace) for which the Cesàro averages $\mathcal{M}_n(T)x$ converges.

Much of modern works occurs to studies the uniform ergodicity for individual operators (see e.g. [2] [8, Ch.II]). An interesting basic result needed in this regard is the uniform ergodic theorem of M. Lin in 1974 [10], who proved that for $T \in \mathcal{B}(\mathcal{X})$ satisfying $\lim_{n\to\infty} \|T^n\|\backslash n = 0$, T is uniformly ergodic if and only if 1 is a simple pole of the resolvent of T, if and only if the range of $(I-T)$ is closed. This result has been generalized by Mbekhta and Zemánek in 1993 [15], in a way that the last condition of the M. Lin's theorem was replaced with the following: there exists an integer $k \geq 1$ such that the range $(I-T)^k\mathcal{X}$ is closed. A number of studies have followed, among those we point out [1, 4, 9, 24].

The classical uniform ergodic theorem for C_0-semigroup of bounded linear operators on a Banach space $\mathcal{X}$, goes back to M. Lin in 1974 [11], he shows that for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\lim_{t\to\infty} \|T(t)\|\backslash t = 0$, $\{T(t)\}_{t\geq 0}$ is uniformly ergodic if and only if the range of its infinitesimal generator A is closed. In [19], S.Y. Shaw proved that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is uniformly ergodic if and only if it satisfies:

(i) the infinitesimal generator A has a closed range,
(ii) the resolvent function $\mathcal{R}(\lambda, A)$ of A, exists for $\lambda > 0$,
(iii) $\lim_{t\to\infty} \frac{\|T(t)\mathcal{R}(\lambda,A)\|}{t} = 0$ for some $\lambda > 0$.

Further conditions equivalent to the uniform convergence of the Cesàro averages $\mathcal{C}(t)$, have been obtained more recently by several authors, see e.g. [5, 7, 19] and the references therein. On the mean ergodicity for C_0-semigroup of bounded linear operators on a Banach space $\mathcal{X}$, there exists an extensive bibliography, see e.g. [6, 14, 18, 20].

This paper is organized as follows. In Section 2, we give some definitions and fundamental properties concerning C_0-semigroups of bounded linear operators on a Banach space $\mathcal{X}$. Also, we present the uniform ergodic theorem for an individual operator T and for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$. In Section 3, we study the uniform ergodicity for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ via the discrete ergodicity of a bounded linear operator $T(t_0)$, for $t_0 > 0$. We show that for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\lim_{t\to\infty} \|T(t)\|\backslash t = 0$, the Cesàro averages $\mathcal{C}(t)$ of $\{T(t)\}_{t\geq 0}$ converges uniformly as $t \to \infty$ if and only if the

Cesàro means $\mathcal{M}_n\big(T(t_0)\big)$ of the operator $T(t_0)$, for some $t_0 > 0$, converges uniformly as $n \to \infty$, which is not suitable in the case of strong convergence. As a consequence, we give a theorem which can be considered as a version of the Gelfand-Hille theorem. Next, we are also interested to investigate the strong convergence of the Cesàro averages of $\{T(t)\}_{t\geq 0}$. More precisely, we show that for an infinitesimal generator A of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ satisfying the following:

(i) $\sup_{t\geq 0} \|\mathcal{C}(t)\| < \infty$, and
(ii) $\lim\limits_{t\to\infty} \|T(t)x\| \backslash t = 0$, for every $x \in \mathcal{X}$.

If either of the following hold

(1) The descent $des(A)$ of A is finite.
(2) $R(A^n)$ is closed for some $n > 1$, or
(3) $R(A^j) + N(A^k)$ is closed for some positive integers $j, k \geq 1$.

Then $\{T(t)\}_{t\geq 0}$ is mean ergodic.

2. Preliminaries

A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space $\mathcal{X}$ is called a C_0-semigroup or a strongly continuous semigroup of operators if

(1) $T(0) = I$,
(2) $T(t+s) = T(t)T(s)\,;\ \ \forall t, s \geq 0$,
(3) $\lim\limits_{t\to 0} T(t)x = x;\ \forall x \in \mathcal{X}$.

$\{T(t)\}_{t\geq 0}$ has a unique infinitesimal generator A defined in domain $D(A)$ by

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t};\ \forall x \in D(A),$$

with $D(A) = \{x \in \mathcal{X} :\ \lim\limits_{t\to 0} \dfrac{T(t)x - x}{t}$ exists$\}$.
In this case, we have the following properties [3, Ch. I] , [16, Ch. I and II]:

(1) $T(t)$ is a bounded linear operator on $\mathcal{X}$ for all $t \geq 0$.
(2) A is closed and $\overline{D(A)} = \mathcal{X}$.
(3) For all $x \in \mathcal{X}$ and $t \geq 0$,

$$\int_0^t T(s)xds \in D(A) \ \text{ and } \ A\int_0^t T(s)xds = T(t)x - x. \tag{2.4}$$

(4) For all $x \in D(A)$ and $t \geq 0$,

$$T(t)x \in D(A) \ \text{ and } \ T'(t) = AT(t)x = T(t)Ax. \tag{2.5}$$

Recall that, if $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on a Banach space $\mathcal{X}$ and A be their infinitesimal generator, then the resolvent function of A is the Laplace transform of $T(t)$ (see e.g. [16, Paragraph p.25]), that's mean

$$\mathcal{R}(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)dt. \tag{2.6}$$

Now, we give the following lemma, which may play an important role in the sequel.

Lemma 2.1. [12, Lemma 5.2] *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on a Banach space $\mathcal{X}$, and let A be their infinitesimal generator. Then the following relations hold:*

(1) $R(A) = (\lambda\mathcal{R}(\lambda, A) - I)\mathcal{X}$.
(2) $N(A) = \{x \in \mathcal{X} :\ \lambda\mathcal{R}(\lambda, A)x = x\} = \mathcal{F}\{T(t)\}$,

where $\mathcal{F}\{T(t)\} = \{x \in \mathcal{X} :\ T(t)x = x;\ t \geq 0\}$, the set of fixed points of $T(t)$.

Let A be a closed linear operator on a Banach space $\mathcal{X}$, with domain $D(A) \subset \mathcal{X}$. The smallest non-negative integer p such that $N(A^p) = N(A^{p+1})$ is called the ascent of A and denoted by $asc(A)$. If such an integer does not exist, we set $asc(A) = \infty$. Similarly, the smallest non-negative integer q such that $R(A^q) = R(A^{q+1})$ is called the descent of A and denoted by $des(A)$. If such an integer does not exist, we set $des(A) = \infty$. If $A \in \mathcal{B}(\mathcal{X})$ and we have $asc(A)$ and $des(A)$ are both finite, then $asc(A) = des(A)$, which is not true if A is a closed linear operator with $D(A) \subsetneq \mathcal{X}$, (see [22, Theorem 6.2]).
For $A \in \mathcal{B}(\mathcal{X})$, we have the following equivalences, see [4, Lemma 1.1]:

$$asc(A) \leq p \iff R(A^p) \cap N(A^j) = \{0\};\ j = 1, 2, ...$$

$$des(A) \leq q \iff \mathcal{X} = R(A^j) + N(A^q);\ j = 1, 2, ...$$

The following Theorem of this section can be deduced from the corresponding classical results of [10] and [15]. Theorems of this nature are referred to in the literature as Ergodic Theorems.

Theorem 2.2. [10, 15] *Let $T \in \mathcal{B}(\mathcal{X})$ such that $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$. Then, the following statements are equivalent:*

(1) *T is uniformly ergodic,*
(2) *$(I - T)^k\mathcal{X}$ is closed for some $k = 1, 2, \dots$,*
(3) *the point 1 is a simple pole of the resolvent function of T,*
(4) *the operator $I - T$ has a finite descent.*

The limit in (1) is the projection P of $\mathcal{X}$ onto $N(I-T)$ along $(I-T)\mathcal{X}$, that is, the Riesz projection corresponding to the simple pole 1 of the resolvent of T.

Recall that, a C_0-semigroup $\{T(t)\}_{t\geq 0} \in \mathcal{B}(\mathcal{X})$, is said to be uniformly Abel ergodic if the Abel averages, defined by

$$\mathcal{A}(\lambda) = \lambda \int_0^\infty e^{-\lambda t} T(t)dt. \tag{2.7}$$

converges uniformly as $\lambda \to 0^+$. Clearly, if $\{T(t)\}_{t\geq 0}$ is uniformly ergodic then is uniformly Abel ergodic, but the reverse is not true, see e.g. [19] and [8, Chapter 2] for more information. Further, we have from identity (2.6), $\mathcal{A}(\lambda) = \lambda\mathcal{R}(\lambda, A)$, for every $\lambda > 0$, where $\mathcal{R}(\lambda, A)$ is the resolvent of A.

Now, we will need the following results which can be considered as a version of the ergodic theorems for semi-groups.

Theorem 2.3. [11, Theorem] *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on $\mathcal{X}$ satisfying $\lim_{t\to\infty} \frac{\|T(t)\|}{t} = 0$, and A be their infinitesimal generator. Then, the following statements are equivalent:*

(1) *$\{T(t)\}_{t\geq 0}$ is uniformly ergodic,*
(2) *$\mathcal{X} = R(A) \oplus N(A)$, with $R(A)$ is closed,*
(3) *the range $R(A)$ of A is closed,*
(4) *for some $\lambda > 0$, the operator $\mathcal{A}(\lambda)$ is uniformly ergodic, that is, the Cesàro averages $\mathcal{M}_n\big(\mathcal{A}(\lambda)\big)$ converges uniformly on $\mathcal{B}(\mathcal{X})$, as $n \to \infty$.*

Moreover, the limit in (1) and (4) is the same, is the projection P of $\mathcal{X}$ onto $N(A)$ along $R(A)$, corresponding to the ergodic decomposition

$$\mathcal{X} = R(A) \oplus N(A).$$

Corollary 2.4. [21, Corollary 3] *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on $\mathcal{X}$ satisfying $\lim_{t\to\infty} \frac{\|T(t)\|}{t} = 0$, and A be their infinitesimal generator. Then, the following statements are equivalent:*

(1) *$\{T(t)\}_{t\geq 0}$ is uniformly ergodic,*
(2) *$\{T(t)\}_{t\geq 0}$ is uniformly Abel ergodic,*
(3) *$\mathcal{X} = R(A) \oplus N(A)$, with $R(A)$ is closed,*

(4) $R(A^k)$ *is closed for some (equivalent all) integer* $k \geq 1$,
(5) $R(A^k) + N(A^m)$ *is closed for some (equivalent all) integers* $k, m \geq 1$.

3. Main results and proofs

3.1. **Uniform ergodicity of C_0-semigroups.** This subsection is devoted to study the relationship between the uniform ergodicity for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ and the discrete ergodicity of a bounded linear operator $T(t_0)$, for some $t_0 > 0$. Furthermore, we give a theorem which can be considered as a version of the Gelfand-Hille theorem.

The first main result of this paper is the following theorem.

Theorem 3.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on $\mathcal{X}$ such that $\lim\limits_{t\to\infty} \frac{\|T(t)\|}{t} = 0$. Then, $\{T(t)\}_{t\geq 0}$ is uniformly ergodic if and only if there exists an operator $T(t_0)$, for $t_0 > 0$, which is uniformly ergodic.*

Proof. Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on $\mathcal{X}$ and A be their infinitesimal generator with domain $D(A) \subset \mathcal{X}$. Let's denote

$$S(t) = \int_0^t T(s)ds \text{ for all } t \geq 0.$$

Next, we assume that $\{T(t)\}_{t\geq 0}$ satisfies $\lim\limits_{t\to\infty} \frac{\|T(t)\|}{t} = 0$, then by M. Lin's theorem [11], the range $R(A)$ of A is closed and we have $\mathcal{X} = R(A) \oplus N(A)$.

Let show that the range $R\big(I - T(t_0)\big)$, for $t_0 > 0$, of the operator $I - T(t_0)$ is closed. Let $(x_n)_{n\geq 0} \subset R\big(I - T(t_0)\big)$, such that $x_n \longrightarrow x$, hence we can write $x_n = (I - T(t_0))u_n = AS(t_0)u_n$, with $u_n \subset \mathcal{X}$ (see [3, Lemma II.1.9]). Since $S(t_0)$ is bounded and $R(A)$ is closed, then $x_n \longrightarrow AS(t_0)u$ for $u \in \mathcal{X}$, which means that $x = AS(t_0)u = (I - T(t_0))u$. Therefore $R(I - T(t_0))$ is closed.

On the other hand, we have $\lim\limits_{t\to\infty} \frac{\|T(t)\|}{t} = 0$, then there exists $\varepsilon > 0$ and $t > 0$ large enough such that

$$\|T(t)\| \leq \varepsilon t.$$

So, we can take $n = \frac{t}{t_0}$, we get

$$\frac{\|T^n(t_0)\|}{n} = t_0 \frac{\|T(nt_0)\|}{nt_0} = t_0 \frac{\|T(t)\|}{t} \leq t_0 \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty} \frac{\|T^n(t_0)\|}{n} = 0$, and Theorem 2.2 implies that the operator $T(t_0)$ is uniformly ergodic.

Conversely, let $T(t_0)$, for $t_0 > 0$, be an operator belongs to the family $\{T(t)\}_{t\geq 0}$, such that $T(t_0)$ is uniformly ergodic, hence by the hypothesis and Theorem 2.2, we obtain $R\big(I - T(t_0)\big)$ is closed. We will show that $R(A)$ is closed, let $z \in \overline{R(A)}$, that is, there exists $(u_n)_{n\geq 0} \subset D(A)$, such that $Au_n \longrightarrow z$, hence

$$\big(I - T(t)\big)u_n = S(t)Au_n, \text{ for all } t \geq 0.$$

It's clear that $S(t) = \int_0^t T(s)ds$ for all $t \geq 0$ is a bounded linear operator, then

$$S(t)Au_n \longrightarrow S(t)z \text{ as } n \to \infty, \text{ for all } t \geq 0.$$

Since the range $R\big(I - T(t_0)\big)$ is closed, and

$$\big(I - T(t_0)\big)u_n = S(t_0)Au_n \longrightarrow S(t_0)z, \text{ as } n \to \infty,$$

then

$$S(t_0)z \in R(I - T(t_0)).$$

Therefore, we can take $y \in D(A)$ such that $S(t_0)z = \big(I - T(t_0)\big)y = S(t_0)Ay$, which yields that $z - Ay \in N\big(S(t_0)\big)$.

Now, let show that $N\big(S(t_0)\big) \subset N(A)$, to infer that $z \in R(A)$. To this end, we show that $\int_0^{t_0} S(s)xds \in D(A)$ and $A\int_0^{t_0} S(s)xds = S(t_0)x - t_0x$. Indeed, for $x \in \mathcal{X}$ and $t_0 > 0$,

$$\begin{aligned}\frac{T(h)-I}{h}\int_0^{t_0} S(s)xds &= \frac{1}{h}\int_0^{t_0} T(h)S(s)xds - \frac{1}{h}\int_0^{t_0} S(s)xds\\ &= \int_0^{t_0}\Big[\frac{1}{h}\int_0^{s} T(h)T(u)xdu - \frac{1}{h}\int_0^{s} T(u)xdu\Big]ds\\ &= \int_0^{t_0}\Big[\frac{1}{h}\int_0^{s} T(h+u)xdu - \frac{1}{h}\int_0^{s} T(u)xdu\Big]ds\\ &= \int_0^{t_0}\Big[\frac{1}{h}\int_h^{s+h} T(u)xdu - \frac{1}{h}\int_0^{s} T(u)xdu\Big]ds\\ &= \int_0^{t_0}\Big[\frac{1}{h}\int_s^{s+h} T(u)xdu - \frac{1}{h}\int_0^{h} T(u)xdu\Big]ds.\end{aligned}$$

Since $\frac{1}{h}\int_s^{s+h} T(u)xdu - \frac{1}{h}\int_0^{h} T(u)xdu$ converges to $T(s)x - x$ when $h \to 0$, we have that

$$A\int_0^{t_0} S(s)xds = \int_0^{t_0} T(s)xds - \int_0^{t_0} xds,$$

which means that

$$A\int_0^{t_0} S(s)xds = S(t_0)x - t_0x.$$

Therefore, we can easily deduce that $N\big(S(t_0)\big) \subset R(A)$. Hence $z \in R(A)$, which yields that $R(A)$ is closed, and Theorem 2.3 implies that $\{T(t)\}_{t\geq 0}$ is uniformly ergodic. □

The following result is an immediate consequence of the previous theorem, Theorem 2.2 and Corollary 2.4.

Corollary 3.2. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on $\mathcal{X}$ satisfying $\lim\limits_{t\to\infty} \frac{\|T(t)\|}{t} = 0$, and A be their infinitesimal generator. Then, the following statements are equivalent:*

1. *$\{T(t)\}_{t\geq 0}$ is uniformly ergodic,*
2. *$\{T(t)\}_{t\geq 0}$ is uniformly Abel ergodic,*
3. *there exists $t_0 > 0$ such that the range of the operator $\big(I - T(t_0)\big)^k$ is closed for some integer $k \geq 1$,*
4. *there exists $t_0 > 0$ such that the operator $I - T(t_0)$ has a finite descent,*
5. *there exists $t_0 > 0$ such that $\mathcal{X} = R\big(I - T(t_0)\big) \oplus N\big(I - T(t_0)\big)$,*
6. *there exists $t_0 > 0$ such that 1 is a simple pole of the resolvent $T(t_0)$,*
7. *the point 0 is a simple pole of the resolvent $\mathcal{R}(.,A)$ of A,*
8. *the infinitesimal generator A has a finite descent.*

The next Lemma can be considered a version of the Gelfand-Hille theorem. Thus, we give in the Theorem 3.4 a new version of the Gelfand-Hille theorem corresponding to the C_0-semigroups.

Lemma 3.3. [15, Corollaire 2] *Given an operator $T \in \mathcal{B}(\mathcal{X})$, with $\sigma(T) = \{1\}$. If T is uniformly ergodic, then $T = I$.*

Theorem 3.4. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on a complex Banach space $\mathcal{X}$. If there exists an operator $T(t_0)$, for some $t_0 > 0$, such that $T(t_0)$ be uniformly ergodic and $\sigma\big(T(t_0)\big) = \{1\}$, then $T(t) = I$, for all $t \geq 0$.*

Proof. We assume that $T(t_0)$, for $t_0 > 0$, be an operator uniformly ergodic, Theorem 3.1 implies that the C_0-semigroup $\{T(t)\}_{t\geq 0}$ is uniformly ergodic and the limit of Cesàro averages $\mathcal{C}(t)$ is the projection P of $\mathcal{X}$ onto $N(A)$ along $R(A)$, corresponding to ergodic decomposition $\mathcal{X} = R(A) \oplus N(A)$. From Lemma 2.1, we have

$$R(A) = (\lambda\mathcal{R}(\lambda, A) - I)\mathcal{X} \text{ and } N(A) = \cap_{t\geq 0} N\big(I - T(t)\big) = \mathcal{F}\{T(t)\}.$$

Now, we assume that $\sigma(T(t_0)) = \{1\}$, then Lemma 3.3 implies that $T(t_0) = I$. Therefore

$$\mathcal{X} = N\big(I - T(t_0)\big) = R(A) \oplus \cap_{t\geq 0} N\big(I - T(t)\big). \tag{3.8}$$

Next, we set y belongs to $N\big(I - T(t_0)\big) \cap R(A)$. Since $y \in N\big(I - T(t_0)\big)$, then $T(t_0)y = y$, it follows that $T(t_0)^n y = y$ for all $n \geq 0$, hence $T(nt_0)y = y$. Therefore $\lim\limits_{n\to\infty} T(nt_0)y = y$, hence

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t T(s)yds = y. \tag{3.9}$$

Since $y \in R(A)$ and $R(A) = N(P)$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t T(s)yds = 0. \tag{3.10}$$

Thus, (3.9) and (3.10) implies that $y = 0$, hence $N\big(I - T(t_0)\big) \cap R(A) = \{0\}$, which means that $R(A) = \{0\}$ by the decomposition (3.8). Then

$$\mathcal{X} = \cap_{t\geq 0} N\big(I - T(t)\big).$$

Therefore, $T(t) = I$ for all $t \geq 0$, and the proof is finished. □

3.2. **Mean ergodicity of C_0-semigroups.** In general, the uniform ergodicity in all the results considered above cannot be replaced by the mean ergodicity. The proof of the uniform ergodicity often requires proving first the mean ergodicity (in order to obtain the ergodic decomposition). Our purpose of this subsection it to study the strong convergence of the Cesàro averages of $\{T(t)\}_{t\geq 0}$, so that we give some sufficient conditions implying that $\{T(t)\}_{t\geq 0}$ is mean ergodic.

In terms of the ergodic decomposition, we give the following lemma which will be widely used in the sequel.

Lemma 3.5. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on $\mathcal{X}$ such that $\lim\limits_{t\to\infty} \frac{\|T(t)x\|}{t} = 0$, for every $x \in \mathcal{X}$, with A be the infinitesimal generator. If $y \in R(A)$ and $z \in N(A)$, then*

$$\frac{1}{t}\int_0^t T(s)(y+z)ds = z + O\Big(\frac{1}{t}\Big), \quad \textit{as } t \to \infty.$$

Proof. Let A be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ with domain $D(A) \subset \mathcal{X}$, and let $\mathcal{C}(t)$ be the Cesàro averages defined in (1.2).

For $z \in N(A)$, hence Lemma 2.1 implies that $T(t)z = z$. It follows that

$$\mathcal{C}(t)z = \frac{1}{t}\int_0^t T(s)zds = z \text{ for all } t \geq 0.$$

Let $y \in R(A)$, then there exists $w \in D(A)$ such that $y = Aw$. Then, we get

$$\mathcal{C}(t)y = \frac{1}{t}\int_0^t T(s)yds = \frac{1}{t}\int_0^t T(s)Awds = \frac{1}{t}\int_0^t \frac{d}{ds}[T(s)w]ds = \frac{1}{t}[T(t)w - w].$$

Therefore

$$\Big\|\frac{1}{t}\int_0^t T(s)yds\Big\| \leq \Big\|\frac{T(t)w}{t}\Big\| + \Big\|\frac{w}{t}\Big\| \leq C^{ste}\Big\|\frac{w}{t}\Big\|.$$

Then

$$\frac{1}{t}\int_0^t T(s)(y+z)ds = z + O\Big(\frac{1}{t}\Big), \text{ as } t \to \infty.$$

□

We denote by $\mathcal{X}_{me}$ the subspace of all $x \in \mathcal{X}$ such that the Cesàro averages $\mathcal{C}(t)x$ converges strongly on $\mathcal{B}(\mathcal{X})$, in the case where $\mathcal{X}_{me}$ equal to the space $\mathcal{X}$, then we get that C_0-semigroup $\{T(t)\}_{t\geq 0}$ is mean ergodic. The following theorem gives a characterization of the subspace $\mathcal{X}_{me}$.

Theorem 3.6. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on $\mathcal{X}$ and A be its infinitesimal generator. If $T(t)$ satisfies the following conditions:*

(1) $\sup_{t\geq 0}\|\mathcal{C}(t)\| < \infty$, *and*
(2) $\lim\limits_{t\to\infty} \frac{\|T(t)x\|}{t} = 0$, *for every* $x \in \mathcal{X}$.

Then, $\mathcal{X}_{me}$ is closed and $T(t)$-invariant subspace for all $t \geq 0$. Moreover,

$$\mathcal{X}_{me} \;=\; \overline{R(A)} \oplus N(A).$$

Proof. Let A be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ with domain $D(A) \subset \mathcal{X}$, and let $\mathcal{X}_{me}$ be a subspace of $\mathcal{X}$ defined by:

$$\mathcal{X}_{me} := \Big\{\ x \in \mathcal{X} : \lim_{t\to\infty} \mathcal{C}(t)x \text{ exists}\Big\}.$$

First, we have each of the assumptions (1) or (2) alone implies that

$$R(A) \cap N(A) = \{0\}.$$

And by Lemma 3.5, we observe that

$$\overline{R(A)} \oplus N(A) \subset \mathcal{X}_{me}.$$

For reverse inclusion, the first hypothesis implies that $\mathcal{X}_{me}$ is closed and $T(t)$-invariant subspace of $\mathcal{X}$ for all $t \geq 0$. Let us show that any $x \in \mathcal{X}_{me}$ is belong to $\overline{R(A)} + N(A)$. To do that, let $x \in \mathcal{X}_{me}$, then there exists an operator $P \in \mathcal{B}(\mathcal{X})$, for which $\lim\limits_{t\to\infty} \|\mathcal{C}(t)x - Px\| = 0$. Hence, we have the follows:

$$\begin{aligned}
\big(I - T(t)\big)Px &= \big(I - T(t)\big)\lim_{s\to\infty}\mathcal{C}(s)x \\
&= \lim_{s\to\infty}\mathcal{C}(s)x - \lim_{s\to\infty}T(t)\mathcal{C}(s)x \\
&= \lim_{s\to\infty}\mathcal{C}(s)x - \lim_{s\to\infty}\frac{1}{s}\int_0^s T(t)T(k)xdk \\
&= \lim_{s\to\infty}\mathcal{C}(s)x - \lim_{s\to\infty}\frac{1}{s}\int_0^s T(t+k)xdk \\
&= \lim_{s\to\infty}\mathcal{C}(s)x - \lim_{s\to\infty}\frac{1}{s}\int_t^{s+t} T(\nu)xd\nu \\
&= 0.
\end{aligned}$$

Therefore

$$T(t)Px = Px, \text{ for all } t \geq 0.$$

So, we get

$$\mathcal{C}(t)Px = Px, \text{ for all } t \geq 0.$$

Hence

$$P^2x = Px = T(t)Px = PT(t)x, \text{ for all } t \geq 0.$$

Now, we suppose that $\mathcal{X}_{me} = \mathcal{X}$, which means that $\{T(t)\}_{t\geq 0}$ is mean ergodic. Then,

$$P^2 = P = T(t)P = PT(t), \text{ for all } t \geq 0.$$

In particular, P is a projection corresponding to the following decomposition

$$\mathcal{X} = R(P) \oplus N(P).$$

Since $N(A) = \{x \in \mathcal{X} : \ T(t)x = x, \ t \geq 0\}$, it is easy to check that

$$R(P) = N(A).$$

Next, let's show that $N(P) = \overline{R(A)}$. Indeed, as shown above in Lemma 3.5, that for every $y \in R(A)$, we have $\mathcal{C}(t)y \to 0$, as $t \to \infty$, which means that $R(A) \subset N(P)$, then we get $\overline{R(A)} \subset N(P)$.

For the inverse inclusion, let $x \in N(P)$ and we denote by $S(t) = \int_0^t T(u)du$. Then, we have

$$\lim_{t\to\infty} \mathcal{C}(t)x = \lim_{t\to\infty} \frac{S(t)x}{t} = 0.$$

Fix $a > 0$, and using integration by parts, we obtain the following inequality:

$$\begin{aligned}
\|\mathcal{A}(\lambda)x\| &= \|\lambda\mathcal{R}(\lambda, A)x\| \\
&\leq \Big\|\lambda^2 \int_0^\infty e^{-\lambda t} S(t)x dt\Big\| \\
&\leq \lambda^2 \int_0^a e^{-\lambda t}\|S(t)x\|dt + \lambda^2 \int_a^\infty e^{-\lambda t} t\Big\|\frac{S(t)x}{t}\Big\|dt \\
&\leq \lambda^2 a \sup_{t\leq a}\|S(t)x\| + \sup_{t>a}\Big\|\frac{S(t)x}{t}\Big\|dt.
\end{aligned}$$

Since $\lim\limits_{t\to\infty} \dfrac{S(t)x}{t} = 0$, then it easy to see from the above estimation that

$$\lim_{\lambda\to 0^+} \lambda\mathcal{R}(\lambda, A)x = 0.$$

It follows that

$$\lim_{t\to\infty} \mathcal{C}(t)x = \lim_{\lambda\to 0^+} \lambda\mathcal{R}(\lambda, A)x = 0.$$

And the following identity:

$$A\big(\mathcal{R}(\lambda, A)\big)x = \lambda\mathcal{R}(\lambda, A)x - x, \ \text{ for every } x \in \mathcal{X},$$

implies that

$$\lim_{\lambda\to 0^+} A\big(\mathcal{R}(\lambda, A)x\big) = x.$$

Therefore, $x \in \overline{R(A)}$, hence $N(P) \subset \overline{R(A)}$ and the equality holds.
Finally, we deduce that $\mathcal{X}_{me} = \overline{R(A)} \oplus N(A)$, and the proof is finished. □

As a consequence of the previous theorem, we get the following corollary.

Corollary 3.7. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on $\mathcal{X}$ and A be their infinitesimal generator with domain $D(A) \subset \mathcal{X}$. We assume that $\{T(t)\}_{t\geq 0}$ satisfies the following conditions:*

(1) $\sup_{t\geq 0}\|\mathcal{C}(t)\| < \infty$, *and*
(2) $\lim\limits_{t\to\infty} \frac{\|T(t)x\|}{t} = 0$, *for every* $x \in \mathcal{X}$.

If the Cesàro averages $\mathcal{C}(t)x$ converges strongly (resp. weakly) for all $x \in D(A)$, then $\{T(t)\}_{t\geq 0}$ is mean (resp. weakly) ergodic.

Proof. From the closure of $\mathcal{X}_{me}$ and density of the domain $D(A)$ of the infinitesimal generator A, we deduce the proof. □

Theorem 3.8. *Let A be the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$, and assume that $T(t)$ satisfies the following conditions:*

(1) $\sup_{t\geq 0}\|\mathcal{C}(t)\| < \infty$, *and*
(2) $\lim\limits_{t\to\infty} \frac{\|T(t)x\|}{t} = 0$, *for every* $x \in \mathcal{X}$.

If descent $des(A)$ of A is finite, then $\{T(t)\}_{t\geq 0}$ is mean ergodic.

Proof. Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space $\mathcal{X}$ and A be their infinitesimal generator. As shown above, in Theorem 3.6, that if $T(t)$ satisfies the conditions (1) and (2), then $\mathcal{X}_{me}$ be closed and $T(t)$-invariant subspace. Moreover, we have each of the assumptions (1) or (2) alone implies that

$$R(A) \cap N(A) = \{0\},$$

which yields that $asc(A) \leq 1$.
Now, we suppose that descent $des(A)$ of A is finite, then there exists an integer $n \geq 1$ such that $R(A^n) = R(A^{n+1})$. So, to prove that $\mathcal{X} = R(A) \oplus N(A)$ it suffices to show $R(A^k) = \big(\lambda\mathcal{R}(\lambda, A) - I\big)^k \mathcal{X}$ for all integer $k \geq 1$.
Indeed, the equality $\big(\lambda\mathcal{R}(\lambda, A) - I\big) = A\mathcal{R}(\lambda, A)$, shows that the assertion is true for $k = 1$. Assuming that $R(A^k) = \big(\lambda\mathcal{R}(\lambda, A) - I\big)^k \mathcal{X}$ for some $k > 1$, hence

$$\begin{aligned}\big(\lambda\mathcal{R}(\lambda, A) - I\big)^{k+1} &= \big(\lambda\mathcal{R}(\lambda, A) - I\big)\big(\lambda\mathcal{R}(\lambda, A) - I\big)^k \\ &= A\mathcal{R}(\lambda, A)A^k\mathcal{R}(\lambda, A)^k,\end{aligned}$$

which implies that $R(A^{k+1}) = \big(\lambda\mathcal{R}(\lambda, A) - I\big)^{k+1}\mathcal{X}$.
Therefore

$$R(A^k) = \big(\lambda\mathcal{R}(\lambda, A) - I\big)^k\mathcal{X}, \text{ for all integer } k \geq 1.$$

Since $R(A^n) = R(A^{n+1})$, then $\big(\lambda\mathcal{R}(\lambda, A) - I\big)^n\mathcal{X} = \big(\lambda\mathcal{R}(\lambda, A) - I\big)^{n+1}\mathcal{X}$, which means that $des\big(\lambda\mathcal{R}(\lambda, A) - I\big) < \infty$. In addition, from $\big(\lambda\mathcal{R}(\lambda, A) - I\big)$ is a bounded linear operator on $\mathcal{X}$ and $asc\big(\lambda\mathcal{R}(\lambda, A) - I\big) = asc(A) \leq 1$, then $des\big(\lambda\mathcal{R}(\lambda, A) - I\big)$ and $asc\big(\lambda\mathcal{R}(\lambda, A) - I\big)$ are both equals. Consequently,

$$\mathcal{X} = \big(\lambda\mathcal{R}(\lambda, A) - I\big)\mathcal{X} \oplus N\big(\lambda\mathcal{R}(\lambda, A) - I\big).$$

From the Lemma 2.1, we get $\mathcal{X} = R(A) \oplus N(A)$.
Next, we apply Theorem 3.6, we get $\mathcal{X} = \mathcal{X}_{me}$, which means that $T(t)$ is mean ergodic. This completes the proof. ☐

We end this section with the following result.

Theorem 3.9. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on a Banach space $\mathcal{X}$, and let A be their infinitesimal generator. Suppose that $\{T(t)\}_{t\geq 0}$ satisfies the following conditions:*

(i) $\sup_{t\geq 0}\|\mathcal{C}(t)\| < \infty$, *and*
(ii) $\{\frac{T(t)x}{t}\}_{t\geq 0}$ *converges to* 0 *in some operator topology.*

If either of the following hold

(1) $R(A^n)$ *is closed for some* $n > 1$, *or*
(2) $R(A^j) + N(A^k)$ *is closed for some positive integers* $j, k \geq 1$.

Then, $\mathcal{X}$ is the direct sum of the closed subspaces $R(A)$ and $N(A)$. Moreover, the Cesàro averages $\mathcal{C}(t)$ converges in some operator topology.

We need the following auxiliary result to prove this theorem.

Lemma 3.10. *Let A be a closed linear operator with domain $D(A) \subset \mathcal{X}$ such that $asc(A) = d < \infty$. If either of the following hold:*

(i) $R(A^n)$ *is closed for some* $n > d$, *or*
(ii) $R(A^j) + N(A^k)$ *is closed for some positive integers* j, k *with* $j + k = n \geq d$.

Then, $R(A^n)$ is closed for all $n \geq d$, and $R(A^j) + N(A^k)$ is closed for all integers j, k with $j + k \geq d$.

Proof. Whenever A and B are linear operators on a vector space, we have the following identity:

$$A^{-1}R(AB) = R(B) + N(A).$$

Then, for a linear operator A on a vector space and for some integers $j, k \geq 1$, we have

$$A^{-k}R(A^jA^k) = R(A^j) + N(A^k).$$

Then, we infer the following properties:

(a) If $R(A^n)$ is closed, so is $R(A^j) + N(A^k)$ whenever $j + k = n$.

(b) For $n \geq d$, the range $R(A^n)$ is closed whenever $R(A^n) + N(A^m)$ is closed for some $m \geq 1$.

Assume that A be a closed linear operator on X with domain $D(A) \subset \mathcal{X}$ such that $asc(A) = d < \infty$, then

$$R(A^d) \cap N(A^m) = \{0\}, \text{ for all } m = 1, 2, ...$$

Now, we separate the hypothesis. Let $R(A^n)$ is closed for some $n > d$, hence by (a) we get, $R(A^j) + N(A^k)$ is closed whenever $j + k = n$. So take $j = n - 1$ and $k = 1$, then by (b), $R(A^{n-1})$ is closed. Therefore, by induction $R(A^j)$ is closed for all $d \leq j \leq n$. Since $R(A^{n-1}) \cap N(A) = \{0\}$, then the restriction of A to the closed invariant subspace $R(A^{n-1})$ is one to one. Thus the restriction is a Banach space isomorphism from the closed subspace $R(A^{n-1})$ onto the closed subspace $R(A^n)$. It carries the subspace $R(A^n)$ onto $R(A^{n+1})$, which must be also closed. Hence $R(A^n)$ is closed for all $n \geq d$.

Next, suppose $R(A^n) + N(A^m)$ is closed for some $n > d$ and $m \geq 1$. Hence, by (b) we get $R(A^n)$ is closed for some $n > d$. Again, we applied the first argument, we deduce that $R(A^n)$ is closed for all $n \geq d$, so is $R(A^n) + N(A^m)$ for all $n \geq d$ and $m \geq 1$. □

Proof. of Theorem 3.9: Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space $\mathcal{X}$ and A be their infinitesimal generator. we have each of the assumptions (i) or (ii) alone implies that $R(A) \cap N(A) = \{0\}$, which yields that $asc(A) \leq 1$. From Lemma 3.10, we have if (1) or (2) holds, then $R(A^n)$ is closed for all $n \geq 1$ and $R(A^j) + N(A^k)$ is closed for all positive integers $j, k \geq 1$.

On the other hand, we have the adjoint $T^*(t)$ of C_0-semigroup $T(t)$ is a semigroup not necessary strongly continuous, and its generator is exactly the adjoint of A. Since $T(t)$ satisfies the conditions (i) and (ii), then so is $T^*(t)$. Therefore

$$R(A^*) \cap N(A^*) = \{0\}.$$

Consequently

$$asc(A^*) \leq 1.$$

Since $R(A^n)$ is closed for all $n \geq 1$, then we get

$$R(A^2) =^{\perp} \left(N({A^*}^2)\right) =^{\perp} \left(N(A^*)\right) = R(A).$$

Finally, we apply the Theorem 3.6 and Theorem 3.8 to complete the proof. □

Acknowledgments

We gratefully acknowledge the constructive comments of the referee concerning this paper.

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Received 26/07/2020; Revised 12/03/2021