POLARIZATION INEQUALITY IN COMPLEX NORMED SPACES

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ABSTRACT. We introduce a product in all complex normed vector spaces, which generalizes the inner product of complex inner product spaces. Naturally the question occurs whether the polarization inequality of line (3.1) is fulfilled. We show that the polarization inequality holds for the product from Definition 1.1.

This also yields a new proof of the Cauchy-Schwarz inequality in complex inner product spaces, which does not rely on the linearity of the inner product. The proof depends only on the norm in the vector space.

1. INTRODUCTION

We deal with vector spaces X over the complex field $\mathbb{C}$, provided with a norm $\| \cdot \|$.

As a motivation we begin with the special case of an inner product space $(X, < \cdot \,|\, \cdot >)$. The inner product $< \cdot \,|\, \cdot >$ generates a norm by $\|\vec{x}\| = \sqrt{< \vec{x} \,|\, \vec{x} >}$, for all $\vec{x} \in X$. By the same token it is well known that the inner product can be expressed by this norm, namely for $\vec{x}, \vec{y} \in X$ we can write

$$(1.1) \qquad < \vec{x} \,|\, \vec{y} > = \frac{1}{4} \left[\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 + \mathbf{i} \left(\|\vec{x} + \mathbf{i}\,\vec{y}\|^2 - \|\vec{x} - \mathbf{i}\,\vec{y}\|^2 \right) \right],$$

where the symbol '$\mathbf{i}$' means the imaginary unit.

We now assume only that X is provided with a norm $\| \cdot \|$ which defines the topology on X, and we use an idea in [3] to define a continuous product $< \cdot \,|\, \cdot >: \ X \times X \to \mathbb{C}$.

Definition 1.1. Let $\vec{x}, \vec{y}$ be two arbitrary elements of X. In the case of $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$ we set $< \vec{x} \,|\, \vec{y} > := 0$, and if both $\vec{x}, \vec{y} \neq \vec{0}$ (i.e. $\|\vec{x}\| \, \|\vec{y}\| > 0$) we define the complex number

$< \vec{x} \,|\, \vec{y} > :=$

$$\|\vec{x}\| \, \|\vec{y}\| \frac{1}{4} \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \mathbf{i} \left(\left\| \frac{\vec{x}}{\|\vec{x}\|} + \mathbf{i}\, \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \mathbf{i}\, \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right) \right].$$

The product from Definition 1.1 opens the possibility to define a generalized 'angle' both in real normed spaces, see [4], and in complex normed spaces, see [5]. In this paper we turn our focus on the product. We prove the polarization inequality, further we notice some properties of the product.

Let $(X, \|\cdot\|)$ be an arbitrary complex normed vector space. In Definition 1.1 we defined a continuous product $< \cdot \,|\, \cdot >$ on X. This is just an inner product in the case that the norm $\| \cdot \|$ generates this product by the equation of line (1.1).

Generally, for spaces $X \neq \{\vec{0}\}$, the codomain of the product from Definition 1.1 is the entire complex plane $\mathbb{C}$, i.e. we have a surjective map $< \cdot \,|\, \cdot >: X^2 \to \mathbb{C}$. If we restrict the domain of the product $< \cdot \,|\, \cdot >$ on unit vectors of X, it is easy to see that the codomain

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changes into the 'complex square' $\{r + \mathbf{i}\,s \in \mathbb{C} \mid -1 \leq r, s \leq +1\}$. We can improve this statement: Actually the codomain is the complex unit circle $\{r + \mathbf{i}\,s \in \mathbb{C} \mid r^2 + s^2 \leq 1\}$. This is a consequence of the polarization inequality.

First we show that for a proof of this inequality we can restrict our research on the two dimensional complex vector space $\mathbb{C}^2$, provided with all possible norms.

2. General properties

Let $(X, \| \cdot \|)$ be an arbitrary complex vector space provided with a *norm* $\| \cdot \|$, this means that there is a continuous map $\| \cdot \|\colon X \longrightarrow \mathbb{R}^+ \cup \{0\}$ which fulfills the following axioms $\|z\,\vec{x}\| = |z|\,\|\vec{x}\|$ ('absolute homogeneity'), $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ ('triangle inequality'), and $\|\vec{x}\| = 0$ only for $\vec{x} = \vec{0}$ ('positive definiteness'), for $\vec{x}, \vec{y} \in X$ and $z \in \mathbb{C}$.

Assume that the complex vector space X is provided with a norm $\| \cdot \|$, and further there is a product $< \cdot \mid \cdot >\colon X \times X \to \mathbb{C}$. We say that the triple $(X, \| \cdot \|, < \cdot \mid \cdot >)$ satisfies the *Polarization Inequality* if and only if for all $\vec{x}, \vec{y} \in X$ there is the inequality $|< \vec{x} \mid \vec{y} >| \leq \|\vec{x}\|\,\|\vec{y}\|$.

It is well known that a complex normed space $(X, \|\cdot\|)$ where X is additionally provided with an inner product fulfills the polarization inequality, see for instance [2], p. 307.

Proposition 2.1. *For all vectors $\vec{x}, \vec{y} \in (X, \| \cdot \|)$ and for real numbers r the product $< \cdot \mid \cdot >$ of Definition 1.1 has the following properties.*

(a) $< \vec{x} \mid \vec{y} > = \overline{< \vec{y} \mid \vec{x} >}$ *(conjugate symmetry),*

(b) $< \vec{x} \mid \vec{x} > \geq 0$, *and* $< \vec{x} \mid \vec{x} > = 0$ *only for* $\vec{x} = \vec{0}$ *(positive definiteness),*

(c) $< r\,\vec{x} \mid \vec{y} > = r < \vec{x} \mid \vec{y} > = < \vec{x} \mid r\,\vec{y} >$ *(homogeneity for real numbers),*

(d) $< r\,\mathbf{i}\,\vec{x} \mid \vec{y} > = r\,\mathbf{i} < \vec{x} \mid \vec{y} > = < \vec{x} \mid -r\,\mathbf{i}\,\vec{y} >$ *(homogeneity for pure imaginary numbers),*

(e) $\|\vec{x}\| = \sqrt{< \vec{x} \mid \vec{x} >}$ *(the norm can be expressed by the product).*

(f) *For a pair $\vec{x}, \vec{y} \in X$ of unit vectors, i.e. $\|\vec{x}\| = 1 = \|\vec{y}\|$, it holds that both the real part and*

     the imaginary part of $< \vec{x} \mid \vec{y} >$ are in the interval $[-1, 1]$.

Proof. We use Definition 1.1, and the proofs for (a) and (b) are easy. For positive $r \in \mathbb{R}$ the point (c) is trivial. We can prove $< -\vec{x} \mid \vec{y} > = - < \vec{x} \mid \vec{y} > = < \vec{x} \mid -\vec{y} >$, and (c) follows immediately. The point (d) is similar to (c), and (e) is clear. The point (f) is proven by the triangle inequality, for instance for the real part we have

$$|\text{real part of } < \vec{x} \mid \vec{y} >| = \left| \|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 \right| /4 \leq \|\vec{x} + \vec{y}\|^2 /4 \leq 2^2/4 = 1.$$

$\square$

The statement (f) in the above proposition means, that $\{< \vec{x} \mid \vec{y} > \mid \vec{x}, \vec{y} \in X, \|\vec{x}\| = 1 = \|\vec{y}\|\}$ is a subset of the 'complex square' $\{r + \mathbf{i}\,s \in \mathbb{C} \mid -1 \leq r, s \leq +1\}$. Hence, for unit vectors $\vec{x}, \vec{y}$ there is the estimate $|< \vec{x} \mid \vec{y} >| \leq \sqrt{2}$.

Now we notice a few facts about the general product $< \cdot \mid \cdot >$ from Definition 1.1.

Lemma 2.2. *In a complex normed space $(X, \| \cdot \|)$ for $\vec{x}, \vec{y} \in X$ and real φ there are identities*

$$< e^{\mathbf{i}\,\varphi}\,\vec{x} \mid \vec{x} > = e^{\mathbf{i}\,\varphi} < \vec{x} \mid \vec{x} > \quad \text{and} \quad < e^{\mathbf{i}\,\varphi}\,\vec{x} \mid e^{\mathbf{i}\,\varphi}\,\vec{y} > = < \vec{x} \mid \vec{y} >,$$

where $< \cdot \mid \cdot >$ is the product of Definition 1.1.

Proof. To prove the first equation take an unit vector $\vec{x}$, i.e. $\|\vec{x}\| = 1$, write $e^{\mathbf{i}\,\varphi} = \cos(\varphi) + \mathbf{i}\,\sin(\varphi)$, and use Definition 1.1. The second identity comes directly from Definition 1.1. $\square$

Corollary 2.3. *For an unit vector $\vec{x} \in X$ we have that the set $\{< e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{x} > \;|\; \varphi \in [0, 2\pi]\}$ is the complex unit circle, since $< e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{x} > = e^{\mathbf{i}\varphi} < \vec{x} \,|\, \vec{x} > = e^{\mathbf{i}\varphi}$.*

Remark 2.4. The next example shows that in a complex normed space $(X, \|\cdot\|)$ generally we have the inequality $< e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{y} > \neq e^{\mathbf{i}\varphi} < \vec{x} \,|\, \vec{y} >$. This statement seems to be 'probable', but we need an example, which we yield in the proof of the next lemma.

This inequality means, that the set of products $\{< e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{y} > \;|\; \varphi \in [0, 2\pi]\}$ commonly does not generate a proper Euclidean circle (with radius $|< \vec{x} \,|\, \vec{y} >|$) in $\mathbb{C}$.

If we take $\varphi \in \{\pi, \pi/2, \pi\,3/2\}$, however, we get with Proposition 2.1 three identities

$$< -\vec{x} \,|\, \vec{y} > = -< \vec{x} \,|\, \vec{y} >, \quad < \mathbf{i}\,\vec{x} \,|\, \vec{y} > = \mathbf{i} \,< \vec{x} \,|\, \vec{y} >,$$
$$\text{and} \quad < -\mathbf{i}\,\vec{x} \,|\, \vec{y} > = -\mathbf{i} \,< \vec{x} \,|\, \vec{y} > .$$

Lemma 2.5. *Let $(X, \|\cdot\|)$ be a complex normed space, and let $< \cdot \,|\, \cdot >$ be the product of Definition 1.1. We have generally*
$$< e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{y} > \neq e^{\mathbf{i}\varphi} < \vec{x} \,|\, \vec{y} >, \quad \text{even their moduli are different.}$$

Proof. We use the most simple non-trivial example of a complex normed space, let $(X, \|\cdot\|) := (\mathbb{C}^2, \|\cdot\|_\infty)$, where for two complex numbers $r + \mathbf{i}\,s$, $v + \mathbf{i}\,w \in \mathbb{C}$ we have its norm $\|\cdot\|_\infty$ by

$$\left\| \begin{pmatrix} r + \mathbf{i}\,s \\ v + \mathbf{i}\,w \end{pmatrix} \right\|_\infty = \max\left\{ \sqrt{r^2 + s^2}, \sqrt{v^2 + w^2} \right\} .$$

The following calculations are easy, but laborious. We define two unit vectors $\vec{x}, \vec{y}$ of $(\mathbb{C}^2, \|\cdot\|_\infty)$,

$$\vec{x} := \frac{1}{4} \begin{pmatrix} 1 + \mathbf{i}\,\sqrt{15} \\ 2 + \mathbf{i}\,2 \end{pmatrix} \quad \text{and} \quad \vec{y} := \frac{1}{4} \begin{pmatrix} 2 + \mathbf{i} \\ 3 + \mathbf{i}\,\sqrt{7} \end{pmatrix} .$$

Some calculations yield the complex number

$$< \vec{x} \,|\, \vec{y} > = \frac{1}{64} \left(19 + 4\sqrt{7} + 2\sqrt{15} + \mathbf{i} \left[7 - 4\sqrt{7} + 4\sqrt{15} \right] \right) \approx 0.583 + \mathbf{i}\,0.186 .$$

We choose $e^{\mathbf{i}\varphi} := 1/2 \left(1 + \mathbf{i}\,\sqrt{3} \right)$ from the complex unit circle, and we get approximately $e^{\mathbf{i}\varphi} < \vec{x} \,|\, \vec{y} > \approx 0.130 + \mathbf{i}\,0.598$. After that we take the unit vector

$$e^{\mathbf{i}\varphi} \vec{x} = \frac{1}{8} \begin{pmatrix} 1 - \sqrt{45} + \mathbf{i} \left[\sqrt{3} + \sqrt{15} \right] \\ 2 - 2\sqrt{3} + \mathbf{i} \left[2 + 2\sqrt{3} \right] \end{pmatrix} , \text{ and we compute the product } < e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{y} >,$$

$$< e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{y} > = (p + \mathbf{i}\,q)/64 \approx 0.113 + \mathbf{i}\,0.628, \text{ where } p \text{ and } q \text{ abbreviate real numbers}$$

$$p = 11 + 2\left(\sqrt{7} + \sqrt{21} - \sqrt{45} \right) - 5\sqrt{3} + \sqrt{15}, \quad q = 8 + 2\left(4\sqrt{3} - \sqrt{7} + \sqrt{15} + \sqrt{21} \right) + \sqrt{45} .$$

This proves the inequality $< e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{y} > \neq e^{\mathbf{i}\varphi} < \vec{x} \,|\, \vec{y} >$, and the lemma is confirmed. $\square$

The above lemma suggests the following conjecture. One direction is trivial.

Conjecture 2.6. *In a complex normed space $(X, \|\cdot\|)$ for all $\vec{x}, \vec{y} \in X$, $\varphi \in \mathbb{R}$ it holds*
$$< e^{\mathbf{i}\varphi} \vec{x} \,|\, \vec{y} > = e^{\mathbf{i}\varphi} < \vec{x} \,|\, \vec{y} >$$
if and only if its product $< \cdot \,|\, \cdot >$ from Definition 1.1 is actually an inner product, i.e. $(X, < \cdot \,|\, \cdot >)$ is an inner product space.

3. THE POLARIZATION INEQUALITY

In this section we deal with the polarization inequality. It turns into the Cauchy-Schwarz inequality in the case that the pair $(X, < \cdot \,|\, \cdot >)$ is actually an inner product space. Let X be a complex normed space, let $\|\cdot\|$ be a norm on X and $< \cdot \,|\, \cdot >$ be the product from Definition 1.1. We ask whether in the triple $(X, \|\cdot\|, < \cdot \,|\, \cdot >)$ the inequality

$$(3.1) \qquad | < \vec{x}\,|\,\vec{y} > | \;\leq\; \|\vec{x}\|\,\|\vec{y}\|$$

is fulfilled for all $\vec{x}, \vec{y} \in X$. The answer is positive.

Theorem 3.1. *The polarization inequality in line* (3.1) *holds in all complex vector spaces X, provided with a norm $\|\cdot\|$ and the product $< \cdot \,|\, \cdot >$ from Definition 1.1.*

Remark 3.2. The theorem is the main contribution of this paper. The proof of the Cauchy-Schwarz inequality in inner product spaces is well documented in many books about functional analysis by using the linearity of the inner product, see for instance [6], p. 204. This new proof of the polarization inequality depends only on the norm in the vector space.

Proof. First we need a lemma, which shows that for a complete answer it suffices to investigate the complex vector space $\mathbb{C}^2$, provided with all possible norms.

Lemma 3.3. *The following two statements* (1) *and* (2) *are equivalent.*
(1) There exists a complex normed vector space $(X, \|\cdot\|)$ and two vectors $\vec{a}, \vec{b} \in X$ with

$$(3.2) \qquad | < \vec{a}\,|\,\vec{b} > | \;>\; \|\vec{a}\|\,\|\vec{b}\| \;.$$

(2) There is a norm $\|\cdot\|$ on $\mathbb{C}^2$ and two unit vectors $\vec{x}, \vec{y} \in \mathbb{C}^2$ with

$$(3.3) \qquad | < \vec{x}\,|\,\vec{y} > | \;>\; 1 \;.$$

Proof. (1) $\Leftarrow$ (2) Trivial.
(1) $\Rightarrow$ (2) Easy. Let us consider the two-dimensional subspace U of X which is spanned by the linear independent vectors $\vec{a}, \vec{b}$. This space U is isomorphic to $\mathbb{C}^2$. We take the norm from X on U. We normalize $\vec{a}, \vec{b}$, i.e. we define unit vectors $\vec{x} := \vec{a}/\|\vec{a}\|$ and $\vec{y} := \vec{b}/\|\vec{b}\|$. Hence the inequality (3.2) turns into (3.3). $\qquad\square$

The lemma means, that we can restrict our investigations on the complex vector space $\mathbb{C}^2$. By a transformation of coordinates we state that instead of the linear independent unit vectors $\vec{x}, \vec{y}$ of inequality (3.3) we set $(1,0) := \vec{x}$ and $(0,1) := \vec{y}$. With Definition 1.1 the product $< (1,0)|(0,1) >$ has the presentation

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\middle|\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$(3.4)$$

$$= \; \frac{1}{4}\,\left[\, \left\|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|^2 - \left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\|^2 + \mathbf{i}\left(\left\|\begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix}\right\|^2 - \left\|\begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix}\right\|^2\right) \right] \;.$$

We take four suitable real numbers s, t, v, w, and we define four positive values

$$(3.5)\quad \left\|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| =: \frac{1}{s}\;, \quad \left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\| =: \frac{1}{t}\;, \quad \left\|\begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix}\right\| =: \frac{1}{v}\;, \quad \left\|\begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix}\right\| =: \frac{1}{w}\;,$$

or, equivalently, we have four unit vectors $(s, s), (t, -t), (v, \mathbf{i}\,v), (w, -\mathbf{i}\,w)$, i. e.

$$1 = \|(s,s)\| = \|(t,-t)\| = \|(v,\mathbf{i}\,v)\| = \|(w,-\mathbf{i}\,w)\|.$$

Hence the product $< (1,0)|(0,1) >$ changes into

$$(3.6) \quad \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \frac{1}{4} \left[\left(\frac{1}{s}\right)^2 - \left(\frac{1}{t}\right)^2 + \mathbf{i} \left(\left(\frac{1}{v}\right)^2 - \left(\frac{1}{w}\right)^2 \right) \right].$$

Further, instead of the polarization inequality $| < (1,0)|(0,1) > | \leq 1$, for an easier handling we can deal with the equivalent inequality

$$(3.7) \quad \begin{aligned} & \left| 4 \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \right|^2 \\ & = \left[\left(\frac{1}{s}\right)^2 - \left(\frac{1}{t}\right)^2 \right]^2 + \left[\left(\frac{1}{v}\right)^2 - \left(\frac{1}{w}\right)^2 \right]^2 \leq 16. \end{aligned}$$

Lemma 3.4. *All four numbers s, t, v, w are greater or equal $1/2$.*

Proof. For instance to show $1/2 \leq s$ use the equation $(s, s) = s(1,0) + s(0,1)$. Apply the triangle inequality, and note $\|(s,s)\| = 1$, and also $\|(0,1)\| = 1 = \|(1,0)\|$. $\square$

The next lemma means, that we can assume that both the real part and the imaginary part of $< (1,0)|(0,1) >$ are positive.

Lemma 3.5. *Without restriction of generality we assume $s < t$ and $v < w$.*

Proof. In the case of $s = t$, the first summand of the middle term in line (3.7) is zero. From Lemma 3.4 follows $1/v \leq 2$. Hence $|4 < (1,0)|(0,1) > |^2 \leq (1/v)^4 \leq (2)^4 = 16$, it holds (3.7).

In the case of $s > t$, i.e. $1/s < 1/t$, i.e. we have a negative real part of $< (1,0)|(0,1) >$, we consider instead $< (-1,0)|(0,1) >$. By Proposition 2.1(c), we get a positive real part. With a transformation of coordinates we rename $-(1,0)$ into $(1,0)$, to get a representation $< (1,0)|(0,1) >$ with positive real part. In the case that the imaginary part of $< (1,0)|(0,1) >$ is still negative, we take the product $< (0,1)|(1,0) >$. Now, by Proposition 2.1(a), also the imaginary part is positive. We make a second transformation of coordinates, and in new coordinates we call this $< (1,0)|(0,1) >$.

The cases $v = w$ and $v > w$ work in a similar manner. $\square$

The following propositions Proposition 3.6 and Proposition 3.7 collect general properties of the product $< (1,0)|(0,1) >$ from line (3.4). The proofs always rely on the triangle inequality of a normed space, which is equivalent to the fact that its unit ball is convex.

Please consider now the next equation, which we will use in Proposition 3.6.

$$(3.8) \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = [(1-a) - \mathbf{i}\, b] \begin{pmatrix} 1 \\ 0 \end{pmatrix} + [(1-b) + \mathbf{i}\, a] \begin{pmatrix} 0 \\ 1 \end{pmatrix} + [a + \mathbf{i}\, b] \begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix},$$

which is true for arbitrary $a, b \in \mathbb{R}$. We will use also the corresponding equation of the following line.

$$(3.9) \quad (1, \mathbf{i}) = [(1-a) + \mathbf{i}\, b] (1,0) + [\mathbf{i}(1-b) + a] (0,1) + [a - \mathbf{i}\, b] (1,-1).$$

The next proposition looks strange, but it will give the deciding hint for the proof.

Proposition 3.6. *We get for each $b \in \mathbb{R}$ the following two inequalities:*

$$(3.10) \quad \frac{1}{s} \leq 2\sqrt{1 + 2b^2 - 2b} + \sqrt{2}\,|b|\,\frac{1}{w}, \quad \frac{1}{v} \leq 2\sqrt{1 + 2b^2 - 2b} + \sqrt{2}\,|b|\,\frac{1}{t}.$$

Proof. From the line (3.8) we get $1/s \leq \sqrt{(1-a)^2 + b^2} + \sqrt{(1-b)^2 + a^2} + \sqrt{a^2 + b^2}/w$, which is true for arbitrary real numbers a, b. If we choose $b = a$, it follows the first inequality in Proposition 3.6. The second inequality uses the corresponding equation in line (3.9). $\square$

The Proposition 3.6 has an important consequence.

Proposition 3.7. *If* $\frac{1}{2}\sqrt{2} \le w$ *it holds inequality* (A), *and in the case of* $\frac{1}{2}\sqrt{2} \le t$ *it holds inequality* (B), *where*

$$(A): \ \left(\frac{1}{s}\right)^2 \ \le \ 2 + \frac{\sqrt{4\,w^2 - 1}}{w^2} \quad and \quad (B): \ \left(\frac{1}{v}\right)^2 \ \le \ 2 + \frac{\sqrt{4\,t^2 - 1}}{t^2} \ .$$

Proof. To prove inequality (A), we consider again Proposition 3.6, and we investigate the right hand side of the first inequality in line (3.10). For all constants $w > 1/2$ we define a function $\mathcal{R}(b)$, $b \in \mathbb{R}$,

$$(3.11) \qquad\qquad \mathcal{R}(b) \ := \ 2\sqrt{1 + 2\,b^2 - 2\,b} \ + \ \sqrt{2}\,|b|\frac{1}{w} \ .$$

Obviously we get the limits $\lim_{b \to +\infty} \mathcal{R}(b) = \lim_{b \to -\infty} \mathcal{R}(b) = +\infty$ and since the parabola $1 + 2\,b^2 - 2\,b$ has only positive values, we state that $\mathcal{R}$ has a codomain of positive numbers,

$$\mathcal{R} : \mathbb{R} \to \mathbb{R}^+ \ .$$

By Proposition 3.6 it holds $1/s \le \mathcal{R}(b)$ for all b, hence we are interested in minimums of $\mathcal{R}$ to get an estimate for $1/s$ as small as possible. Since $\mathcal{R}(-b) > \mathcal{R}(b)$ for all positive b, the minimum must occur at a non negative b. Therefore, we consider the function $\mathcal{R}$ for non negative b. The search for a minimum is the standard method, we have

$$\mathcal{R}'(b) \ = \ \frac{4\,b - 2}{\sqrt{1 + 2\,b^2 - 2\,b}} + \frac{\sqrt{2}}{w} \quad \text{for all} \quad b \ge 0 \ .$$

In the case of $\frac{1}{2}\sqrt{2} < w$ the equation $\mathcal{R}'(b_E) = 0$ has one positive solution b_E,

$$b_E \ = \ \frac{1}{2}\left[1 - \frac{1}{\sqrt{4\,w^2 - 1}}\right] \quad (\text{We have } w > 1/2 \text{ by the lemmas 3.4 and 3.5 }).$$

Recall that we are looking for positive $b_E\,'s$, hence we are investigating the positive part in the definition (3.11) of $\mathcal{R}(b)$. The condition $0 \le b_E$ holds if $\frac{1}{2}\sqrt{2} \le w$. We add this as an assumption, hence in this proposition we assume $\frac{1}{2}\sqrt{2} \le w, t$.

As an intermediate step we mention that for the expression $1 + 2\,b^2 - 2\,b$ for $b = b_E$ we get the term

$$\frac{2\,w^2}{4\,w^2 - 1} \ .$$

Finally we get a number of $\mathcal{R}$ at b_E, we have

$$(b_E, \mathcal{R}(b_E)) \ = \ \left(\frac{1}{2}\left[1 - \frac{1}{\sqrt{4\,w^2 - 1}}\right], \ \frac{\sqrt{2}}{2\,w}\left[1 + \sqrt{4\,w^2 - 1}\right]\right) \ .$$

By Proposition 3.6 we get an estimate for $1/s$, but actually we are more interested in an estimate for $(1/s)^2$. We calculate

$$(\mathcal{R}(b_E))^2 \ = \ 2 + \frac{\sqrt{4\,w^2 - 1}}{w^2} \ ,$$

which finishes the proof of Proposition 3.7, since the proof of inequality (B) works in a similar manner.   □

The above propositions may be a useful tool for further computations. For our purpose it will be sufficient. With Proposition 3.7 we are able to do the final stroke. We are still proving Theorem 3.1, i.e. we try to confirm the polarization inequality (3.7). To prove the theorem we need to distinguish between three cases (Case a), (Case b), (Case c); only the third will be difficult.

(Case a): Let both t, w be in the closed interval $[1/2, \sqrt{2}/2]$. Hence
$|4 < (1,0)|(0,1) > |^2 = [(1/s)^2 - (1/t)^2]^2 + [(1/v)^2 - (1/w)^2]^2 \leq 2 [4 - (2/\sqrt{2})^2]^2 = 2 [2]^2 = 8$.

(Case b): Let $1/2 < t \leq \sqrt{2}/2 < w$ or the contrary $1/2 < w \leq \sqrt{2}/2 < t$.
We use the estimate (A) or (B) from Proposition 3.7, and we compute

$$(3.12) \quad \left| 4 \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \right|^2 \leq \left[2 + \frac{\sqrt{4 w^2 - 1}}{w^2} - \left(\frac{2}{\sqrt{2}} \right)^2 \right]^2 + \left[2^2 - \left(\frac{1}{w} \right)^2 \right]^2$$

$$(3.13) \qquad = \frac{4 w^2 - 1}{w^4} + \left[16 + \frac{1}{w^4} - \frac{8}{w^2} \right] = \frac{4}{w^2} + 16 - \frac{8}{w^2} ,$$

and it is trivial that the last sum is less than 16, hence the polarization inequality is confirmed for (Case b). Before we deal with the last case (Case c), one more lemma is necessary.

Lemma 3.8. *Let* $\;|t|, |w| \geq \frac{1}{2}$.
The following two inequalities (C) *and* (D) *are equivalent, and both are true.*

$$(\mathsf{C}): \quad \left[2 + \frac{\sqrt{4 w^2 - 1}}{w^2} - \left(\frac{1}{t} \right)^2 \right]^2 + \left[2 + \frac{\sqrt{4 t^2 - 1}}{t^2} - \left(\frac{1}{w} \right)^2 \right]^2 \;\leq\; 16,$$

$$(\mathsf{D}): \quad (2 t^2 - 1) \sqrt{4 w^2 - 1} + (2 w^2 - 1) \sqrt{4 t^2 - 1} \;\leq\; 4 t^2 w^2.$$

Proof. Starting with (C), the proof of the equivalence is straightforward.
The last step is to confirm the second inequality (D) for all $|t|, |w| \geq \frac{1}{2}$. This needs two substitutions. The first is $p := 4 t^2 - 1$ and $z := 4 w^2 - 1$. The inequality (D) leads to

$$(3.14) \qquad \left(\frac{p+1}{2} - 1 \right) \sqrt{z} + \left(\frac{z+1}{2} - 1 \right) \sqrt{p} \;\leq\; \frac{1}{4} (p z + p + z + 1)$$

$$(3.15) \qquad \Longleftrightarrow \quad p \sqrt{z} - \sqrt{z} + z \sqrt{p} - \sqrt{p} \;\leq\; \frac{1}{2} (p z + p + z + 1) .$$

The second substitution is $h := \sqrt{p}$ and $k := \sqrt{z}$. It follows the equivalent inequalities

$$(3.16) \qquad h^2 k - k + k^2 h - h \;\leq\; \frac{1}{2} (h^2 k^2 + h^2 + k^2 + 1)$$

$$(3.17) \quad \Longleftrightarrow \quad h^2 \left(k - \frac{1}{2} k^2 - \frac{1}{2} \right) + h (k^2 - 1) - \left(k + \frac{1}{2} k^2 + \frac{1}{2} \right) \;\leq\; 0 .$$

We multiply it by (-2), and we get

$$(3.18) \qquad h^2 \left(-2 k + k^2 + 1 \right) - 2 h (k^2 - 1) + \left(2 k + k^2 + 1 \right) \;\geq\; 0$$

$$(3.19) \quad \Longleftrightarrow \quad [h (k - 1) - (k + 1)]^2 \;\geq\; 0 .$$

Obviously, the last inequality is true. Hence, both inequalities (C) and (D) in the lemma are also correct, for real t, w with $|t|, |w| \geq \frac{1}{2}$. □

Remark 3.9. In the above Lemma 3.8 in the second inequality (D) it occurs equality if and only if $h (k - 1) = k + 1$, or equivalently if and only if the two variables t and w fulfill the relation

$$t = \frac{w \sqrt{2}}{\sqrt{4 w^2 - 1} - 1} , \quad \text{for} \;\; |w| \neq \frac{1}{2} \sqrt{2} \;\; \text{and} \;\; |t|, |w| \geq \frac{1}{2} .$$

Further we remark that in this formula the variables t and w can be exchanged. We remark furthermore that in inequality (D), for $|t| = \frac{1}{2} \sqrt{2}$ or $|w| = \frac{1}{2} \sqrt{2}$ both sides of (D) have a constant difference of 1.

Now we regard the last case.

(Case c): Let $\sqrt{2}/2 < t, w$.

From Proposition 3.7 we have the estimates (A) and (B), hence we get the inequality

$$
\left| 4 \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \right|^2
$$
$$
\leq \quad \left[2 + \frac{\sqrt{4\,w^2 - 1}}{w^2} - \left(\frac{1}{t} \right)^2 \right]^2 + \left[2 + \frac{\sqrt{4\,t^2 - 1}}{t^2} - \left(\frac{1}{w} \right)^2 \right]^2.
$$

With Lemma 3.8 this yields the last step to prove the polarization inequality, since from inequality (C) in Lemma 3.8 follows the polarization inequality, i.e. Theorem 3.1 finally is confirmed.   Note that for two linear dependent vectors $\vec{x}, \vec{y} \in X$ the polarization inequality is trivial. $\qquad\square$

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