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FIXED POINTS OF COMPLEX SYSTEMS WITH ATTRACTIVE INTERACTION

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The paper is dedicated to memory of prominent Ukrainian mathematician Myroslav Gorbachuk

Abstract. We study the behavior of complex dynamical systems describing an attractive interaction between two opponents. We use the stochastic interpretation and describe states of systems in terms of probability distributions (measures) and their densities. For the time evolution we derive specific non-linear difference equations which generalize the well-known Lotka-Volterra equations. Our results state the existence of fixed points (equilibrium states) for various kinds of attractive interactions. Besides, we present an explicit description of the limiting distributions and illustrate abstract results by several examples.

1. Introduction

Let $\mathcal{A}$ denote a complex system consisting of a big set of agents $\mathcal{A} = \{a_i\}_{i=1}^n$, $n \geq \infty$, which live on a common resource space Ω. States of the agents a are described by stochastic distributions (probability measures) μ on Ω. Values $\mu(A)$, $A \subseteq \Omega$, characterize the probabilities for the agent a to be in A. In particular models $\mu(A)$ may have various meanings: a probability of occupation, a power of population, etc. Let $\mathcal{M}(\Omega)$ denote the set of all measures corresponding to $\mathcal{A}$. We assume that the system $\mathcal{A}$ is interactive and each agent a changes its state under a certain positive or negative interaction with other agents. Let $\circledast$ denote some law of interaction between agents. We call $\circledast$ a *conflict interaction* since it perturbs the free evolution of agents. The triple $\{\Omega, \ \mathcal{M}(\Omega), \circledast\}$ is called [16, 19] a *dynamical system of conflict.*

Here we consider a discrete time evolution of states which is written in a symbolic form as follows:

$$\mu^N \xrightarrow{\circledast} \mu^{N+1}, \quad N = 0, 1, \dots, \tag{1.1}$$

where $\mu^0 = \mu$ stands for an initial state. We are interested in studying the behavior of the trajectories μ^N, $N \to \infty$, under a certain law of conflict interaction $\circledast$ considered as a transformation in space of measures.

We will study trajectories (1.1) produced by the difference equation of the form

$$\mu^{N+1} = \frac{1}{z^N}[\mu^N + \theta^N \cdot \mu^N + \tau^N],$$

where $\theta^N = \theta(\mu^N)$ and θ is a real-valued multiplicative functional on $\mathcal{M}(\Omega)$, and τ^N is a sequence of positive measures on Ω. Here z^N denotes the normalizing denominator. We suppose that θ^N is responsible for the independent time evolution of each agent, and τ^N describes the conflict interaction.

In this paper we continue to develop mathematical tools for studying complex dynamical systems that describe the phenomenon of conflict interaction between opponents which are associated with agents a.

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In the previous works (see [2, 5, 14, 15, 16, 18]) for a description of opponent states we mainly used discrete probability measures (stochastic vectors $p, r \in \mathbb{R}^n_+,\ n > 1$). In these cases the evolution equations are rather simple but non-linear, and present some generalization of the vector form of the Lotka-Volterra difference equations

$$p_i^{N+1} = p_i^N(1 + \theta^N \pm r_i^N), \quad r_i^{N+1} = r_i^N(1 + \theta^N \pm p_i^N), \quad N = 0, 1, \dots,$$

where $p_i^N, r_i^N,\ i = 1, 2, \dots, n$, stands for coordinates of the stochastic vectors $p^N, r^N \in \mathbb{R}^n_+$ that describe states of the opponents at time $t = N$, and $\theta^N = (p^N, r^N)$ is the inner product in $\mathbb{R}^n_+$. The signs $\pm$ correspond to attractive or repulsive interactions, respectively. In both cases the theorem of conflict was proved (see also [5, 15, 16] and [19]). This theorem establishes existence of fixed points (equilibrium states) for any trajectory of the system.

There were discovered two difficulties. The first problem was concerned with a complete description of the limiting states for systems with attractive interaction. And the second one was that we were not able to go to arbitrary not necessarily discrete measures. For the case of repulsive interaction we reached a certain success [17, 18] using the improved system of evolution equations in terms of the probability measures

$$\mu^{N+1} = \mu^N(1 + \theta^N) - \tau^N, \tag{1.2}$$

where θ^N has the meaning similar to the above, and τ^N fixes the local confrontation between the opponents at time $t = N$. The existence of compromise ω-limit states for the repulsive conflict interaction between opponents with any couple of starting probability measures $\mu, \nu \in \mathcal{M}(\Omega)$ was proved in [18].

In this paper we establish a similar result in the case of an attractive interaction, i.e., for a positive sign in front of the measure τ^N in (1.2). Moreover, we give an explicit description of limiting states μ^∞ and present particular examples.

The influence of the publications [4, 7, 8, 9, 10, 12, 13, 20, 21, 22, 23] has affected on our constructions.

2. Complex systems with attractive internal interaction

At first we consider an abstract case of a complex dynamical system with an internal attractive interaction between couples of agents. We start with some preparations.

Let Ω be a metric space and $\mathcal{R}$ be a σ-algebra of the Borel subsets of Ω. Denote by $\mathcal{M}^+(\Omega)$ the family of all σ-additive finite positive measures on Ω. And the subset of positive probability measures on Ω is denoted by $\mathcal{M}^+_1(\Omega)$.

To construct a dynamical system, we define the conflict transformation $\circledast$ as follows. For a pair of probability measures $\mu, \nu \in \mathcal{M}^+_1(\Omega)$ we put into correspondence a new pair of probability measures μ^1, ν^1 according to the rule

$$\begin{aligned} \mu^1(A) &\equiv (\mu \circledast \nu)(A) = \tfrac{1}{z}[\mu(A) + \theta \cdot \mu(A) + \tau(A)], \quad \forall A \in \mathcal{R}, \\ \nu^1(A) &\equiv (\nu \circledast \mu)(A) = \tfrac{1}{z}[\nu(A) + \theta \cdot \nu(A) + \tau(A)], \end{aligned}$$

where $\theta = \theta(\mu, \nu)$ is an arbitrary positive form of μ, ν and $\tau(A)$ is a positive measure from $\mathcal{M}^+(\Omega)$. It is easy to check that the normalizing denominator $z = 1 + \theta + \mathcal{T}$, where $\mathcal{T} := \tau(\Omega)$. Due to the construction $\mu^1, \nu^1 \in \mathcal{M}^+_1(\Omega)$.

Since we consider pairs of probability measures μ, ν from $\mathcal{M}^+_1(\Omega) \times \mathcal{M}^+_1(\Omega)$ one may denote the conflict dynamical system with attraction as the triple $\{\Omega, \mathcal{M}^+_1(\Omega) \times \mathcal{M}^+_1(\Omega), \circledast\}$. The iteration of the above defined transformation $\circledast$ determines the trajectory in terms of couples of measures,

$$\{\mu^N, \nu^N\} \xrightarrow{\circledast} \{\mu^{N+1}, \nu^{N+1}\}, \quad N = 0, 1, \dots, \tag{2.1}$$

where $\mu^0 = \mu, \nu^0 = \nu$ and

$$\begin{aligned} \mu^{N+1}(A) &= \tfrac{1}{z^N}[\mu^N(A) + \theta^N \cdot \mu^N(A) + \tau^N(A)], \\ \nu^{N+1}(A) &= \tfrac{1}{z^N}[\nu^N(A) + \theta^N \cdot \nu^N(A) + \tau^N(A)]. \end{aligned} \tag{2.2}$$

Here $\theta^N = \theta(\mu^N, \nu^N)$, $\tau^N \in \mathcal{M}^+(\Omega)$, and

$$z^N = 1 + \theta^N + \mathcal{T}^N, \quad \mathcal{T}^N = \tau^N(\Omega).$$

Now we are able to present the main result of this section.

Theorem 2.1. *Let a conflict dynamical system $\{\Omega, \mathcal{M}_1^+(\Omega) \times \mathcal{M}_1^+(\Omega), \circledast\}$ be defined by the system of equations* (2.2) *where θ is a bounded positive quadratic form on $\mathcal{M}_1^+(\Omega)$ and τ^N is a sequence of positive measures on Ω which satisfy the following conditions:*

$$\frac{\tau^N(A)}{\tau^N(\Omega)} = \frac{\tau(A)}{\tau(\Omega)}, \quad A \in \mathcal{R}, \quad N = 0, 1, \ldots, \tag{2.3}$$

$$\tau^N(\Omega) \le \tau^{N+1}(\Omega), \quad N = 0, 1, \ldots \tag{2.4}$$

Then each trajectory (2.1) *converges to a fixed point $\{\mu^\infty, \nu^\infty\}$ such that*

$$\mu^\infty(A) = \nu^\infty(A) = \frac{\tau(A)}{\tau(\Omega)}. \tag{2.5}$$

Proof. In fact we have to prove the existence of the limits

$$\mu^\infty(A) = \lim_{N\to\infty} \mu^N(A), \quad \nu^\infty(A) = \lim_{N\to\infty} \nu^N(A), \quad A \in \mathcal{R},$$

such that

$$\mu^\infty(A) = \nu^\infty(A) = \frac{\tau(A)}{\mathcal{T}}, \quad A \in \mathcal{R}.$$

Denote $\tau_{\text{norm}}(A) := \frac{\tau(A)}{\mathcal{T}}$ and $d^0(A) := \mu(A) - \tau_{\text{norm}}(A)$. Using equations (2.2) and (2.3) we get

$$\begin{aligned} d^1(A) = \mu^1(A) - \tau_{\text{norm}}(A) &= \mu^1(A) - \frac{\tau(A)}{\mathcal{T}} \\ &= \frac{1}{z}\left[\mu(A)(1+\theta) + \tau(A) - \frac{\tau(A)(1+\theta+\mathcal{T})}{\mathcal{T}}\right] \\ &= \frac{1+\theta}{z}\left(\mu(A) - \frac{\tau(A)}{\mathcal{T}}\right) = \frac{1+\theta}{1+\theta+\mathcal{T}}\left(\mu(A) - \tau_{\text{norm}}(A)\right). \end{aligned}$$

Denote $k_0 := \frac{1+\theta}{1+\theta+\mathcal{T}}$. Thus we can write

$$d^1(A) = k_0 d^0(A),$$

and, by induction,

$$d^{N+1}(A) = \mu^{N+1}(A) - \frac{\tau(A)}{\mathcal{T}} = k_N \cdots k_0 \cdot d^0(A),$$

where obviously $k_N = \frac{1+\theta^N}{1+\theta^N+\mathcal{T}^N} < 1$.

We can also write

$$k_N = \frac{1}{1 + \frac{\mathcal{T}^N}{1+\theta^N}}.$$

Recall that by the assumptions of the theorem, $0 < \mathcal{T}^N$, $\mathcal{T}^N \le \mathcal{T}^{N+1}$ and $\theta^N \not\to \infty$. Hence we have $\frac{\mathcal{T}^N}{1+\theta^N} \not\to 0$. So $k_N \not\to 1$ and therefore

$$\prod_{N=0}^{\infty} k_N = 0.$$

Due to the above equality the limit $\lim_{N\to\infty} d^N(A)$ exists and, moreover,

$$\lim_{N\to\infty} d^N(A) = \lim_{N\to\infty} \prod_{N=0}^{\infty} k_N d^N(A) = 0.$$

Hence

$$\lim_{N\to\infty} \mu^N(A) = \frac{\tau(A)}{\mathcal{T}} = \tau_{\text{norm}}(A).$$

In a similar way one can show that $\lim_{N\to\infty} \nu^N(A) = \nu^\infty(A) = \frac{\tau(A)}{\mathcal{T}} = \tau_{\text{norm}}(A)$. □

Remark. We note that by (2.5) the limit fixed point is the same for all starting couples μ_0, ν_0, i.e., it does not depend an the initial states.

3. Examples

In this section we construct examples of conflict dynamical systems with internal attraction which illustrate the results of Theorem 2.1.

Recall that Ω is a metric space and $\mathcal{R}$ is a σ-algebra of Borel subsets of Ω, $\mathcal{M}^+(\Omega)$ is family of all σ-additive finite positive measures on Ω. A subset of positive probability measures on Ω is denoted by $\mathcal{M}_1^+(\Omega)$.

Here we construct several examples of conflict dynamical system with attraction $\{\Omega, \mathcal{M}_1^+(\Omega) \times \mathcal{M}_1^+(\Omega), \circledast\}$, where the transformation $\circledast$ is given by the equations (2.2)

$$\begin{aligned}\mu^{N+1}(A) &= \tfrac{1}{z^N}[\mu^N(A) + \theta^N \cdot \mu^N(A) + \tau^N(A)],\\ \nu^{N+1}(A) &= \tfrac{1}{z^N}[\nu^N(A) + \theta^N \cdot \nu^N(A) + \tau^N(A)],\end{aligned}$$

where $\theta^N = \theta(\mu^N, \nu^N)$ is an arbitrary positive form of μ^N, ν^N and τ^N will be defined in different ways.

Example 3.1. Consider a simplest case where the measures τ^N do not depend on N, i.e., $\tau^N = \tau$ in (2.2). This is a trivial case and obviously all trajectories of the conflict dynamical system $\{\Omega, \mathcal{M}_1^+(\Omega) \times \mathcal{M}_1^+(\Omega), \circledast\}$ converge to the same limiting state $\{\frac{\tau}{\tau(\Omega)}, \frac{\tau}{\tau(\Omega)}\}$.

Example 3.2. Let μ, ν be arbitrary probability measures from $\mathcal{M}_1^+(\Omega)$. Assume $\mu \neq \nu$. Consider a signed measure $\omega := \mu - \nu \in \mathcal{M}(\Omega)$. Here $\mathcal{M}(\Omega)$ denotes the family of all σ-additive finite signed measures on Ω. According to measure theory [3] each signed measure ω determines the Hahn decomposition of Ω onto two parts

$$\Omega = \Omega_+ \cup \Omega_-, \quad \Omega_+ \cap \Omega_- = \emptyset, \quad \Omega_+, \Omega_- \in \mathcal{R} \tag{3.1}$$

with the properties

$$\forall A_+ \subseteq \Omega_+, \quad \omega(A_+) \geq 0, \quad \forall A_- \subseteq \Omega_-, \quad \omega(A_-) \leq 0 \quad (A_+, A_- \in \mathcal{R}).$$

In other terms,

$$\mu(A_+) \geq \nu(A_+), \quad A_+ \subseteq \Omega_+, \quad \mu(A_-) \leq \nu(A_-), \quad A_- \subseteq \Omega_-.$$

See more details in [3].

Now we define the measures τ^N as follows:

$$\tau^N(A) := \mu^N(A_-) + \nu^N(A_+), \quad A \in \mathcal{R}, \quad A_- = A \cap \Omega_-, \quad A_+ = A \cap \Omega_+. \tag{3.2}$$

And

$$z^N = 1 + \theta^N + \mathcal{T}^N, \quad \mathcal{T}^N = \tau^N(\Omega).$$

In this example we assume that $0 \leq \theta^N \leq 1$. Then it is easy to see that $0 < z^N \leq 1$ and $0 \leq \mathcal{T}^N \leq 1$ for any $N = 0, 1, \ldots$

Let us show that the Hahn decomposition (3.1) of Ω does not depend on N, i.e., it is the same for any pair $\{\mu^N, \nu^N\}$, $N = 0, 1, \dots$ Indeed, let $A_+ \subseteq \Omega_+$ be fixed. Then $\omega(A_+) = \mu(A_+) - \nu(A_+) \geq 0$. And for $N = 1$ we have

$$\omega^1(A_+) = \mu^1(A_+) - \nu^1(A_+) = \frac{1}{z}(1+\theta)\omega(A_+) \geq 0,$$

since $z > 0$, $\theta > 0$, and $\omega(A_+) \geq 0$. In the same way we find that $\omega^N(A_+) \geq 0$ for any $N = 1, 2, \dots$ Similarly, one can check that $\omega^N(A_-) \leq 0$ for any subset $A_- \subset \Omega_-$. Thus,

$$\begin{aligned} &\mu^N(A_+) \geq \nu^N(A_+), \quad A_+ \subseteq \Omega_+, \\ &\mu^N(A_-) \leq \nu^N(A_-), \quad A_- \subseteq \Omega_-, \quad N = 0, 1, \dots \end{aligned} \tag{3.3}$$

Proposition 3.1. *The conflict dynamical system $\{\Omega, \mathcal{M}_1^+(\Omega) \times \mathcal{M}_1^+(\Omega), \circledast\}$ determined by the system of equations* (2.2) *with measures τ^N defined by* (3.2) *has two subsets of invariant states:*

(1) *The subset of couples $\{\mu, \nu\}$ such that μ, ν are mutually singular,*

$$\Gamma_\perp = \{\{\mu, \nu\} \in \mathcal{M}_1^+(\Omega) \times \mathcal{M}_1^+(\Omega) \ \mid \ \mu \perp \nu\}.$$

(2) *The subset of couples $\{\mu, \nu\}$ such that μ, ν are identical,*

$$\Gamma_= = \{\{\mu, \nu\} \in \mathcal{M}_1^+(\Omega) \times \mathcal{M}_1^+(\Omega) \ \mid \ \mu = \nu\}.$$

Proof. If the measures μ, ν are mutually singular, $\mu \perp \nu$, then due to the Hahn decomposition, $\mu(\Omega_+) = 1$, and $\mu(A_-) = 0$, $\forall A_- \subseteq \Omega_-$. And $\nu(\Omega_-) = 1$, $\nu(A_+) = 0$, $\forall A_+ \subseteq \Omega_+$. Therefore

$$\tau(A) = \mu(A_-) + \nu(A_+) = 0, \quad \forall A \in \mathcal{R}, \quad \text{and} \quad \mathcal{T}^0 \equiv \mathcal{T} = \tau(\Omega) = 0.$$

From equations (2.2) due to (3.3) we have $\mu^{N+1}(A) = \mu^N(A)$, $\nu^{N+1}(A) = \nu^N(A)$, $\forall A \in \mathcal{R}$, for any $N = 0, 1, \dots$ It proves the invariance property of the set $\Gamma_\perp$ with respect to mapping (2.2).

If the measures μ, ν are identical, $\mu = \nu$, then it is obvious that for any $A \in \mathcal{R}$, $\tau(A) = \mu(A) = \nu(A)$, $\mathcal{T} = \tau(\Omega) = 1$. And from equations (2.2) and (3.3) we have $\mu^{N+1}(A) = \mu^N(A)$, $\nu^{N+1}(A) = \nu^N(A)$ for any $N = 0, 1, \dots$ □

Now we can describe all limit states of this dynamical system.

Theorem 3.1. *Let a triple $\{\Omega, \mathcal{M}_1^+(\Omega) \times \mathcal{M}_1^+(\Omega), \circledast\}$ be a conflict dynamical system given by equations* (2.2) *with τ^N defined by* (3.2). *Then every trajectory* (2.1) *starting with a couple of mutually non-singular measures, $\{\mu, \nu\} \notin \Gamma_\perp$, converges to a fixed point $\{\mu^\infty, \nu^\infty\} \in \Gamma_=$.*

Moreover, the limiting measures μ^∞, ν^∞ admit a description in terms of the Hahn decomposition of the starting measures,

$$\mu^\infty(A) = \nu^\infty(A) = \frac{\mu(A_-) + \nu(A_+)}{\mu(\Omega_-) + \nu(\Omega_+)}.$$

Proof. Let us prove that the measures τ^N defined by (3.2) satisfy conditions (2.4), (2.3) of Theorem 2.1.

Let $A \in \mathcal{R}$. Then by (3.1) we can write $A = A_+ \cup A_-$, where $A_+ = A \cap \Omega_+$, $A_- = A \cap \Omega_-$. By definition,

$$\begin{aligned} \tau^{N+1}(A) &= \mu^{N+1}(A_-) + \nu^{N+1}(A_+) \\ &= \frac{1}{z^N}\left[(\mu^N(A_-) + \nu^N(A_+))(1+\theta^N) + \tau^N(A_-) + \tau^N(A_+)\right] \\ &= \frac{1}{z^N}\left[\tau^N(A)(1+\theta^N) + \tau^N(A)\right]. \end{aligned}$$

Thus

$$\tau^{N+1}(A) = \frac{2+\theta^N}{1+\theta^N+\mathcal{T}^N}\tau^N(A).$$

So, $\tau^{N+1}(A) > \tau^N(A)$ since, by construction, $0 < \mathcal{T}^N = \tau(\Omega) < 1$ for all N, and moreover, $\mathcal{T}^{N+1} > \mathcal{T}^N$ because

$$\mathcal{T}^{N+1} = \tau^{N+1}(\Omega) = \frac{2+\theta^N}{1+\theta^N+\mathcal{T}^N}\mathcal{T}^N,$$

which proves inequality (2.4). And we also get

$$\frac{\tau^{N+1}(A)}{\mathcal{T}^{N+1}} = \frac{\tau^N(A)}{\mathcal{T}^N}, \quad N = 0, 1, \ldots,$$

which proves the equality (2.3).

Thus conditions (2.3) and (2.4) in Theorem 2.1 hold. So by Theorem 2.1 for any couple of measures $\{\mu, \nu\} \notin \Gamma_\perp$ the trajectory (2.1) converges to a fixed point $\{\mu^\infty, \nu^\infty\}$,

$$\mu^\infty(A) = \nu^\infty(A) = \frac{\tau(A)}{\tau(\Omega)} = \frac{\mu(A_-)+\nu(A_+)}{\mu(\Omega_-)+\nu(\Omega_+)}.$$

Obviously the limit couple μ^∞, ν^∞ belongs to $\Gamma_=$. □

Example 3.3. Let us define the sequence of measures τ^N as follows:

$$\tau^N(A) = \alpha\mu^N(A) + \beta\nu^N(A), \quad A \in \mathcal{R}, \quad N = 0, 1, \ldots,$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$ and $\alpha^2 + \beta^2 \neq 0$.

Then

$$\mathcal{T}^N = \tau^N(\Omega) = \alpha\mu^N(\Omega) + \beta\nu^N(\Omega) = \alpha + \beta, \quad N = 0, 1, \ldots$$

Thus, $\mathcal{T}^N = \mathcal{T}$ which means that condition (2.4) in Theorem 2.1 holds.

Let us prove that the measures τ^N satisfy condition (2.3) too. In fact, we have to show that $\tau^N(A) = \tau(A)$, since $\mathcal{T}^N = \mathcal{T} = \alpha + \beta$. By definition we have

$$\begin{aligned}\tau^1(A) = \alpha\mu^1(A) + \beta\nu^1(A) &= \frac{1}{z}\left((\alpha\mu(A)+\beta\nu(A))(1+\theta) + (\alpha+\beta)\tau(A)\right)\\ &= \frac{1}{1+\theta+\mathcal{T}}\left(\tau(A)(1+\theta+\alpha+\beta)\right) = \tau(A).\end{aligned}$$

Therefore, by induction, $\tau(A) = \tau^N(A)$ for any $N = 0, 1, \ldots$ Now applying Theorem 2.1 we can state that for any couple of measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$ the trajectory (2.1) converges to the fixed point $\{\mu^\infty, \nu^\infty\}$,

$$\mu^\infty(A) = \nu^\infty(A) = \frac{\tau(A)}{\tau(\Omega)} = \frac{\alpha\mu(A)+\beta\nu(A)}{\alpha+\beta}, \quad A \in \mathcal{R}.$$

Example 3.4. Starting with a couple of measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$, fix two non-empty sets $\mathcal{M} \subset \operatorname{supp}\mu$, $\mathcal{N} \subset \operatorname{supp}\nu$ such that $\mathcal{M} \cap \mathcal{N} = \emptyset$. Further,

$$\tau^0(A) = \tau(A) = \mu(\mathcal{M} \cap A) + \nu(\mathcal{N} \cap A),$$

and using (2.2) define the sequence of measures $\tau^N, N \geq 1$ as follows:

$$\tau^N(A) = \mu^N(\mathcal{M} \cap A) + \nu^N(\mathcal{N} \cap A).$$

Let us show that this sequence satisfies conditions (2.3) and (2.4). Indeed, by definition,

$$\begin{aligned}\tau^1(A) &= \mu^1(\mathcal{M} \cap A) + \nu^1(\mathcal{N} \cap A)\\ &= \frac{1}{z}\left(\mu(\mathcal{M} \cap A)(1+\theta) + \tau(\mathcal{M} \cap A) + \nu(\mathcal{N} \cap A)(1+\theta) + \tau(\mathcal{N} \cap A)\right)\\ &= \frac{1}{z}\left((\mu(\mathcal{M} \cap A) + \nu(\mathcal{N} \cap A))(1+\theta) + \tau(\mathcal{M} \cap A) + \tau(\mathcal{N} \cap A)\right).\end{aligned}$$

However,

$$\tau(\mathcal{M} \cap A) = \mu(\mathcal{M} \cap (\mathcal{M} \cap A)) + \nu(\mathcal{N} \cap (\mathcal{M} \cap A)) = \mu(\mathcal{M} \cap A),$$

since $\mathcal{N} \cap \mathcal{M} = \emptyset$ and $\nu(\mathcal{N} \cap (\mathcal{M} \cap A)) = 0$. Similarly we get

$$\tau(\mathcal{N} \cap A) = \nu(\mathcal{N} \cap A),$$

since $\mu(\mathcal{M} \cap (\mathcal{N} \cap A)) = 0$. Hence,

$$\tau^1(A) = \frac{1}{z}(2+\theta)\tau(A) = \frac{2+\theta}{1+\theta+\mathcal{T}}\tau(A).$$

Because of

$$\mathcal{T}^1 = \tau^1(\Omega) = \frac{2+\theta}{1+\theta+\mathcal{T}}\mathcal{T},$$

we obtain the equality

$$\frac{\tau^1(A)}{\mathcal{T}^1} = \frac{\tau(A)}{\mathcal{T}}$$

and the inequality $\tau^1(\Omega) \equiv \mathcal{T}^1 \geq \tau(\Omega) \equiv \mathcal{T}$ too. By the induction conditions, (2.3) and (2.4) hold for all N. Due to Theorem 2.1 the sequence of measures μ^N, ν^N converges to a fixed point $\{\mu^\infty, \nu^\infty\}$,

$$\mu^\infty(A) = \nu^\infty(A) = \frac{\mu(\mathcal{M} \cap A) + \nu(\mathcal{N} \cap A)}{\mu(\mathcal{M} \cap \Omega) + \nu(\mathcal{N} \cap \Omega)}.$$

4. The case of exponential growth of θ^N

This section was inspired by the Myroslav Gorbachuk's particular interest to various problems that are concerned with exponential operator semigroups [11]. It is well known that the corresponding functionals after their extension to the complex plane have an exponential growth with respect to time. This property will be used here. Namely, in this section we will investigate the behavior of trajectories of the conflict dynamical system with attractive interaction in the situation where the values of the global evolution indicator θ^N have exponential growth as $N \to \infty$. As above, $\circledast$ denotes the conflict transformation. We are interesting in convergence of the trajectories

$$\{\mu^N, \nu^N\} \xrightarrow{\circledast} \{\mu^{N+1}, \nu^{N+1}\}, \quad N = 0, 1, \dots,$$

where the measures μ^{N+1}, ν^{N+1} are defined by the recurrent law as in (2.2),

$$\mu^{N+1} = \frac{1}{z^N}[\mu^N(1+\theta^N) + \tau^N], \quad \nu^{N+1} = \frac{1}{z^N}[\nu^N(1+\theta^N) + \tau^N],$$

starting with an arbitrary couple of different measures $\mu^0 = \mu, \nu^0 = \nu$ from $\mathcal{M}_1^+(\Omega)$.

We recall that $\theta^N = \theta(\mu^N, \nu^N)$ denotes the multiplicative indicator of global evolution. It shows the changes of some essential characteristic for agents of the complex system $\mathcal{A}$. Usually θ^N is presented by values of a non-negative real-valued functional which characterizes the power or the rate of growth of $\mu^N(A)$ or $\nu^N(A)$ at time N. There are various ways for definition of θ (see [1, 6]).

In this section we assume that values of the global evolution indicator approach infinity, $\theta^N \longrightarrow \infty$ as $N \longrightarrow \infty$.

Below we will assume that additive factors of the attractive interaction τ^N are presented by some sequence of positive measures depending on μ^N, ν^N in such a way that the conditions of Theorem 2.1 are satisfied.

We say that $\theta^N = \theta(\mu^N, \nu^N)$ has exponential growth with $N \to \infty$ if θ^N tends to infinity so quickly that $\frac{1+\theta^N}{1+\theta^N+\mathcal{T}^N} \to 1$ and the following series converges to some non-zero finite value:

$$c := \prod_{N=0}^{\infty} k_N > 0, \tag{4.1}$$

where

$$k_N := \frac{1+\theta^N}{1+\theta^N+\mathcal{T}^N} \equiv 1 - \frac{\mathcal{T}^N}{1+\theta^N+\mathcal{T}^N}.$$

Theorem 4.1. *Let the global evolution indicator $\theta^N = \theta(\mu^N, \nu^N)$ have an exponential growth and satisfy condition (4.1), and the sequence τ^N be defined by* (3.2). *Then, for each couple of measures μ, ν from $\mathcal{M}_1^+(\Omega)$, trajectory* (2.1) *of the conflict dynamical system* (2.2) *converges to a fixed point $\{\mu^\infty, \nu^\infty\}$,*

$$\mu^\infty(A) = \lim_{N\to\infty} \mu^N(A), \quad \nu^\infty(A) = \lim_{N\to\infty} \nu^N(A), \quad A \in \mathcal{R},$$

where

$$\begin{aligned} \mu^\infty(A) &= c\mu(A) + (1-c)\tfrac{\tau(A)}{\mathcal{T}}, \\ \nu^\infty(A) &= c\nu(A) + (1-c)\tfrac{\tau(A)}{\mathcal{T}}. \end{aligned} \tag{4.2}$$

Proof. Above it has been proved that τ^N satisfy condition (2.3) of Theorem 2.1, that is,

$$\frac{\tau^N(A)}{\tau^N(\Omega)} \equiv \frac{\tau^N(A)}{\mathcal{T}^N} = \frac{\tau(A)}{\mathcal{T}}, \quad A \in \mathcal{R}, \quad N = 0, 1, \ldots$$

Using this fact and denoting

$$d^0 := \mu(A) - \tau(A)/\mathcal{T}$$

due to (2.2) we have

$$d^1 = \mu^1(A) - \frac{\tau(A)}{\mathcal{T}} = \frac{1+\theta}{1+\theta+\mathcal{T}} \left(\mu(A) - \frac{\tau(A)}{\mathcal{T}} \right) = k_0 \cdot d^0,$$

where we recall that $k_0 = \frac{1+\theta}{1+\theta+\mathcal{T}}$. By induction one can write

$$d^{N+1} = \mu^{N+1}(A) - \frac{\tau(A)}{\mathcal{T}} = k_N \cdots k_0 \cdot d^0, \quad k_N = \frac{1+\theta^N}{1+\theta^N+\mathcal{T}^N}.$$

Passing to infinity we get

$$d^\infty = \lim_{N\to\infty} d^N = \mu^\infty(A) - \frac{\tau(A)}{\mathcal{T}} = \prod_{N=0}^{\infty} k_N \cdot d^0.$$

Thus, due to (4.1),

$$\mu^\infty(A) - \frac{\tau(A)}{\mathcal{T}} = c(\mu(A) - \frac{\tau(A)}{\mathcal{T}}),$$

and, therefore,

$$\mu^\infty(A) = c\mu(A) + (1-c)\frac{\tau(A)}{\mathcal{T}},$$

which proves the first equation in (4.2). In the similar way one can prove the second equation, i.e., that

$$\nu^\infty(A) = c\nu(A) + (1-c)\frac{\tau(A)}{\mathcal{T}}.$$

$\square$

We remark that now the limit measures μ^∞, ν^∞ are distinct. But if the global evolution indicator θ^N does not have the exponential growth with $N \to \infty$ then $\prod_{N=0}^{\infty} k_N = c = 0$ and also $\lim_{N\to\infty} d^N = 0$. In such a case, $\mu^\infty(A) - \frac{\tau(A)}{\mathcal{T}} = 0$ and $\mu^\infty(A) = \frac{\tau(A)}{\mathcal{T}} = \nu^\infty(A)$. This result is the same as in Theorem 2.1.

5. Conflict dynamical systems in terms of densities

Put $\Omega = [0,1]$ and let $\mathcal{R}$ be the Borel σ-algebra of subsets of $[0,1]$. Consider a couple of probability measures $\mu, \nu \in \mathcal{M}_1^+([0,1])$, and assume that μ, ν are absolutely continuous with respect to the Lebesgue measure λ. Let $\rho(x), \sigma(x) \geq 0$ be densities defined as Radon-Nikodym derivatives of μ, ν with respect to λ. Thus we have

$$\mu(A) = \int_A \rho(x)\, d\lambda(x), \quad \nu(A) = \int_A \sigma(x)\, d\lambda(x), \quad A \in \mathcal{R}.$$

Besides we will assume that the densities $\rho(x), \sigma(x)$ are continuous, $\rho, \sigma \in C([0,1])$. Under these assumptions we can construct the conflict dynamical system with attractive interaction directly in terms of point-wise evolutions of the densities

$$\begin{cases} \rho^{N+1}(x) = \frac{1}{z^N}[\rho^N(x)(1+\theta^N) + \tau^N(x)], \\ \sigma^{N+1}(x) = \frac{1}{z^N}[\sigma^N(x)(1+\theta^N) + \tau^N(x)], \quad x \in [0,1], \quad N = 0, 1, \dots, \end{cases} \tag{5.1}$$

where $\rho^0 = \rho$, $\sigma^0 = \sigma$. Here θ^N is a bounded sequence of real-valued functionals, $\tau^N(x)$ are positive continuous functions, $\tau^N \in C([0,1])$, and

$$z^N = 1 + \theta^N + \mathcal{T}^N,$$

where

$$\mathcal{T}^N = \int_{[0,1]} \tau^N(x)\, d\lambda(x).$$

We denote the conflict dynamical system with attraction in terms of densities by $\{\Omega, C_1^+([0,1]) \times C_1^+([0,1]), \circledast\}$, where $C_1^+([0,1]) \times C_1^+([0,1])$ stands for the set of couples of positive continuous functions $\sigma(x), \rho(x)$ which are normed by 1,

$$\int_{[0,1]} \rho(x)\, d\lambda(x) = \int_{[0,1]} \sigma(x)\, d\lambda(x) = 1.$$

The system of difference equations (5.1) defines the conflict transformation $\circledast$ which generates the trajectories

$$\{\rho^N, \sigma^N\} \xrightarrow{\circledast} \{\rho^{N+1}, \sigma^{N+1}\}, \quad N = 0, 1, \dots, \tag{5.2}$$

where all pairs of functions belong to $C_1^+([0,1]) \times C_1^+([0,1])$.

Theorem 5.1. *Let the conflict dynamical system $\{\Omega, C_1^+([0,1]) \times C_1^+([0,1]), \circledast\}$ be defined by system (5.1), where θ^N is a bounded sequence of real-valued positive functionals on $C_1^+([0,1])$ and τ^N is a sequence of positive continuous functions on $[0,1]$ that satisfy the following conditions ($\tau^0 \equiv \tau$, $\mathcal{T}^0 \equiv \mathcal{T}$):*

$$\frac{\tau^N(x)}{\mathcal{T}^N} = \frac{\tau(x)}{\mathcal{T}}, \quad N = 0, 1, \dots, \tag{5.3}$$

$$\mathcal{T}^N \leq \mathcal{T}^{N+1}, \quad N = 0, 1, \dots \tag{5.4}$$

Then each trajectory (5.2) converges to a fixed point $\{\rho^\infty, \sigma^\infty\}$,

$$\rho^\infty(x) = \lim_{N\to\infty} \rho^N(x), \quad \sigma^\infty(x) = \lim_{N\to\infty} \sigma^N(x),$$

such that

$$\rho^{\infty}(x) = \sigma^{\infty}(x) = \frac{\tau(x)}{\mathcal{T}}, \quad x \in [0,1]. \tag{5.5}$$

Proof. Denote $d(x) := \rho(x) - \frac{\tau(x)}{\mathcal{T}}$. It follows from equations (5.1) and assumption (5.3) that

$$d^1(x) := \rho^1(x) - \frac{\tau(x)}{\mathcal{T}} = \frac{1+\theta}{1+\theta+\mathcal{T}}\left(\rho(x) - \frac{\tau(x)}{\mathcal{T}}\right),$$

where $\frac{1+\theta}{1+\theta+\mathcal{T}} < 1$, since $\tau(x) \geq 0$ and $\mathcal{T} > 0$. By induction,

$$d^{N+1}(x) := \rho^{N+1}(x) - \frac{\tau(x)}{\mathcal{T}} = \frac{1+\theta^N}{1+\theta^N+\mathcal{T}^N}\left(\rho^N(x) - \frac{\tau(x)}{\mathcal{T}}\right).$$

Denoting

$$k_N = \frac{1+\theta^N}{1+\theta^N+\mathcal{T}^N},$$

we can write

$$d^{N+1}(x) = k_N \cdots k_1 k_0 d(x),$$

where it is obvious that $k_N < 1$. Rewriting k_N in the form

$$k_N = \frac{1}{1+\frac{\mathcal{T}^N}{1+\theta^N}}$$

we observe that $\prod_{N=0}^{\infty} k_N = 0$, since θ^N is bounded, $\mathcal{T}^N$ is non-decreasing, $\frac{\mathcal{T}^N}{1+\theta^N} \not\to 0$, and $k_N \not\to 1$.

This proves existence of the zero limits $\lim\limits_{N\to\infty} d^N(x)$ for all points $x \in [0,1]$

$$\lim_{N\to\infty} d^N(x) = \lim_{N\to\infty} k_N \cdot \ldots \cdot k_1 \cdot k_0 d(x) = \prod_{N=0}^{\infty} k_N d(x) = 0.$$

Therefore,

$$\lim_{N\to\infty} \rho^N(x) = \frac{\tau(x)}{\mathcal{T}}.$$

In a similar way we prove that

$$\lim_{N\to\infty} \sigma^N(x) = \frac{\tau(x)}{\mathcal{T}}.$$

□

The next example illustrates Theorem 5.1.

Example 5.1. Let $\mu, \nu \in \mathcal{M}_1^+([0,1])$ and assume that μ, ν are absolutely continuous with respect to the Lebesgue measure λ. Let $\rho(x), \sigma(x) \geq 0$ be densities defined as the Radon-Nikodym derivatives of μ, ν with respect to λ.

In terms of given densities, the Hahn decomposition corresponding to the charge $\omega = \mu - \nu$ may be defined as follows:

$$[0,1] = [0,1]_+ \cup [0,1]_-,$$

where

$$[0,1]_+ = \{x \in [0,1] | \rho(x) > \sigma(x)\}, \quad [0,1]_- = \{x \in [0,1] | \rho(x) \leq \sigma(x)\}.$$

Now we can define a more specific conflict dynamical system $\{[0,1], C_1^+([0,1]) \times C_1^+([0,1]), \divideontimes\}$, with the transformation $\divideontimes$ given by equations (5.1), where

$$\theta^N = \int_{[0,1]} \sqrt{\rho^N(x)\sigma^N(x)}\, d\lambda(x),$$

$$\tau^N(x) = min\left\{\sigma^N(x), \rho^N(x)\right\} = \begin{cases} \sigma^N(x), \ x \in [0,1]_+, \\ \rho^N(x), \ x \in [0,1]_-, \end{cases} \tag{5.6}$$

and

$$z^N = 1 + \theta^N + \mathcal{T}^N, \quad \mathcal{T}^N = \int_{[0,1]} \tau^N(x)\, d\lambda(x). \tag{5.7}$$

We observe that the Hahn decompositions corresponding to $\omega^N = \mu^N - \nu^N$ does not depend on N,

$$[0,1]^N_+ = \{x \in [0,1] | \rho^N(x) > \sigma^N(x)\} = [0,1]_+,$$

$$[0,1]^N_- = \{x \in [0,1] | \rho^N(x) \le \sigma^N(x)\} = [0,1]_-, \quad N = 0, 1, \dots$$

Indeed, by (5.1)

$$\rho^1(x) - \sigma^1(x) = (\rho(x) - \sigma(x))(\theta + 1)\frac{1}{z}, \quad x \in [0,1].$$

Therefore, if $\rho(x) > \sigma(x)$ then $\rho^1(x) > \sigma^1(x)$ and, by induction, $\rho^N(x) > \sigma^N(x)$ for any $N = 1, 2, \dots$ It proves that $[0,1]^N_+ = [0,1]_+$. In a similar way we get $[0,1]^N_- = [0,1]_-$. Now we prove that condition (5.3) of Theorem 5.1 holds for $\tau^N(x)$ given by (5.6). Indeed using (5.6) we get

$$\tau^1(x) = \begin{cases} \frac{1}{z}(\sigma(x)(\theta+1) + \tau(x)), \ x \in [0,1]_+, \\ \frac{1}{z}(\rho(x)(\theta+1) + \tau(x)), \ x \in [0,1]_-. \end{cases}$$

Hence

$$\tau^1(x) = \frac{1}{z}(\tau(x)(\theta+1) + \tau(x)) = \frac{2+\theta}{z}\tau(x).$$

Then, by (5.7),

$$\mathcal{T}^1 = \int_{[0,1]} \tau^1(x) d\lambda(x) = \int_{[0,1]} \frac{2+\theta}{z}\tau(x) d\lambda(x) = \frac{2+\theta}{z} \cdot \mathcal{T}.$$

Therefore,

$$\frac{\tau^1(x)}{\mathcal{T}^1} = \frac{\frac{2+\theta}{z}\tau(x)}{\frac{2+\theta}{z} \cdot \mathcal{T}} = \frac{\tau(x)}{\mathcal{T}}.$$

By induction,

$$\mathcal{T}^{N+1} = \frac{2+\theta}{1+\theta+\mathcal{T}^N} \cdot \mathcal{T}^N$$

and

$$\frac{\tau^{N+1}(x)}{\mathcal{T}^{N+1}} = \frac{\tau^N(x)}{\mathcal{T}^N} = \frac{\tau(x)}{\mathcal{T}},$$

which proves (5.3). And since $\mathcal{T}^N \le 1$, we have $\mathcal{T}^N \le \mathcal{T}^{N+1}$ which proves (5.4).

So, by Theorem 5.1, $\rho^\infty(x) = \frac{\tau(x)}{\mathcal{T}}$, $\sigma^\infty(x) = \frac{\tau(x)}{\mathcal{T}} = \min\{\rho(x), \sigma(x)\}/\mathcal{T}$.

Finally we illustrate the results of Theorem 5.1 by computer simulation. We present two particular examples. Below, with thick lines Figures 1 and 2 exhibit the limit densities $\rho^\infty(x)$, $\sigma^\infty(x)$ for trajectories of conflict dynamical system with two sets of different starting densities $\rho(x)$, $\sigma(x)$:

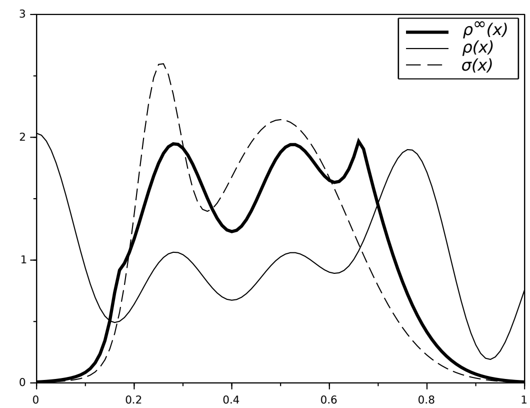

FIGURE 1. $\rho(x) = \frac{1}{2.95}(3 + \cos 9x + \cos 17x + \cos 24x)$, $\sigma(x) = \frac{1}{1.95}(2 + \cos 10x)$.

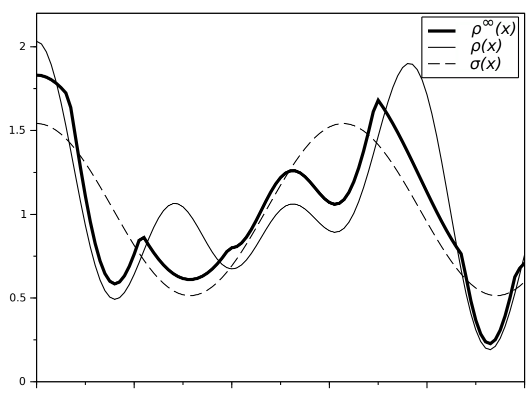

FIGURE 2. $\rho(x) = \frac{1}{2.95}(3 + \cos 9x + \cos 17x + \cos 24x)$, $\sigma(x) = \frac{1}{0.466}(e^{-\frac{(x-0.5)^2}{0.04}} + e^{-\frac{(x-0.25)^2}{0.004}})$.

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