Methods of Functional Analysis and Topology
Vol. 23 (2017), no. 4, pp. 346–366

SPECTRAL PROPERTIES AND STABILITY OF A NONSELFADJOINT EULER–BERNOULLI BEAM

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Abstract. In this note we study the spectral properties of an Euler–Bernoulli beam model with damping and elastic forces applying both at the boundaries as well as along the beam. We present results on completeness, minimality, and Riesz basis properties of the system of eigen- and associated vectors arising from the nonselfadjoint spectral problem. Within the semigroup formalism it is shown that the eigenvectors have the property of forming a Riesz basis, which in turn enables us to prove the uniform exponential decay of solutions of the particular system considered.

This paper was corrected on October 11, 2024. Lemma 5 had the assumption that $\alpha_0, \alpha_1 \geq 0$. *From this assumption it was stated that* A *was boundedly invertible, which is not true. To ensure that* A *has a closed bounded inverse, the lemma now is proven under the stronger assumption that* $\alpha_0, \alpha_1 > 0$. *All results derived in the paper satisfy this assumption. Additionally, several typos and misprints have also been corrected.*

1. Introduction

The study of stability and stabilisation problems in infinite dimensions is certainly an interesting but also challenging mathematical task. The challenges are twofold. On the one hand, there is more than one way to extend the idea of stability from finite-dimensional to infinite-dimensional spaces, depending on the choice of norm induced by the space considered. On the other hand, the spectral mapping property which holds for systems with bounded system operators (that is, bounded generating operators of their corresponding semigroups) does not hold in general for infinite-dimensional systems with unbounded system operators. Unfortunately, this means that infinite-dimensional systems do not necessarily have spectrum-determined growth. So merely requiring the spectrum of the infinitesimal generator to lie in the open left half-plane does not always guarantee that the semigroup is stable, let alone exponentially stable, when it is strongly continuous, in contrast to the case when it is for example uniformly continuous. This fact has been known for more than half a century and goes back at least to an implicit counterexample of Hille and Phillips [13, p. 665], and many other counterexamples have appeared since. In this respect it should be noted that in addition to being of interest in its own right, the spectrum-determined growth condition – which actually holds for a rather large class of semigroups – is integral to many major aspects of infinite-dimensional systems. These include, for example, such aspects as controllability and observability, as well as, in particular, stabilisability. A good introductory treatment of these subjects may be found in the book by Curtain and Zwart [5].

One situation, however, where it is justified to relate the spectrum of the system operator to that of the semigroup arises when the system operator is a discrete spectral operator (in the terminology of Dunford [6, Chapters XVIII to XX]) whose system of

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2010 *Mathematics Subject Classification.* 34B07, 34L10, 34L20, 35B40.

Key words and phrases. Euler–Bernoulli beam, nonselfadjoint operator, operator pencil, completeness, minimality, Riesz basis, series expansion, exponential stability.

eigen- and associated vectors (in short root vectors) form an unconditional or Riesz basis, that is, a basis equivalent to an orthonormal basis for the underlying Hilbert space. Miloslavskii in [24], and redeveloped in [25], and Röh in [33] were among the first to note the connection between the stability properties of infinite-dimensional systems and the Riesz basis property of the root vectors.

In general, the system operator is not necessarily a discrete spectral operator. In fact, in most of the literature dealing with other approaches to the problems of stability and stabilisability (using the rather sophisticated multiplier or Liapunov methods for example), only very basic mathematical properties of the system operator are verified for the purposes of unique solvability or well-posedness of the associated Cauchy problem: it is maximal dissipative, and hence is closed and densely defined, and usually has a compact resolvent. Yet, these preliminary results would also prove useful in obtaining Riesz basis results from that point on and then use a purely spectral approach to such problems. We recall from [6, Theorem XVIII.2.32 or Corollary XVIII.2.33] that in order that a closed linear operator, A, say, in a Hilbert space be a discrete spectral operator, the following properties are needed: (i) A has a compact resolvent; and (ii) the root vectors of A form a Riesz basis (with or without parentheses) for the underlying Hilbert space. To show that A is a discrete spectral operator, as will be seen, and as is well known for so-called "properly spaced" spectra, one essentially only needs verification of (i) and hence that A has purely discrete spectrum, which lies in a strip of finite height parallel to the real axis, and is separated by a uniform gap, in the sense that for eigenvalues λ_n of A,

$$\sup \operatorname{Im} \lambda_n < \infty, \qquad \inf_{n \neq m} |\lambda_n - \lambda_m| > 0, \tag{1.1}$$

and that the root vectors of A are minimal complete. Note that sequences of eigenvalues having the properties in (1.1) are called interpolating.

Let us describe the system we study in this paper. We consider a damped Euler–Bernoulli beam of unit length on an elastic foundation and with a combination of elastic and damping effects in the boundary conditions. Specifically, we consider the initial/boundary-value problem consisting of the partial differential equation

$$\frac{\partial^4}{\partial s^4} w(s,t) + \gamma^2 w(s,t) + 2\gamma \frac{\partial}{\partial t} w(s,t) + \frac{\partial^2}{\partial t^2} w(s,t) = 0 \tag{1.2}$$

governing beam vibration, together with boundary conditions

$$\left. \frac{\partial^2}{\partial s^2} w(s,t) \right|_{s=0} = 0, \tag{1.3}$$

$$\left. \left(\frac{\partial^3}{\partial s^3} w(s,t) + \alpha_0 w(s,t) + \beta_0 \frac{\partial}{\partial t} w(s,t) \right) \right|_{s=0} = 0, \tag{1.4}$$

$$\left. \frac{\partial^2}{\partial s^2} w(s,t) \right|_{s=1} = 0, \tag{1.5}$$

$$\left. \left(\frac{\partial^3}{\partial s^3} w(s,t) - \alpha_1 w(s,t) - \beta_1 \frac{\partial}{\partial t} w(s,t) \right) \right|_{s=1} = 0 \tag{1.6}$$

and initial conditions

$$w(s,0) = h_0(s), \qquad \left. \frac{\partial}{\partial t} w(s,t) \right|_{t=0} = h_1(s), \tag{1.7}$$

where the functions h_0, h_1 are assumed given and sufficiently smooth (as specified later in Section 6). Here $s \in [0,1]$, $t \geq 0$, and $w(s,t)$ represents the transverse displacement of the beam at a point s and time t. The parameter γ determines both the external viscous damping and elastic foundation, and the pairs α_0, β_0 and α_1, β_1 represent combinations of elastic and viscous damping effects at the boundary points $s = 0$ and $s = 1$, respectively. It is interesting to note that (1.2) is a fourth-order version of the classical telegrapher's equation [2, p. 192]. Obviously an important source of difficulty here comes from the viscoelastic boundary conditions (1.4) and (1.6).

The initial/boundary-value problem (1.2)–(1.7) corresponds to the interesting and relevant case for stabilisability when it can be associated with an energy-dissipative system. Indeed, if we consider the total vibrational energy of the beam at time t

$$E(t) = \frac{1}{2}\left\{\int_0^1 \left[\left(\frac{\partial^2}{\partial s^2} w(s,t)\right)^2 + \gamma^2 w(s,t)^2 \right.\right.$$
$$\left.\left. + \left(\frac{\partial}{\partial t} w(s,t)\right)^2\right] ds + \alpha_0 w(0,t)^2 + \alpha_1 w(1,t)^2\right\},$$

which is the sum of the kinetic and potential energies, then, for smooth solutions of (1.2)–(1.7), a straightforward computation using integration by parts verifies the energy-dissipation relation

$$\frac{d}{dt}E(t) = -2\gamma \int_0^1 \left(\frac{\partial}{\partial t} w(s,t)\right)^2 dt - \beta_0 \left(\frac{\partial}{\partial t} w(s,t)\right)^2\Bigg|_{s=0} - \beta_1 \left(\frac{\partial}{\partial t} w(s,t)\right)^2\Bigg|_{s=1},$$

and we have $E(t)$ nonincreasing with time if we assume that $\beta_0, \beta_1, \gamma \geq 0$. However, the question of whether or not in that case $E(t) \to 0$ (if at all) at a uniform exponential rate as t tends to infinity is much more difficult to resolve but can be intimately related to analysis of the spectral problem associated with (1.2)–(1.7).

A substantial portion of our work will be devoted, as well as to conclude information about the spectrum of the spectral problem, but also to addressing the study of the geometric properties of the root vectors connected with the operator-theoretic or abstract version of (1.2)–(1.7) in state-space form. This requires an involved analysis because, since the system is dissipative, the system operator will not be selfadjoint in general. As indicated previously, analysing the spectrum of the system operator for verification of the interpolation properties in (1.1), essentially, is only one step (albeit a crucial one) in the verification that it has a Riesz basis property and thus is a discrete spectral operator in the state space. Once a thorough analysis of the spectrum – that is, existence, location, and, in particular, asymptotics of eigenvalues – is completed, in actuality there still remains the problem of establishing that the root vectors are minimal complete and form a Riesz basis for the whole of the state space. We omit here the standard details of the definitions of completeness, minimality, Riesz bases or unconditional bases, and so on, which can be found, for example, in [8, 18, 29].

In this paper we follow an argument to prove the Riesz basis property which avoids the need for deriving asymptotic expansions of the root vectors as required when working with the familiar theorem of Bari [8, Theorem VI.2.3]. It is possible to use Bari's theorem, a line that has been pursued by many investigators in the field during the past two decades or so, such as Conrad and Morgül [1], Cox and Zuazua [3, 4], Guo [11, 12], Rao [32], and Xu and Feng [36], to name only a few. However, we feel that it will be interesting to see how we can prove the Riesz basis property not by working with asymptotic expansions of the root vectors, but by using knowledge of the asymptotics of the eigenvalues. Such proof method is indeed possible, and we demonstrate this by making use of a less well-known theorem, proven in different ways for dissipative bounded linear operators, by Glazman and, in a paper in Doklady, Mukminov in the 1950's, as well as in extended form by Markus [21]. The sufficient conditions given in those papers, probably being the earliest abstract results on Riesz basis properties, will be seen for the specific system under consideration here to be a more direct route to verification of the Riesz basis property, since the system operator possesses a structure that is related directly to the well-known spectral properties of a compact selfadjoint operator in the specialised nondissipative case. The idea is to some degree, in fact, related to the rationale behind Kato's results [14, Section V.4]. For additional details, see the book by Gohberg and Krein [7, Sections VI.4 and 6] and the recent survey [35] of Shkalikov.

An essential prerequisite for our undertaking is the verification of completeness of the root vectors, which usually is the challenging part. Here, however, we exploit the aforementioned specific structure of the system operator to prove completeness, and we will explore in detail exactly how this can be accomplished in Section 5. The key ingredient there is provided by a version of a well-known theorem of Keldysh which was published in his famous 1951 paper. Properly exploited, it enables us to establish in a rather straightforward way a completeness result for the system operator in the state space. (An English translation of Keldysh's original theorem is contained in the Appendix of the text [22] by Markus, and further discussion on other versions of his results can be found, for example, in [16, 31] and in [7, Sections V.8 to 10].)

The rest of the paper is structured as follows. Section 2 treats some preliminary material including relevant definitions, while our essential starting point is Section 3, where we pose the abstract spectral problems and define the underlying spaces and operators relevant to them. This is an important first step which we take in a way that will have advantages for the study of the spectrum later. For example, instead of working on the spectral problem only in state-space form, we choose to additionally work in another Hilbert product space. In doing so, we can invoke some of the theory of quadratic operator pencils in the development in Section 4 of proofs for many results on spectral properties (with only minor redundancy in the results). The state space is particularly suited to studying the problems, to be considered in Section 5, of completeness, minimality, and Riesz basis properties, which in turn are ultimately used in Section 6 to solve the problem of stability of solutions of the initial/boundary-value problem (1.2)–(1.7) in its semigroup formulation. So, where it is relevant, we will link the findings of Section 4 to the spectral problem in the state space. The results of Section 6 are the culmination of the results of the foregoing sections.

There are a few interesting papers which are close in spirit to this paper. Here, we must particularly note the works of Gomilko and Pivovarchik [9, 10], Möller and Pivovarchik [27], Pivovarchik [30], and a series of works of Möller and Zinsou, many of which are listed in [37]. Apart from some methodological differences, the main difference between our work and these papers lies in the type of system we consider. Although, from a mathematical point of view, it is a rather special system (due primarily to the parameters in the partial differential equation (1.2) being all constants), it has a prominent place in the engineering literature, mainly as the so-called "half-car" or "bridge" models, which describe the structural dynamics of suspended vehicles and bridges. An example is the early paper [19] by the author and others. That paper, however, lacks rigour in that it does not justify mathematically the familiar method used there of series expansion in root vectors of solutions to initial/boundary-value problems. It is clear that in the end this comes down to establishing conditions for the convergence, in some sense, of the series expansion, and here lies the need for study of the geometric properties of the root vectors. To date, as far as we know, no rigorous mathematical study has been conducted that adequately addresses these problems (and the underlying ones) in the context of the present system. We shall give such a study in this paper, and we show in Section 6 how to apply our results to rigorously establish the uniform exponential decay of the solutions.

2. Definitions and preliminaries

To begin with, let us assume a separable solution to the initial/boundary-value problem (1.2)–(1.7) described in the Introduction, for some spectral parameter ω. We specify the relationship $\omega = \lambda + i\gamma$ between ω and another spectral parameter λ, and obtain, on making the substitution

$$w(s,t) = e^{i(\lambda+i\gamma)t} w(\lambda, s),$$

the boundary-eigenvalue problem

$$w^{(4)}(\lambda, s) - \lambda^2 w(\lambda, s) = 0, \tag{2.1}$$

$$w''(\lambda, 0) = 0, \tag{2.2}$$

$$w^{(3)}(\lambda, 0) + (\alpha_0 - \beta_0\gamma + i\beta_0\lambda) w(\lambda, 0) = 0, \tag{2.3}$$

$$w''(\lambda, 1) = 0, \tag{2.4}$$

$$w^{(3)}(\lambda, 1) - (\alpha_1 - \beta_1\gamma + i\beta_1\lambda) w(\lambda, 1) = 0. \tag{2.5}$$

For convenience, we define

$$\theta_0(\lambda) = \alpha_0 - \beta_0\gamma + i\beta_0\lambda, \quad \theta_1(\lambda) = \alpha_1 - \beta_1\gamma + i\beta_1\lambda \tag{2.6}$$

in the boundary conditions (2.3) and (2.5) and unless otherwise specified, it is understood that

$$\alpha_0, \alpha_1 > 0, \quad \beta_0, \beta_1, \gamma \geq 0, \quad \alpha_0 > \beta_0\gamma, \quad \alpha_1 > \beta_1\gamma$$

throughout.

The boundary-eigenvalue problem (2.1)–(2.5) has λ-dependent boundary conditions. It is impossible therefore to recast it abstractly as a spectral problem for a linear operator in $L_2(0,1)$. It does, however, fit in the abstract framework if one considers its operator-theoretic formulation in an extended product-space context. This is the route that we will follow, and we wish to do this in two spaces each endowed with appropriate topologies. In one product space we will fit (2.1)–(2.5) into the setting of quadratic operator pencils which are nonmonic (that is, the leading operator coefficient of the pencil will not be the identity operator), and the same problem but with λ replaced by $\omega - i\gamma$ will be fit into the setting of linear monic operator pencils or, equivalently, linear operators in the other product space. The spaces as well as the operators in them basic to the abstract formulations are precisely defined in the next section.

Let us recall now, for completeness, some standard definitions of the spectral theory of operator pencils with generally unbounded operator coefficients as a convenience for the reader. A good account of the spectral theory of operator pencils, with many application examples from mechanics, is given by Möller and Pivovarchik in their recent text [26]; refer there and to [22] for equivalent definitions.

Definition 1. *Let $\lambda \mapsto P(\lambda)$ be a mapping from $\mathbb{C}$ (or some nonempty subset thereof) into the set of closed linear operators in Hilbert space. A number $\lambda \in \mathbb{C}$ is said to belong to $\varrho(P)$, the resolvent set of P, provided $P(\lambda)$ has a closed bounded inverse, that is, provided $P(\lambda)$ is boundedly invertible. We call $P^{-1}(\lambda) := (P(\lambda))^{-1}$ the resolvent of $P(\lambda)$. The complement of $\varrho(P)$ is the spectrum of P and is denoted by $\sigma(P)$. If a number $\lambda_0 \in \mathbb{C}$ has the property that* $\ker P(\lambda_0) \neq \{0\}$ *then it is called an eigenvalue of P and there exists an eigenvector $x_0 \neq 0$ corresponding to λ_0 such that $P(\lambda_0)x_0 = 0$.*

Definition 2. *A sequence of vectors $\{x_r\}_{r=0}^{h-1}$ is said to form a chain, of length h, consisting of an eigenvector x_0 of P corresponding to an eigenvalue λ_0 and the vectors $x_1, x_2, \ldots, x_{h-1}$ associated with it, or, for brevity, simply a chain of root vectors of P*

corresponding to λ_0, *if*

$$\sum_{l=0}^{r} \frac{1}{l!} \frac{d^l}{d\lambda^l} P(\lambda)\Big|_{\lambda=\lambda_0} x_{r-l} = 0, \quad r = 0, 1, \ldots, h-1.$$

The geometric multiplicity of an eigenvalue λ_0 *is the number of linearly independent eigenvectors in a system of chains of root vectors of* P *corresponding to* λ_0 *and is defined as* $\dim\ker P(\lambda_0)$. *The algebraic multiplicity of an eigenvalue is the maximum value of the sum of the lengths of chains corresponding to the linearly independent eigenvectors. We call an eigenvalue* λ_0 *simple if its geometric and algebraic multiplicities are equal and* $\dim\ker P(\lambda_0) = 1$.

Definition 3. *If an eigenvalue* λ_0 *of* P *is an isolated point in* $\sigma(P)$ *and* $P(\lambda_0)$ *is a Fredholm operator, then we call* λ_0 *a normal eigenvalue. The set of all normal eigenvalues is denoted by* $\sigma_0(P)$.

Remark 1. *In the case of a linear operator pencil* $P(\lambda) = \lambda I - A$ *a chain of root vectors of* P *corresponding to an eigenvalue* λ_0 *coincides with a chain of root vectors of the operator* A *corresponding to the same eigenvalue. Then obviously within the context of* A *in the standard abstract spectral problem the definitions carry over.*

For Riesz basis properties, a suitable indexing of the eigenvalues is crucial. This leads to the following definition.

Definition 4. *A sequence of eigenvalues is said to be properly enumerated if it is a sequence of complex numbers* $\{\lambda_n \in \mathbb{C}\}$ *which are counted properly – that is, such that*

(i) *their multiplicities are taken into account;*
(ii) $\lambda_{-n} = -\overline{\lambda_n}$ *when* $\operatorname{Re}\lambda_n \neq 0$; *and*
(iii) $\operatorname{Re}\lambda_{n+1} \geq \operatorname{Re}\lambda_n$.

As we have already mentioned in the Introduction, prerequisite conditions for the verification of the Riesz basis property of root vectors are completeness and minimality. These can be deduced readily from the following two results, the first of which is due to Keldysh [7, Theorem X.4.1] on the completeness property. For a proof of the second result, on the minimality property, see, for example, [20, Lemma 2.4]. We shall invoke both later in Section 5.

Lemma 1. *Let* K *be a compact selfadjoint operator on a Hilbert space* X *with* $\ker K = \{0\}$. *Let the sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *represent the eigenvalues* K, *and assume*

$$\sum_{n=1}^{\infty} |\lambda_n|^p < \infty \tag{2.7}$$

for some $p \geq 1$. *Suppose further that* S *is a compact operator such that* $I+S$ *is invertible. Then the root vectors of the operator*

$$A = K(I+S) \tag{2.8}$$

are complete in X.

Lemma 2. *Let* A *be a compact operator on* X *and* $\ker A = \{0\}$. *Then the root vectors of* A *are minimal in* X.

The principal tool in establishing the Riesz basis property of the eigenvectors in Section 5 will be the individual theorems of Glazman and Mukminov. The following is a basic version of their result (see [8, p. 328] or [18, p. 213]).

Lemma 3. *Let A be a bounded dissipative operator on a Hilbert space X and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of its eigenvalues such that*

$$\sum_{\substack{n,m=1\\ n\neq m}}^{\infty} \frac{\operatorname{Im}\lambda_n \operatorname{Im}\lambda_m}{\left|\lambda_n - \overline{\lambda_m}\right|^2} < \infty. \tag{2.9}$$

Then the corresponding eigenvectors in X form a Riesz basis (in fact, a Bari basis) for its closed linear hull.

We close this section with our first result, which essentially guarantees the existence of an infinite number of normal eigenvalues of the boundary-eigenvalue problem (2.1)–(2.5). Its proof rests entirely on results proven by Mennicken and Möller in [23, Sections 7.2 and 7.3], and the reader is referred there for further information.

Proposition 1. *The boundary-eigenvalue problem* (2.1)–(2.5) *under the spectral transformation $\lambda \mapsto \mu^2$ is Birkhoff regular in the sense of* [23, Definition 7.3.1].

Proof. We assume the change from λ to μ^2 in (2.1)–(2.5) and consider (2.6) as

$$\theta_0(\mu) = \alpha_0 - \beta_0\gamma + i\beta_0\mu^2, \quad \theta_1(\mu) = \alpha_1 - \beta_1\gamma + i\beta_1\mu^2.$$

First of all, note that the differential equation (2.1) has associated with it a characteristic function of degree four, defined by [23, (7.1.4)], which takes here the form $\pi(\rho) = \rho^4 - 1$. The zeros of π are i^{k-1}, $k = 1, 2, 3, 4$, and it is easily verified that the assumptions for [23, Theorem 7.2.4.A] are satisfied. Thus there is a 4×4 transformation matrix which we can choose to be

$$C(s,\mu) = \operatorname{diag}(1, \mu, \mu^2, \mu^3)\,(i^{(j-1)(k-1)})_{j,k=1}^{4}.$$

Consequently, in view of [23, (7.3.1)], we have for the boundary matrices

$$W^{(0)}(\mu) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \theta_0(\mu) & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu) = \begin{pmatrix} \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$W^{(1)}(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\theta_1(\mu) & 0 & 0 & 1 \end{pmatrix} C(1,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{pmatrix},$$

where

$$\eta_j := \theta_0(\mu) + (-i)^{j-1}\mu^3, \quad \zeta_j := -\theta_1(\mu) + (-i)^{j-1}\mu^3, \quad j = 1, 2, 3, 4.$$

From [23, Definition 7.3.1 and Theorem 7.3.2], choosing $C_2(\mu) = \operatorname{diag}(\mu^2, \mu^3, \mu^2, \mu^3)$, we have that

$$C_2^{-1}(\mu)\,W^{(0)}(\mu) = W_0^{(0)} + O(\mu^{-1}), \quad C_2^{-1}(\mu)\,W^{(1)}(\mu) = W_0^{(1)} + O(\mu^{-1}),$$

where

$$W_0^{(0)} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

For Birkhoff regularity, we must require the nonsingularity of the Birkhoff matrices (I here being the identity matrix)

$$W_0^{(0)}\Delta_j + W_0^{(1)}(I - \Delta_j), \quad j = 1, 2, 3, 4,$$

in which the Δ_j are, according to [23, Proposition 4.1.7 and Definition 7.3.1], 4×4 diagonal matrices whose diagonal elements consist of two consecutive ones, followed by two consecutive zeros in a cyclic arrangement: $\Delta_1 := \operatorname{diag}(1,1,0,0)$, $\Delta_2 := \operatorname{diag}(0,1,1,0)$, $\Delta_3 := \operatorname{diag}(0,0,1,1)$, and $\Delta_4 := \operatorname{diag}(1,0,0,1)$. The Birkhoff matrices can all be verified to be nonsingular, and therefore the proposition is established. □

3. Abstract spectral problems

In this section our intention is to recast the boundary-eigenvalue problem (2.1)–(2.5) abstractly as operator pencils acting in spaces which we denote as X and Y. We define X to be the state space $H^2(0,1) \times L_2(0,1)$ of two-component vectors, $H^m(0,1)$ the usual Sobolev–Hilbert spaces of order $m \in \mathbb{N}_0$ related to $L_2(0,1)$, and we define Y as the space $L_2(0,1) \times \mathbb{C}^2$ of three-component vectors. With these definitions it is possible to pose our spectral problems in X and Y.

3.1. **The spectral problem in X.** The state space X is a Hilbert space under the energy-motivated norm $\|(w,v)\|_X$ induced by the inner product

$$((w,v),(\tilde{w},\tilde{v}))_X := (w,\tilde{w})_2 + (v,\tilde{v})_0$$

for any $(w,v),(\tilde{w},\tilde{v}) \in X$, where

$$(w,\tilde{w})_2 = \int_0^1 \left(w''(s)\overline{\tilde{w}''(s)} + \gamma^2 w(s)\overline{\tilde{w}(s)}\right) ds + \alpha_0 w(0)\overline{\tilde{w}(0)} + \alpha_1 w(1)\overline{\tilde{w}(1)},$$

which for $\gamma \geq 0$ is well defined because $\alpha_0, \alpha_1 > 0$. Throughout we denote by $(\cdot,\cdot)_0$ the usual inner product in $L_2(0,1)$ and write $\|\cdot\|_0$ for the resulting norm.

Define the linear operator

$$(Ax)(s) = i(-v(s), w^{(4)}(s)), \quad x \in D(A), \quad s \in [0,1], \tag{3.1}$$

with domain

$$D(A) = \left\{ x = (w,v) \in X \;\middle|\; \begin{array}{c} w \in H^4(0,1),\ v \in H^2(0,1), \\ w''(0) = 0,\ w^{(3)}(0) + \alpha_0 w(0) + \beta_0 v(0) = 0, \\ w''(1) = 0,\ w^{(3)}(1) - \alpha_1 w(1) - \beta_1 v(1) = 0 \end{array} \right\}, \tag{3.2}$$

and further define

$$(Bx)(s) = i(0, \gamma^2 w(s) + 2\gamma v(s)), \quad x \in X, \quad s \in [0,1]. \tag{3.3}$$

Since B, as thus defined, is a bounded linear operator on X,

$$D(A+B) = D(A)$$

and we may regard $A+B$ for $\gamma > 0$ as a bounded linear perturbation of A. The operator $i(A+B)$ is the system operator, to which we will come back in Section 6.

The formulation of the abstract spectral problem in X now requires that we employ the substitution $\lambda = \omega - i\gamma$ in (2.1)–(2.5) to obtain the following boundary-eigenvalue problem with spectral parameter ω:

$$w^{(4)}(\omega,s) + \gamma^2 w(\omega,s) + 2i\gamma\omega w(\omega,s) - \omega^2 w(\omega,s) = 0, \tag{3.4}$$

$$w''(\omega,0) = 0, \tag{3.5}$$

$$w^{(3)}(\omega,0) + (\alpha_0 + i\beta_0\omega)w(\omega,0) = 0, \tag{3.6}$$

$$w''(\omega,1) = 0, \tag{3.7}$$

$$w^{(3)}(\omega,1) - (\alpha_1 + i\beta_1\omega)w(\omega,1) = 0. \tag{3.8}$$

With the linear operator pencil

$$P(\omega) = \omega I - (A+B), \quad D(P(\omega)) = D(A), \tag{3.9}$$

the spectral problem

$$P(\omega)\,x = [\omega I - (A+B)]\,x = 0, \quad x \in D(A), \quad \omega \in \mathbb{C}, \tag{3.10}$$

can be checked to be equivalent to the boundary-eigenvalue problem (3.4)–(3.8).

The following two lemmas will be used to connect the spectral properties of $P(\omega)$ given by (3.9) or, equivalently, of the operator $A+B$ with those of a quadratic operator pencil (definition in the next section) acting in the space Y. (Notice, for use later in Section 6, that Lemma 5 means effectively that the Cauchy problem associated with (1.2)–(1.7) is well posed in X.)

Lemma 4. *The eigenvalues, including multiplicities, of the boundary-eigenvalue problem* (3.4)–(3.8) *coincide with those of the operator $A+B$. Further, the following relation holds between a chain of root functions $w_0, w_1, \ldots, w_{h-1}$ of* (3.4)–(3.8) *corresponding to an eigenvalue ω_0 and a chain of root vectors $x_0, x_1, \ldots, x_{h-1}$ of $A+B$ corresponding to the same eigenvalue:*

$$x_r = (w_r, v_r), \quad v_r = i\omega_0 w_r + i w_{r-1}, \quad r = 0, 1, \ldots, h-1. \tag{3.11}$$

Proof. It suffices to verify the relations (3.11) (see [28, Section I.2] and [34, Lemma 1.4]). Let ω_0 be an eigenvalue of $A+B$ with a corresponding chain formed by the root vectors $x_0, x_1, \ldots, x_{h-1}$. Recall the spectral problem (3.10), and note that, according to Definition 2,

$$P(\omega_0)\,x_r + x_{r-1} = [\omega_0 I - (A+B)]\,x_r + x_{r-1} = 0, \quad r = 0, 1, \ldots, h-1,$$

where $x_{-1} := 0$, or equivalently in coordinates,

$$\begin{aligned} \omega_0 w_r(\omega_0, s) + i v_r(\omega_0, s) + w_{r-1}(\omega_0, s) &= 0, \\ \omega_0 v_r(\omega_0, s) - i w_r^{(4)}(\omega_0, s) - i\gamma^2 w_r(\omega_0, s) - 2i\gamma v_r(\omega_0, s) + v_{r-1}(\omega_0, s) &= 0 \end{aligned}$$

together with the boundary conditions

$$\begin{aligned} w_r''(\omega_0, 0) &= 0, \\ w_r^{(3)}(\omega_0, 0) + \alpha_0 w_r(\omega_0, 0) + \beta_0 v_r(\omega_0, 0) &= 0, \\ w_r''(\omega_0, 1) &= 0, \\ w_r^{(3)}(\omega_0, 1) - \alpha_1 w_r(\omega_0, 1) - \beta_1 v_r(\omega_0, 1) &= 0 \end{aligned}$$

for $r = 0, 1, \ldots, h-1$. The lemma follows from these relations. $\square$

Lemma 5. *The following assertions hold:*

(i) *$A+B$ is maximal dissipative for $\beta_0, \beta_1, \gamma > 0$, and selfadjoint when $\beta_0 = \beta_1 = \gamma = 0$; and*

(ii) *$A+B$ has a compact inverse.*

Proof. To prove assertion (i), we show that $A+B$ is dissipative and boundedly invertible. We first observe that, for $x \in D(A)$,

$$((A+B)\,x, x)_X = -i\,(v, w)_2 + i\,(w^{(4)} + \gamma^2 w + 2\gamma v, v)_0.$$

An elementary computation using integration by parts shows that

$$i\,(w^{(4)} + \gamma^2 w, v)_0 = i\,(w, v)_2 + i\beta_0\,|v(0)|^2 + i\beta_1\,|v(1)|^2.$$

Hence, on rearranging,

$$((A+B)\,x, x)_X = i\,(w, v)_2 - i\,(v, w)_2 + 2i\gamma\,\|v\|_0^2 + i\beta_0\,|v(0)|^2 + i\beta_1\,|v(1)|^2.$$

It is clear that we have

$$i\,(w, v)_2 - i\,(v, w)_2 = 2\,\mathrm{Im}\,(v, w)_2.$$

Thus

$$\operatorname{Im}\left(\left(A+B\right)x,x\right)_X = 2\gamma \left\|v\right\|_0^2 + \beta_0 \left|v\left(0\right)\right|^2 + \beta_1 \left|v\left(1\right)\right|^2 \geq 0, \tag{3.12}$$

the inequality a consequence of $\beta_0, \beta_1, \gamma \geq 0$. So we have that $A+B$ is dissipative. It is symmetric when $\beta_0 = \beta_1 = \gamma = 0$. To finally complete the proof of assertion (i), we now show that $A+B$ is boundedly invertible (and thus closed). We do this in two steps.

Step 1. For $\tilde{x} \in X$, $x \in D\left(A\right)$ let us consider the problem $Ax = \tilde{x}$. Equivalently in coordinates,

$$-iv\left(s\right) = \tilde{w}\left(s\right), \tag{3.13}$$

$$iw^{(4)}\left(s\right) = \tilde{v}\left(s\right), \tag{3.14}$$

$$w''\left(0\right) = 0, \tag{3.15}$$

$$w^{(3)}\left(0\right) + \alpha_0 w\left(0\right) + \beta_0 v\left(0\right) = 0, \tag{3.16}$$

$$w''\left(1\right) = 0, \tag{3.17}$$

$$w^{(3)}\left(1\right) - \alpha_1 w\left(1\right) - \beta_1 v\left(1\right) = 0. \tag{3.18}$$

By integrating the differential equation (3.14) four times we obtain

$$w\left(s\right) = w\left(0\right) + sw'\left(0\right) + \frac{s^2}{2!}w''\left(0\right) + \frac{s^3}{3!}w^{(3)}\left(0\right) - i\int_0^s \frac{\left(s-r\right)^3}{3!}\tilde{v}\left(r\right)dr.$$

The boundary conditions (3.15)–(3.18) together with (3.13) imply

$$\begin{aligned}
w''\left(0\right) &= 0,\\
\alpha_0 w\left(0\right) + w^{(3)}\left(0\right) &= -i\beta_0 \tilde{w}\left(0\right),\\
w''\left(0\right) + w^{(3)}\left(0\right) &= i\int_0^1 \left(1-r\right)\tilde{v}\left(r\right)dr,\\
-\alpha_1 w\left(0\right) - \alpha_1 w'\left(0\right) - \frac{\alpha_1}{2!}w''\left(0\right) + \left(1-\frac{\alpha_1}{3!}\right)w^{(3)}\left(0\right)&\\
&= i\int_0^1 \left[1 - \frac{\alpha_1}{3!}\left(1-r\right)^3\right]\tilde{v}\left(r\right)dr - i\beta_1 \tilde{w}\left(1\right).
\end{aligned}$$

The coefficients of $w\left(0\right)$, $w'\left(0\right)$, $w''\left(0\right)$, $w^{(3)}\left(0\right)$ form the coefficient matrix of this system of algebraic equations and a direct computation (recalling that $\alpha_0, \alpha_1 > 0$) shows that

$$\det\begin{pmatrix} 0 & 0 & 1 & 0\\ \alpha_0 & 0 & 0 & 1\\ 0 & 0 & 1 & 1\\ -\alpha_1 & -\alpha_1 & -\frac{\alpha_1}{2!} & 1-\frac{\alpha_1}{3!}\end{pmatrix} = \alpha_0\alpha_1 \neq 0.$$

So we have that there is a unique solution which is given by the vector formed by $w\left(0\right)$, $w'\left(0\right)$, $w''\left(0\right)$, $w^{(3)}\left(0\right)$. Thus we have obtained the uniquely-determined nonzero function $w \in H^4\left(0,1\right)$. In particular, we observe that $x = A^{-1}\tilde{x}$, x being a nontrivial element of $D\left(A\right)$, so A is bijective and it is obviously closed. A well-known consequence of the closed graph theorem is that A^{-1} is a bounded operator. So we see that A is boundedly invertible, completing the first step in the proof of assertion (i). Moreover, we also have that the embeddings $H^4\left(0,1\right) \hookrightarrow H^2\left(0,1\right) \hookrightarrow L_2\left(0,1\right)$ are compact; therefore A^{-1} is compact. This completes the first step in the proof of assertion (ii).

Step 2. The second step involves use of the compactness of A^{-1} to show that the operator $A+B$ possesses also a compact inverse. Let us write $A+B$ in the form

$$A + B = \left(I + BA^{-1}\right)A.$$

Obviously BA^{-1} is a compact operator (recall $A+B$ is for $\gamma > 0$ a bounded linear perturbation of the Fredholm operator A). Therefore, using the compactness of A^{-1} together with that of BA^{-1},

$$(A+B)^{-1} = A^{-1}\left(I + BA^{-1}\right)^{-1}$$

exists and is compact, proving assertion (ii). Thus, $A+B$ is maximal dissipative. The selfadjointness of $A+B$ for $\beta_0 = \beta_1 = \gamma = 0$ then follows from the surjectivity of A. This completes the proof of assertion (i), and thus of the lemma.

□

The implications of Lemma 5 call for some natural comments (in preparation for what follows). Firstly, we see that $0 \in \varrho(A+B)$. So the spectrum of P or, equivalently, of the operator $A+B$ is purely discrete and consists only of normal eigenvalues, $\sigma(A+B) = \sigma_0(A+B)$, which accumulate only at infinity (see [7, Theorem XV.2.3] or [14, Theorem III.6.29]). Moreover, when $A+B$ is selfadjoint – we know from Lemma 5 that this is the case when $\beta_0 = \beta_1 = \gamma = 0$ – the spectral theorem [7, Theorem XVI.5.1] yields the existence of an orthonormal basis for X consisting only of eigenvectors of A corresponding to real eigenvalues. However, when $A+B$ is not selfadjoint, the direct analogue of this theorem is not true, for it is then generally possible to have root vectors which are minimal complete but which are not a basis. We shall return to this matter in Section 5 and close this section with the following obvious proposition.

Proposition 2. *Let $\beta_0, \beta_1, \gamma > 0$. Then $A+B$ can have no purely real eigenvalues.*

Proof. We show that unless $\beta_0 = \beta_1 = \gamma = 0$ there is no nontrivial element of $\ker P(\omega)$ when ω is purely real. To this end, suppose (to reach a contradiction) there is an eigenvalue $\omega_0 \in \mathbb{R}$ and take x_0 to be the corresponding eigenvector, such that

$$(P(\omega_0)x_0, x_0)_X = (\omega_0 I - (A+B)x_0, x_0)_X = 0.$$

We have for the imaginary part

$$\operatorname{Im}((A+B)x_0, x_0)_X = 0;$$

so, by (3.12),

$$2\gamma \|v_0\|_0^2 + \beta_0 |v_0(0)|^2 + \beta_1 |v_0(1)|^2 = 0.$$

If $\beta_0, \beta_1, \gamma > 0$, we then have from the above that $v_0 = 0$. Using that $x_0 = (w_0, v_0)$ with $v_0 = i\omega_0 w_0$ along with the fact that $0 \in \varrho(A+B)$, we can infer that $w_0 = 0$, and thus have $x_0 = 0$. This is contrary to our assumption that x_0 is an eigenvector. The proof is complete. □

3.2. **The spectral problem in Y.** We consider now the spectral problem in the space Y. This is a Hilbert space with the inner product

$$((w,a,c), (\tilde{w},\tilde{a},\tilde{c}))_Y := (w,\tilde{w})_0 + a\bar{\tilde{a}} + c\bar{\tilde{c}}$$

for any $(w,a,c), (\tilde{w},\tilde{a},\tilde{c}) \in Y$. Define the linear operator

$$\begin{aligned}(Gy)(s) = (w^{(4)}(s), w^{(3)}(0) + (\alpha_0 - \beta_0\gamma) w(0), \\ -w^{(3)}(1) + (\alpha_1 - \beta_1\gamma) w(1)), \quad y \in D(G), \quad s \in [0,1],\end{aligned} \tag{3.19}$$

with

$$D(G) = \left\{ y = (w,a,c) \in Y \;\middle|\; \begin{array}{c} w \in H^4(0,1), \\ a = w(0),\ c = w(1),\ w''(0) = 0,\ w''(1) = 0 \end{array} \right\}, \tag{3.20}$$

and define

$$(Cy)(s) = (0, \beta_0 w(0), \beta_1 w(1)), \quad y \in Y, \quad s \in [0,1], \tag{3.21}$$

and

$$(Dy)(s) = (w(s), 0, 0), \quad y \in Y, \quad s \in [0,1]. \tag{3.22}$$

The operators C, D are bounded linear operators on Y and selfadjoint, $C \geq 0$, $D \geq 0$, and the operator G will be studied shortly, where we shall see that it is selfadjoint and G^{-1} exists and is compact.

Consider the quadratic operator pencil

$$L(\lambda) = \lambda^2 D - i\lambda C - G, \quad D(L(\lambda)) = D(G). \tag{3.23}$$

Then the boundary-eigenvalue problem (2.1)–(2.5) takes the abstract form

$$L(\lambda) y = (\lambda^2 D - i\lambda C - G) y = 0, \quad y \in D(G), \quad \lambda \in \mathbb{C}. \tag{3.24}$$

Obviously, if there is a chain of root vectors $y_0, y_1, \ldots, y_{h-1}$ of L corresponding to an eigenvalue λ_0, then there is a corresponding chain formed by the root functions $w_0, w_1, \ldots, w_{h-1}$ of (2.1)–(2.5) corresponding to the same eigenvalue. Thus (3.24) holds if and only if (2.1)–(2.5) holds.

The next proposition will be needed later, in Section 4, in the study of the spectral properties of the operator pencil $L(\lambda)$.

Proposition 3. *The operator D is positive definite when restricted to the domain $D(G)$, that is $D|_{D(G)} > 0$. If $\beta_0, \beta_1 > 0$, then $C|_{D(G)} > 0$.*

Proof. It is obvious that we need only show that we have

$$(Cy, y)_Y > 0, \quad (Dy, y)_Y > 0$$

under the restriction $y\,(\neq 0) \in D(G)$, which we assume henceforth. It follows trivially that if we have $w = 0$, then $y = 0$. Therefore, when $y \neq 0$, then $w \neq 0$. So

$$(Dy, y)_Y = \|w\|_0^2 > 0,$$

proving the first statement. The key step in the proof of the second statement is to notice that $w(0) = w(1) = 0$ implies $w = 0$, and thus $y = 0$. That this is true is proven as follows. We take $\{w_k\}_{k=1}^4$ to be a fundamental system of solutions to (2.1) satisfying

$$w_k^{(m)}(0) = \delta_{k,m+1}, \quad m = 0,1,2,3.$$

Then, because of the boundary condition (2.2), there are three linearly independent solutions, namely w_1, w_2, and w_4. Let

$$w(\mu, s) = \frac{1}{2\mu^3} \sinh \mu s - \frac{1}{2\mu^3} \sin \mu s$$

where $\mu = \sqrt{\lambda}$, $\lambda \neq 0$, and set $w_k(s) = w^{(4-k)}(\mu, s)$, $k = 1,2,3,4$. Now, suppose $w(\lambda, 0) = w(\lambda, 1) = 0$. Then the boundary conditions (2.3) and (2.5) imply $w^{(3)}(\lambda, 0) = w^{(3)}(\lambda, 1) = 0$. We also have that $w''(\lambda, 0) = w''(\lambda, 1) = 0$, and we note that the function w_2 is the only solution satisfying $w(\lambda, 0) = w''(\lambda, 0) = w^{(3)}(\lambda, 0) = 0$ with $w'(\lambda, 0) = 1$. Also, it must satisfy $w_2(1) = w_2''(1) = w_2^{(3)}(1) = 0$, which yields that

$$\frac{1}{\mu} \sinh \mu + \frac{1}{\mu} \sin \mu = \mu \sinh \mu - \mu \sin \mu = \mu^2 \cosh \mu - \mu^2 \cos \mu = 0.$$

This is impossible. It follows therefore that with $\beta_0, \beta_1 > 0$,

$$(Cy, y)_Y = \beta_0 |w(0)|^2 + \beta_1 |w(1)|^2 > 0,$$

and the lemma follows. $\square$

4. Spectral properties of the operator pencil $L(\lambda)$

In Section 3.1 we have verified that since $0 \in \varrho(A+B)$ and $(A+B)^{-1}$ is compact, the spectrum of $A+B$ consists only of normal eigenvalues. In the result in Theorem 1, presented shortly, we deduce for the sake of completeness a corresponding result in the context of the quadratic operator pencil $L(\lambda)$ given by (3.23) (which in fact already follows from Proposition 1). First we prove the following lemmas. Throughout the remainder of this section it will be understood that the operators C, D, G are defined as in Section 3.2 by (3.19)–(3.22).

Lemma 6. *The following assertions hold:*

(i) *G is selfadjoint; and*
(ii) *G has a compact inverse.*

Proof. We only establish the symmetry of G as the rest of the proof is similar to that of Lemma 5. Let us begin by showing that G is densely defined. Assume there is an element $\tilde{y} \in Y$ such that for any $y \in D(G)$

$$(y, \tilde{y})_Y = (w, \tilde{w})_0 + w(0)\,\overline{\tilde{a}} + w(1)\,\overline{\tilde{c}} = 0.$$

Let w be a smooth function such that $w^{(m)}(0) = w^{(m)}(1) = 0$, $m \in \mathbb{N}_0$. It follows from the above that

$$(w, \tilde{w})_0 = 0.$$

Consequently, $\tilde{w} = 0$. Consider the polynomial

$$w(s) = \frac{s^3}{2}(s-2) + \frac{s}{2} + 1,$$

which satisfies $w''(0) = w''(1) = 0$ and $w(0) = w(1) = 1$. Clearly $y \in D(G)$, and, since $\tilde{w} = 0$,

$$(y, \tilde{y})_Y = w(0)\,\overline{\tilde{a}} + w(1)\,\overline{\tilde{c}} = 0;$$

but this then implies that $\tilde{a} = \tilde{c} = 0$. Thus $\tilde{y} = 0$, and we conclude $D(G)^{\perp} = \{0\}$ and G is densely defined in Y.

Now let $y, \tilde{y} \in D(G)$, and note that

$$(Gy, \tilde{y})_Y = (w^{(4)}, \tilde{w})_0 - w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1 + (\alpha_0 - \beta_0\gamma)\,w(0)\,\overline{\tilde{w}(0)} + (\alpha_1 - \beta_1\gamma)\,w(1)\,\overline{\tilde{w}(1)}.$$

We compute

$$(w^{(4)}, \tilde{w})_0 = (w'', \tilde{w}'')_0 + w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1.$$

Hence

$$(w'', \tilde{w}'')_0 = (w^{(4)}, \tilde{w})_0 - w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1. \tag{4.1}$$

Now, the left-hand side of (4.1) is symmetric, so

$$(w'', \tilde{w}'')_0 = (w, \tilde{w}^{(4)})_0 - w(s)\,\overline{\tilde{w}^{(3)}(s)}\Big|_0^1.$$

Equating this to (4.1) gives

$$(w^{(4)}, \tilde{w})_0 = (w, \tilde{w}^{(4)})_0 + w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1 - w(s)\,\overline{\tilde{w}^{(3)}(s)}\Big|_0^1.$$

Combining the results, we obtain

$$\begin{aligned}(Gy,\tilde{y})_Y &= (w^{(4)},\tilde{w})_0 - w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1 + (\alpha_0-\beta_0\gamma)\,w(0)\,\overline{\tilde{w}(0)} + (\alpha_1-\beta_1\gamma)\,w(1)\,\overline{\tilde{w}(1)}\\ &= (w,\tilde{w}^{(4)})_0 - w(s)\,\overline{\tilde{w}^{(3)}(s)}\Big|_0^1 + (\alpha_0-\beta_0\gamma)\,w(0)\,\overline{\tilde{w}(0)} + (\alpha_1-\beta_1\gamma)\,w(1)\,\overline{\tilde{w}(1)}\\ &= (y,G\tilde{y})_Y\,,\end{aligned}$$

and the denseness of the domain $D(G)$ ensures that G is symmetric. □

Lemma 7. *Consider the operator pencil*

$$\tilde{L}(\lambda) = I + i\lambda CG^{-1} - \lambda^2 DG^{-1}, \quad D(\tilde{L}(\lambda)) = Y.$$

associated with $L(\lambda)$ *given by* (3.23)*. Then*

$$\sigma(L) = \sigma(\tilde{L}), \quad \sigma_0(L) = \sigma_0(\tilde{L}).$$

Proof. Since $\tilde{L}(\lambda)$ is bounded, the result follows immediately from [22, Lemma 20.1]. □

Theorem 1. *The spectrum of* L *consists only of normal eigenvalues.*

Proof. Consider the operator pencil

$$-L(\lambda)\,G^{-1} = I + i\lambda CG^{-1} - \lambda^2 DG^{-1}.$$

Since G^{-1}, by virtue of Lemma 6, exists and is compact, the operator on the right-hand side is a Fredholm operator for each fixed $\lambda \in \mathbb{C}$. Putting $\tilde{L}(\lambda) := -L(\lambda)\,G^{-1}$, the theorem then follows from Lemma 7 and a more general perturbation result [7, Corollary XI.8.4] for analytic Fredholm operator functions. □

Remark 2. *Note that the spectra of* P *or, equivalently, of the operator* $A+B$ *and* L*, including their multiplicities, coincide when* $\gamma = 0$*, and for* $\gamma > 0$ *there is a direct correspondence between the two.*

We analyse now the spectrum of L in more detail. We begin with its location by showing that the eigenvalues are located symmetrically with respect to the imaginary axis in the closed upper half-plane, excluding the origin, and that when $\beta_0, \beta_1 > 0$, then they are confined to the open upper half-plane.

Theorem 2. *The spectrum of* L *is symmetric with respect to the imaginary axis and is located in the closed upper half-plane but excluding the origin. In the case when* $\beta_0, \beta_1 > 0$*, the spectrum is confined to the open upper half-plane.*

Proof. Let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of L with corresponding eigenvector y_0. Then

$$L(-\overline{\lambda_0})\,\overline{y_0} = (\overline{\lambda_0^2}D + i\overline{\lambda_0}C - G)\,\overline{y_0} = \overline{(\lambda_0^2 D - i\lambda_0 C - G)\,y_0} = \overline{L(\lambda_0)\,y_0} = 0,$$

and we have that $\overline{y_0}$ is an eigenvector corresponding to an eigenvalue $-\overline{\lambda_0}$. This proves that the spectrum of L is symmetric with respect to the imaginary axis. We take now the inner product with the corresponding y_0 and obtain

$$(L(\lambda_0)\,y_0, y_0)_Y = ((\lambda_0^2 D - i\lambda_0 C - G)\,y_0, y_0)_Y = 0.$$

This we can write out in terms of real and imaginary parts:

$$((\operatorname{Re}\lambda_0)^2 - (\operatorname{Im}\lambda_0)^2)\,(Dy_0,y_0)_Y + \operatorname{Im}\lambda_0\,(Cy_0,y_0)_Y - (Gy_0,y_0)_Y = 0 \tag{4.2}$$

and

$$\operatorname{Re}\lambda_0\,(2\operatorname{Im}\lambda_0\,(Dy_0,y_0)_Y - (Cy_0,y_0)_Y) = 0. \tag{4.3}$$

Consider the case $\operatorname{Re}\lambda_0 = 0$. Then from (4.2),

$$(\operatorname{Im}\lambda_0)^2\,(Dy_0,y_0)_Y + (Gy_0,y_0)_Y = \operatorname{Im}\lambda_0\,(Cy_0,y_0)_Y\,,$$

and since $C \geq 0$, $D \geq 0$, and $G \geq 0$, we find that $\operatorname{Im}\lambda_0 \geq 0$. If $\operatorname{Re}\lambda_0 \neq 0$, then we have from (4.3) that

$$2\operatorname{Im}\lambda_0 \left(Dy_0, y_0\right)_Y = \left(Cy_0, y_0\right)_Y \tag{4.4}$$

and so $\operatorname{Im}\lambda_0 \geq 0$. This proves that the spectrum of L lies in the closed upper half-plane. That the spectrum does not include the origin is obvious in view of the previous results. In fact, suppose the origin belongs to the spectrum. Then from (4.2) we obtain $\left(Gy_0, y_0\right)_Y = 0$, and consequently $Gy_0 = 0$; but then $y_0 = 0$ by Lemma 6, and we have a contradiction (since y_0 is an eigenvector).

Let now $\beta_0, \beta_1 > 0$, and suppose λ_0 is a purely real eigenvalue, $\lambda_0 \in \mathbb{R}$. From (4.3) it follows at once that $\left(Cy_0, y_0\right)_Y = 0$, so $Cy_0 = 0$, again a contradiction by Proposition 3. This proves the last statement of the theorem. $\square$

Remark 3. *It follows from* (4.4) *in the proof of the theorem that, in the case* $\operatorname{Re}\lambda_0 \neq 0$, *there are constants* a, b *such that* $\operatorname{Im}\lambda_0 \in [a, b]$, *where*

$$a := \inf_{y(\neq 0)\in D(G)} \frac{\left(Cy_0, y_0\right)_Y}{2\left(Dy_0, y_0\right)_Y}, \quad b := \sup_{y(\neq 0)\in D(G)} \frac{\left(Cy_0, y_0\right)_Y}{2\left(Dy_0, y_0\right)_Y}$$

which for $\beta_0, \beta_1 > 0$, *using the result of Proposition 3, yields that* $a > 0$ *and* $b < \infty$. *(See also* [26, Lemma 1.4.2]*.)*

4.1. **Asymptotics of eigenvalues.** We pass now to the final task of this section, which is to obtain asymptotic expansions of the eigenvalues (with large modulus) of L. First we need to derive a few intermediate results.

According to Theorem 2 all eigenvalues lie in the upper half-plane and those with nonzero real part occur in pairs λ, $-\overline{\lambda}$. So we need only consider the boundary-eigenvalue problem (2.1)–(2.5) in the first quadrant of the complex plane (corresponding to eigenvalues with $\operatorname{Re}\lambda \geq 0$). The sector $0 \leq \arg\mu \leq \pi/4$ corresponds to this quadrant under the spectral transformation $\lambda \mapsto \mu^2$. Let us use μ^2 in place of λ in (2.1)–(2.5), and let $\{w_k\}_{k=1}^4$ be a fundamental system of (2.1). We have noted in the proof of Proposition 3 that the boundary condition (2.2) demands we consider w_1, w_2, and w_4, that is,

$$w_1(s) = \frac{1}{2}\cosh\mu s + \frac{1}{2}\cos\mu s, \quad w_2(s) = \frac{1}{2\mu}\sinh\mu s + \frac{1}{2\mu}\sin\mu s,$$
$$w_4(s) = \frac{1}{2\mu^3}\sinh\mu s - \frac{1}{2\mu^3}\sin\mu s.$$

Applying the remaining three boundary conditions (2.3)–(2.5) to these yields the reduced characteristic matrix

$$M := \begin{pmatrix} \theta_0(\mu) \\ \frac{\mu^2}{2}(\cosh\mu - \cos\mu) \\ \frac{\mu^3}{2}(\sinh\mu + \sin\mu) - \frac{\theta_1(\mu)}{2}(\cosh\mu + \cos\mu) \end{pmatrix.$$
$$\begin{matrix} 0 \\ \frac{\mu}{2}(\sinh\mu - \sin\mu) \\ \frac{\mu^2}{2}(\cosh\mu - \cos\mu) - \frac{\theta_1(\mu)}{2\mu}(\sinh\mu + \sin\mu) \end{matrix}$$
$$\begin{matrix} 1 \\ \frac{1}{2\mu}(\sinh\mu + \sin\mu) \\ \frac{1}{2}(\cosh\mu + \cos\mu) - \frac{\theta_1(\mu)}{2\mu^3}(\sinh\mu - \sin\mu) \end{matrix}\Bigg),$$

wherein $\theta_0(\mu)$, $\theta_1(\mu)$ are as given in the proof of Proposition 1. Using a determinantal calculation we can write down explicitly the characteristic equation $2\det M = 0$ by defining

$$\phi_0(\mu) = \mu^4(1 - \cos\mu\cosh\mu), \quad \phi_1(\mu) = -\mu(\sin\mu\cosh\mu - \cos\mu\sinh\mu),$$
$$\phi_2(\mu) = \frac{2}{\mu^2}\sin\mu\sinh\mu,$$

giving

$$\phi(\mu) = \phi_0(\mu) + (\theta_0(\mu) + \theta_1(\mu))\,\phi_1(\mu) + \theta_0(\mu)\,\theta_1(\mu)\,\phi_2(\mu) = 0.$$

Clearly we are justified in referring to the squares of the zeros of ϕ as the eigenvalues of the boundary-eigenvalue problem (2.1)–(2.5) or, equivalently, of L. We examine now the asymptotics of the roots of $\phi(\mu) = 0$, and hence of the eigenvalues of L.

Theorem 3. *The spectrum of L consists of an infinite number of normal eigenvalues, these being symmetric about the imaginary axis, and which accumulate only at infinity. For large $n \in \mathbb{N}$, the eigenvalues have asymptotic form*

$$\lambda_n = \mu_n^2, \quad \mu_n = \left(n - \frac{1}{2}\right)\pi + i(\beta_0 + \beta_1)\left[\left(n - \frac{1}{2}\right)\pi\right]^{-1} + O(n^{-2}). \tag{4.5}$$

Proof. The first statement is immediate from Proposition 1 or Theorem 1 and Theorem 2. Now, note that

$$2\mu^{-4}e^{-\mu}\phi_0(\mu) = 2e^{-\mu} - \cos\mu(1 + e^{-2\mu}),$$
$$2\mu^{-4}e^{-\mu}\phi_1(\mu) = -\frac{\sin\mu - \cos\mu}{\mu^3} - \frac{\sin\mu + \cos\mu}{\mu^3}e^{-2\mu},$$
$$2\mu^{-4}e^{-\mu}\phi_2(\mu) = \frac{2}{\mu^6}\sin\mu(1 - e^{-2\mu}).$$

With these, by calculating the asymptotic expansion of $2\mu^{-4}e^{-\mu}\phi(\mu) = 0$, we get

$$\cos\mu + \frac{i(\beta_0 + \beta_1)(\sin\mu - \cos\mu)}{\mu} + O(\mu^{-2}) = 0 \tag{4.6}$$

or

$$\cos\mu + O(\mu^{-1}) = 0. \tag{4.7}$$

This means the characteristic equation takes either of the asymptotic forms (4.6) or (4.7), valid for values of μ in small neighbourhoods of $(n - 1/2)\pi$, large $n \in \mathbb{N}$. Set

$$h(\mu) = 2\mu^{-4}e^{-\mu}\phi(\mu), \quad f(\mu) = \cos\mu.$$

Asymptotically, then,

$$g(\mu) = f(\mu) - h(\mu) = O(\mu^{-1}).$$

Consider f on a contour $\{\mu \in \mathbb{C} \mid |\mu - \hat{\mu}_n| = \pi/4\}$ around $\hat{\mu}_n := (n - 1/2)\pi$, large $n \in \mathbb{N}$, so that $|f| \geq 1/2$ on the contour and the estimate $|g| < |f|$ holds. Within the contour, by Rouche's theorem, there is precisely one zero of f, just as there is precisely one zero of h, namely

$$\mu_n = \hat{\mu}_n + \nu_n, \quad \hat{\mu}_n := \left(n - \frac{1}{2}\right)\pi. \tag{4.8}$$

We note that we can specify the standard identities

$$\cos\mu_n = \cos\hat{\mu}_n\cos\nu_n - \sin\hat{\mu}_n\sin\nu_n = -(-1)^{n-1}\sin\nu_n, \tag{4.9}$$

$$\sin\mu_n = \sin\hat{\mu}_n\cos\nu_n + \cos\hat{\mu}_n\sin\nu_n = (-1)^{n-1}\cos\nu_n. \tag{4.10}$$

Writing μ_n in place of μ in (4.6), substitution of (4.9) and (4.10) in (4.6) yields

$$\sin\nu_n = i(\beta_0 + \beta_1)\frac{\cos\nu_n}{\mu_n} + O(\mu_n^{-2}).$$

Thus, since for large $n \in \mathbb{N}$, $\sin \nu_n \sim \nu_n$ and $\cos \nu_n \sim 1$, we have (recall the definition of $\hat{\mu}_n$ above)

$$\nu_n = i\,(\beta_0 + \beta_1)\left[\left(n - \frac{1}{2}\right)\pi\right]^{-1} + O\,(n^{-2}).$$

The substitution of this into (4.8) leads to (4.5) in the statement of the theorem. The proof is complete. □

The results of this section allow some important conclusions to be drawn as to the spectral properties of the operator pencil $L\,(\lambda)$. Firstly, $\lambda = \mu^2 \in \mathbb{C}$ is an eigenvalue of L if and only if $\phi\,(\mu) = 0$. Hence,

$$\sigma\,(L) = \left\{\lambda = \mu^2 \in \mathbb{C} \;\middle|\; \phi\,(\mu) = 0\right\}.$$

Secondly, for large $n \in \mathbb{N}$, e.g. $n \geq n_0$ for some $n_0 < \infty$,

$$\lambda_n = \mu_n^2 = \left[\left(n - \frac{1}{2}\right)\pi\right]^2 + 2i\,(\beta_0 + \beta_1) + O\,(n^{-1}), \tag{4.11}$$

by (4.5), and thus we have that the λ_n are not purely imaginary eigenvalues of L (it is possible to have only finitely many purely imaginary eigenvalues); hence,

$$\sigma\,(L) = \left\{\lambda_n, -\overline{\lambda_n} \;\middle|\; n \in \mathbb{N},\ n \geq n_0\right\}.$$

We likewise have, for the eigenvalues of $A + B$,

$$\sigma\,(A + B) = \{\omega_n, -\overline{\omega_n} \mid n \in \mathbb{N},\ n \geq n_0\}$$

wherein

$$\omega_n = \lambda_n + i\gamma = \left[\left(n - \frac{1}{2}\right)\pi\right]^2 + i\,[2\,(\beta_0 + \beta_1) + \gamma] + O\,(n^{-1}), \tag{4.12}$$

and $\operatorname{Im}\omega_n = 2\,(\beta_0 + \beta_1) + \gamma$ is the horizontal asymptote of $\sigma\,(A + B)$. For large $n \in \mathbb{N}$, the eigenvalues of $A+B$ can be properly enumerated in the sense of Definition 4. In fact, in closing the section, we note the enumeration is such that in the sequence $\{\omega_{\pm n}\}_{n=1}^{\infty}$, $\omega_{-n} = -\overline{\omega_n}$, we have that

$$\sup \operatorname{Im}\omega_{\pm n} < \infty, \quad \inf_{n \neq m} |\omega_n - \omega_m| > 0, \tag{4.13}$$

which implies the sequence is interpolating.

5. Completeness, minimality, and Riesz basis properties

In this section we return to the state-space setting in X where, in the light of the results obtained so far, we turn our focus to the completeness, minimality, and Riesz basis properties of the root vectors, in particular, the eigenvectors of the operator $A+B$ (with A defined by (3.1), (3.2) and B by (3.3)). We know from the conclusions at the end of Section 4 that for large $n \in \mathbb{N}$ the spectrum of $A+B$ consists of an interpolating sequence $\{\omega_{\pm n}\}_{n=1}^{\infty}$ of not purely imaginary eigenvalues which can be properly enumerated in the sense of Definition 4, and that, therefore, all eigenvalues (except for a finite number of them) are simple. The crucial step now in proving the Riesz basis property of the corresponding eigenvectors $x_{\pm n}$ is to establish that all of the conditions of Lemmas 1 to 3 listed in Section 2 are satisfied, and hence there is in X a unique sequence of vectors, $\{z_{\pm n}\}_{n=1}^{\infty}$, say, such that $x_{\pm n}$, $z_{\pm n}$ are biorthogonal pairs (up to normalisation) in X. This will be the main part of the work in this section, and we start with addressing the questions of completeness and minimality.

Theorem 4. *There exists a sequence of eigenvectors of $A+B$ which is minimal complete in X.*

Proof. Recall from Lemma 5 that $0 \in \varrho(A+B)$ and $(A+B)^{-1}$ is compact. Hence, by Lemma 2, the eigenvectors of $(A+B)^{-1}$ are minimal. Let now $A = H + (A - H)$, where H is the selfadjoint part of A, that is, A with $\beta_0 = \beta_1 = 0$, and write

$$(A+B)^{-1} = [H + (A-H) + B]^{-1}.$$

The selfadjoint operator H is boundedly invertible with compact inverse by Lemma 5. Thus we can express $(A+B)^{-1}$ as

$$(A+B)^{-1} = H^{-1}\left[I + (A-H)\,H^{-1} + BH^{-1}\right]^{-1}.$$

Put

$$\left[I + (A-H)\,H^{-1} + BH^{-1}\right]^{-1} = I + S.$$

An easy computation shows that

$$S = -\left[(A-H)\,H^{-1} + BH^{-1}\right]\left[I + (A-H)\,H^{-1} + BH^{-1}\right]^{-1},$$

which is a compact operator. So we have demonstrated that $(A+B)^{-1}$ can be expressed in the form

$$(A+B)^{-1} = H^{-1}\,(I+S)$$

and that the conditions imposed in Lemma 1 on the operators involved in the factorisation (2.8) hold if we identify K with H^{-1}. The condition (2.7) is also satisfied if we identify the λ_n in (2.7) with λ_n^{-1} using (4.11) in the case $\beta_0 = \beta_1 = 0$ (the same leading terms arise in the asymptotic expansions of the eigenvalues in this case). So the operator $(A+B)^{-1}$ has a sequence of eigenvectors which is minimal complete in X. The statement of the theorem obtains because the eigenvectors of $(A+B)^{-1}$ are also eigenvectors of $A+B$. □

Having proven the completeness and minimality, it now remains only to prove that the eigenvectors of $A+B$ have the property of being a Riesz basis for X. Here is where we use Lemma 3 in conjunction with the result of Theorem 4. We can now state and prove the main result of this section.

Theorem 5. *There exists a sequence of eigenvectors of $A+B$ which forms a Riesz basis for X.*

Proof. To obtain that there exists a sequence of eigenvectors of $A+B$ in X which forms a Riesz basis for its closed linear hull or span (denoted by $\mathrm{Sp}\,(A+B)$ in the following), we first note that $A+B$ is dissipative if and only if the operator $-\,(A+B)^{-1}$ is. Indeed we know from Lemma 5 that $A+B$ is dissipative and boundedly invertible. Thus, with $(A+B)\,x = \tilde{x}$ for $\tilde{x} \in X$, $x \in D\,(A)$,

$$\begin{aligned}&\mathrm{Im}\,(-\,(A+B)^{-1}\,\tilde{x}, \tilde{x})_X \\ &\quad = -\,\mathrm{Im}\,((A+B)^{-1}\,\tilde{x}, \tilde{x})_X = \mathrm{Im}\,(\tilde{x}, (A+B)^{-1}\,\tilde{x})_X = \mathrm{Im}\,((A+B)\,x, x)_X \geq 0.\end{aligned}$$

So $-\,(A+B)^{-1}$ is a bounded dissipative operator on X, which we identify with A in Lemma 3. We can immediately verify by making use of (4.12) that the condition in (2.9), wherein we identify λ_n with $-\omega_n^{-1}$, is satisfied. So all conditions of Lemma 3 are satisfied, and there exists a sequence of eigenvectors of $-\,(A+B)^{-1}$ and hence of $A+B$ in X which forms a Riesz basis for its closed linear span. Hence, by means of Theorem 4, we can infer that $\mathrm{Sp}\,(A+B) = X$ and there exists a sequence of eigenvectors of $A+B$ which forms a Riesz basis for X. □

Remark 4. *The same result could have been achieved if we had invoked a theorem proven by Katsnelson in* [15, Theorem 2.2]*, which states Theorem 5 to be true also under somewhat weaker conditions using the interpolation properties in* (4.13)*; more details can be found in* [8, Section VI.6] *and* [29, Lecture X]. *This was done by Miloslavskii in* [25] *for an application example.*

6. Well-posedness and exponential stability

We have collected all necessary ingredients to solve the problem of stability of solutions of the initial/boundary-value problem (1.2)–(1.7) stated in the Introduction. In particular, in Lemma 5 we established the maximal dissipativity of the operator $A+B$ in the state space X. Using this we have immediately by the Lumer–Phillips theorem that a semigroup formulation of (1.2)–(1.7) will be well posed (see [17, Section I.4.2] for details). This will be our first result.

Let us define $[\mathbf{x}(t)](s) \coloneqq x(s,t) = (w(s,t), v(s,t))$. With $v(s,t) = (\partial w/\partial t)(s,t)$, then,

$$\mathbf{x}(t) = (w(\cdot,t), v(\cdot,t)),$$

the state at each time $t \geq 0$. The initial state $\mathbf{x}(0) = x$, where $x = (h_0(\cdot), h_1(\cdot))$, is assumed sufficiently smooth, that is, $x \in D(A)$. We can rewrite (1.2)–(1.7) as an abstract first-order in time Cauchy problem in X in the form

$$\dot{\mathbf{x}}(t) = i(A+B)\mathbf{x}(t), \quad \mathbf{x}(0) = x, \tag{6.1}$$

Theorem 6. *The Cauchy problem* (6.1) *is well posed in the sense that it has for any $x \in D(A)$ a unique solution $\mathbf{x} \in C^1((0,\infty); X) \cap C([0,\infty); D(A))$ given by*

$$\mathbf{x}(t) = T(t)x, \quad t \geq 0, \tag{6.2}$$

where $T(t)$ is a strongly continuous contraction semigroup on X with infinitesimal generator $i(A+B)$.

We are interested now in what we can infer in regard to the stability of the solutions to the Cauchy problem (6.1). Indeed, with the results now in hand we are ready to establish – in the sense of the norm in X – the uniform exponential decay of the solutions given by (6.2). The following theorem constitutes the final result of the paper.

Theorem 7. *Given $x \in D(A)$, the solutions of* (6.1) *can be represented as norm-convergent series of the form*

$$T(t)x = \sum_{n=1}^{\infty} e^{i\omega_n t}(x, z_n)_X x_n + \sum_{n=1}^{\infty} e^{i\omega_{-n} t}(x, z_{-n})_X x_{-n}, \quad t \geq 0,$$

where $\{\omega_{\pm n}\}_{n=1}^{\infty}$ is a sequence of simple eigenvalues of $A+B$ with eigenvectors $x_{\pm n}$ such that $x_{\pm n}$, $z_{\pm n}$ are biorthogonal pairs (up to normalisation) in X. When $\beta_0, \beta_1, \gamma > 0$, solutions of (6.1) *have the property of uniform exponential decay with rate δ determined by the spectrum of $A+B$, in the sense that* $\sup\{\operatorname{Re} i\omega \mid \omega \in \sigma(A+B)\} \leq -\delta < 0$ *and there exists a constant $M \geq 1$ such that*

$$\|T(t)x\|_X \leq Me^{-\delta t}\|x\|_X, \quad t \geq 0.$$

Proof. The first statement follows from the interpolation properties in (4.13) and Theorem 5. The second statement is then immediate from the spectral results of the previous sections since $A+B$ is a discrete spectral operator and so the spectrum-determined growth condition holds. □

References

1. F. Conrad and Ö. Morgül, *On the stabilization of a flexible beam with a tip mass*, SIAM Journal on Control and Optimization **36** (1998), no. 6, 1962–1986.
2. R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. II: Partial Differential Equations.* John Wiley & Sons, New York, NY, USA, 1962.
3. S. Cox and E. Zuazua, *The rate at which energy decays in a damped string*, Communications in Partial Differential Equations **19** (1994), no. 1-2, 213–243.
4. S. Cox and E. Zuazua, *The rate at which energy decays in a string damped at one end*, Indiana University Mathematics Journal **44** (1995), no. 2, 545–573.

5. R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, vol. 21 of Texts in Applied Mathematics. Springer-Verlag, New York, NY, USA, 1995.
6. N. Dunford and J. T. Schwartz, *Linear Operators. Part III: Spectral Operators*, vol. VII of Pure and Applied Mathematics. John Wiley & Sons, New York, NY, USA, 1971.
7. I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators. Vol. I*, vol. 49 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, Switzerland, 1990.
8. I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, vol. 18 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, USA, 1969.
9. A. Gomilko and V. Pivovarchik, *On bases of eigenfunctions of boundary problem associated with small vibrations of damped nonsmooth inhomogeneous string*, Asymptotic Analysis **20** (1999), no. 3-4, 301–315.
10. A. Gomilko and V. Pivovarchik, *On basis properties of a part of eigenfunctions of the problem of vibrations of a smooth inhomogeneous string damped at the midpoint*, Mathematische Nachrichten **245** (2002), no. 1, 72–93.
11. B.-Z. Guo, *Riesz basis approach to the stabilization of a flexible beam with a tip mass*, SIAM Journal on Control and Optimization **39** (2001), no. 6, 1736–1747.
12. B.-Z. Guo, *Riesz basis property and exponential stability of controlled Euler–Bernoulli beam equations with variable coefficients*, SIAM Journal on Control and Optimization **40** (2002), no. 6, 1905–1923.
13. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, vol. 31 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, USA, 1957.
14. T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics. Springer-Verlag, Berlin, Germany, 1995.
15. V. E. Katsnel′son, *Conditions under which systems of eigenvectors of some classes of operators form a basis*, Functional Analysis and Its Applications **1** (1967), no. 2, 122–132.
16. M. V. Keldysh, *On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators*, Russian Mathematical Surveys **26** (1971), no. 4, 15–44.
17. S. G. Kreĭn, *Linear Differential Equations in Banach Space*, vol. 29 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, USA, 1971.
18. Yu. I. Lyubich, *Linear Functional Analysis.* In *Functional Analysis I*, vol. 19 of Encyclopaedia of Mathematical Sciences, pages 1–283. Springer-Verlag, Berlin, Germany, 1992.
19. M. Mahinzaeim, D. C. Swailes, and J. M. Hale, *Bounded vibration control of distributed parameter systems, with application to a beam on viscoelastic supports*, Proceedings of the 4th ECCOMAS Thematic Conference on Smart Structures and Materials, 2009. Held in Porto, Portugal, July 13–15 (CD-ROM).
20. M. M. Malamud and L. L. Oridoroga, *On the completeness of root subspaces of boundary value problems for first order systems of ordinary differential equations*, Journal of Functional Analysis **263** (2012), no. 7, 1939–1980.
21. A. S. Markus, *A basis of root vectors of a dissipative operator*, Soviet Mathematics. Doklady **1** (1960), no. 1-3, 599–602.
22. A. S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, vol. 71 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, USA, 1988.
23. R. Mennicken and M. Möller, *Non-Self-Adjoint Boundary Eigenvalue Problems*, vol. 192 of North-Holland Mathematics Studies. Elsevier/North-Holland, Amsterdam, The Netherlands, 2003.
24. A. I. Miloslavskii, *Foundation of the spectral approach in nonconservative problems of the theory of elastic stability*, Functional Analysis and Its Applications **17** (1983), no. 3, 233–235.
25. A. I. Miloslavskii, *Stability of certain classes of evolution equations*, Siberian Mathematical Journal **26** (1985), no. 5, 723–735.
26. M. Möller and V. Pivovarchik, *Spectral Theory of Operator Pencils, Hermite–Biehler Functions, and their Applications*, vol. 246 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, Switzerland, 2015.
27. M. Möller and V. N. Pivovarchik, *Spectral properties of a fourth order differential equation*, Zeitschrift für Analysis und ihre Anwendungen **25** (2006), no. 3, 341–366.
28. M. A. Naimark, *Linear Differential Operators. Part I: Elementary Theory of Linear Differential Operators.* Frederick Ungar Publishing, New York, NY, USA, 1967.
29. N. K. Nikol′skiĭ, *Treatise on the Shift Operator. Spectral Function Theory*, vol. 273 of Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, Germany, 1986.

30. V. N. Pivovarchik, *On spectra of a certain class of quadratic operator pencils with one-dimensional linear part.*, Ukrainian Mathematical Journal **59** (2007), no. 5, 766–781.
31. G. V. Radzievskii, *The problem of the completeness of root vectors in the spectral theory of operator-valued functions*, Russian Mathematical Surveys **37** (1982), no. 2, 91–164.
32. B. Rao, *Optimal energy decay rate in a damped Rayleigh beam*, Discrete and Continuous Dynamical Systems **4** (1998), no. 4, 721–734.
33. H. Röh, *Dissipative operators with finite dimensional damping*, Proceedings of the Royal Society of Edinburgh. Section A: Mathematics **91** (1982), no. 3-4, 243–263.
34. A. A. Shkalikov, *Boundary problems for ordinary differential equations with parameter in the boundary conditions*, Journal of Mathematical Sciences **33** (1986), no. 6, 1311–1342.
35. A. A. Shkalikov, *Perturbations of self-adjoint and normal operators with discrete spectrum*, Russian Mathematical Surveys **71** (2016), no. 5, 907–964.
36. G.-Q. Xu and D.-X. Feng, *The Riesz basis property of a Timoshenko beam with boundary feedback and application*, IMA Journal of Applied Mathematics **67** (2002), no. 4, 357–370.
37. B. Zinsou, *Fourth-order Birkhoff regular problems with eigenvalue parameter dependent boundary conditions*, Turkish Journal of Mathematics **40** (2016), no. 4, 864–873.

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Received 02/05/2017; Revised 27/06/2017.