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ON CERTAIN SPECTRAL FEATURES INHERENT TO SCALAR TYPE SPECTRAL OPERATORS

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Abstract. Important spectral features such as the emptiness of the residual spectrum, countability of the point spectrum, provided the space is separable, and a characterization of spectral gap at 0, known to hold for bounded scalar type spectral operators, are shown to naturally transfer to the unbounded case.

1. Introduction

As is known [4, Theorem 8] (see also [5, 8]), a *bounded* linear operator T on a complex Banach space $(X, \|\cdot\|)$ is *spectral* iff it allows a unique *canonical decomposition*,

$$T = S + N,$$

where S is a *scalar type spectral operator* and N is a *quasinilpotent operator* commuting with S. The operators T and S have the same *spectrum* and *spectral measure* $E(\cdot)$, with

$$S = \int_{\sigma(T)} \lambda \, dE(\lambda), \tag{1.1}$$

where $\sigma(\cdot)$ is the *spectrum* of an operator, and are called the *scalar* and *radical parts* of T, respectively.

The operator N being *nilpotent*, T is called of *finite type* (cf. [4, 10]), in which case, in particular, for a bounded scalar type spectral operator ($T = S$, $N = 0$), the *residual spectrum* is *empty* [10, Theorem 4.1] and, provided the space X is *separable*, the *point spectrum* is *countable* [10, Theorem 4.4].

Furthermore, [10, Theorem 3.4] describing the closedness of the range of a bounded spectral operator T on a complex Banach space X, when applied to a bounded scalar type spectral operator ($T = S$, $N = 0$), turns into a characterization of *spectral gap* at 0 acquiring the following form:

Theorem 1.1. ([10, Theorem 3.4], the scalar type case). *For a bounded scalar type spectral operator A on a complex Banach space $(X, \|\cdot\|)$ with $0 \in \sigma(A)$, 0 is an isolated point of the spectrum $\sigma(A)$ iff the range of A is closed.*

The case of an *unbounded* spectral operator T in a complex Banach space $(X, \|\cdot\|)$ appears to be essentially more formidable. Thus, $E(\cdot)$ being the spectral measure of T, the scalar part S of T defined by (1.1) is an unbounded scalar type spectral operator and the radical part $N := T - S$ need not be bounded, let alone quasinilpotent [1, 8].

A natural question is whether the discussed spectral features pass to unbounded spectral operators at least when they are of scalar type, which would include the important class *normal operators*. In this note, we are to show that the emptiness of the residual spectrum, the countability of the point spectrum, provided the space is separable, as well

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as the characterization of spectral gap at 0 are inherent to scalar type spectral operators, bounded or not.

2. Preliminaries

Recall that, the spectrum $\sigma(A)$ of a *closed linear operator* A in a complex Banach space $(X, \|\cdot\|)$ is partitioned into disjoint components, $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$, called the *point*, *continuous*, and *residual spectrum* of A, respectively, as follows:

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \,|\, A - \lambda I \text{ is } \textit{not one-to-one}, \text{ i.e., } \lambda \text{ is an } \textit{eigenvalue} \text{ of } A\},$$

$$\sigma_c(A) = \left\{\lambda \in \mathbb{C} \,|\, A - \lambda I \text{ is } \textit{one-to-one} \text{ and } R(A - \lambda I) \neq X, \text{ but } \overline{R(A - \lambda I)} = X\right\},$$

$$\sigma_r(A) = \left\{\lambda \in \mathbb{C} \,|\, A - \lambda I \text{ is } \textit{one-to-one} \text{ and } \overline{R(A - \lambda I)} \neq X\right\},$$

where I stands for the *identity* operator on X, $R(\cdot)$ is the *range* of an operator, and $\bar{\cdot}$ is the *closure* of a set in $\mathbb{C}$ (see, e.g., [10]).

The properties of spectral operators, spectral measures, and the Borel operational calculus underlying the subsequent discourse are exhaustively delineated in [5, 8]. Here, for the reader's convenience, we give an outline of some particularly important facts.

Recall that a *spectral operator* is a densely defined closed linear operator A in a complex Banach space $(X, \|\cdot\|)$ with an associated *spectral measure* (*resolution of the identity*) $E_A(\cdot)$, i.e., a *strongly σ-additive* operator function, which assigns to each set δ from the σ-algebra $\mathcal{B}$ of *Borel sets* in $\mathbb{C}$ a *projection operator* $E_A(\delta) = E_A^2(\delta)$ on X and has the following properties:

$$E_A(\emptyset) = 0, \quad E_A(\mathbb{C}) = I, \quad E_A(\delta \cap \sigma) = E_A(\delta)E_A(\sigma) = E_A(\sigma)E_A(\delta), \quad \delta, \sigma \in \mathcal{B},$$

where 0 stands for the *zero* operator on X, and

$$\begin{aligned}
&E_A(\delta)X \subseteq D(A), \text{ for each } \textit{bounded} \quad \delta \in \mathcal{B},\\
&E_A(\delta)D(A) \subseteq D(A), \quad AE_A(\delta)f = E_A(\delta)Af, \quad \delta \in \mathcal{B}, \quad f \in D(A),\\
&\sigma(A|E_A(\delta)X) \subseteq \bar{\delta}, \quad \delta \in \mathcal{B},
\end{aligned}$$

where $D(\cdot)$ is the *domain* of an operator and $\cdot|\cdot$ is the *restriction* of an operator (left) to a subspace (right).

Due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [6, 8], i.e.,

$$\exists\, M > 0 \ \ \forall\, \delta \in \mathcal{B}: \ \|E_A(\delta)\| \le M.$$

The notation $\|\cdot\|$ has been recycled here to designate the norm in the space $\mathcal{L}(X)$ of all bounded linear operators on X, such an economy of symbols being rather conventional.

A spectral operator A in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ is said to be of *scalar type* if

$$A = \int\limits_{\mathbb{C}} \lambda \, dE_A(\lambda),$$

which is embedded into the structure of the *Borel operational calculus* associated with such operators [5, 8] and assigning to any Borel measurable function $F : \mathbb{C} \to \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int\limits_{\mathbb{C}} F(\lambda) \, dE_A(\lambda)$$

defined as follows:

$$\begin{aligned}
F(A)f &:= \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)),\\
D(F(A)) &:= \left\{f \in X \,|\, \lim_{n\to\infty} F_n(A)f \ \text{ exists}\right\},
\end{aligned}$$

where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\mathbb{C}\,|\,|F(\lambda)|\le n\}}(\cdot), \quad n \in \mathbb{N},$$

($\chi_\delta(\cdot)$ is the *characteristic function* of a set $\delta \subseteq \mathbb{C}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ is the set of *natural numbers*) and

$$F_n(A) := \int\limits_{\mathbb{C}} F_n(\lambda)\, dE_A(\lambda), \quad n \in \mathbb{N},$$

are *bounded* scalar type spectral operators on X defined in the same manner as for a *normal operator* (see, e.g., [7, 18]).

The spectrum $\sigma(A)$ of a scalar type spectral operator A being the *support* of its spectral measure $E_A(\cdot)$, $\mathbb{C}$ can be replaced with $\sigma(A)$ in the above definitions whenever appropriate [5, 8].

In a complex Hilbert space, the scalar type spectral operators are precisely those similar to the *normal* ones [19].

3. Spectral features inherent to scalar type spectral operators

In [17], the following generalization of the well-known orthogonal decomposition for a normal operator in a complex Hilbert space (see, e.g., [7, 18]) is found:

Theorem 3.1. ([17, Theorem]). *For a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$, the direct sum decomposition*

$$X = \ker A \oplus \overline{R(A)} \tag{3.2}$$

($\ker\cdot$ *is the kernel of an operator) holds with*

$$\ker A = E_A(\{0\})X \quad \text{and} \quad \overline{R(A)} = E_A(\sigma(A)\setminus\{0\})X.$$

Decomposition (3.2) has the following immediate implication generalizing the well-known fact for *normal operators* (see, e.g., [7, 18]).

Corollary 3.1. (Emptiness of Residual Spectrum). *For a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$, $\sigma_r(A) = \emptyset$.*

Proof. Whenever, for $\lambda \in \mathbb{C}$, the scalar type spectral operator $A - \lambda I$ is *one-to-one*, $\ker(A - \lambda I) = \{0\}$, and hence, by (3.2), $\overline{R(A - \lambda I)} = X$, which implies that $\sigma_r(A) = \emptyset$. □

Example 3.1. In l_2, the unbounded linear operator

$$A(x_1, x_2, \dots) = (0, x_1, 2x_2, \dots, nx_{n+1}, \dots)$$

with the domain $D(A) = \{(x_1, x_2, \dots) \in l_2 \,|\, (0, x_1, 2x_2, \dots, nx_{n+1}, \dots) \in l_2\}$ is densely defined and closed, but, by Corollary 3.1, is *not spectral of scalar type* since $0 \in \sigma_r(A)$.

In respect that $\sigma_r(A) = \emptyset$, the proof of [10, Theorem 4.4] can be used verbatim to prove the following

Proposition 3.1. (Countability of Point Spectrum). *For a scalar type spectral operator A in a complex separable Banach space $(X, \|\cdot\|)$, $\sigma_p(A)$ is a countable set.*

Example 3.2. In the separable Banach space $C([a, b], \mathbb{C})$ $(-\infty < a < b < \infty)$ with the *maximum norm*, the differentiation operator

$$C^1[a, b] \ni x \mapsto [Ax](t) = x'(t), \quad a \le t \le b,$$

is densely defined, linear, and closed, but, by Proposition 3.1, *not spectral of scalar type* since $\sigma_p(A) = \mathbb{C}$.

Now, let us stretch Theorem 1.1 to the unbounded case.

Theorem 3.2. (Characterization of Spectral Gap at 0). *For a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $0 \in \sigma(A)$, 0 is an isolated point of the spectrum $\sigma(A)$ iff the range] $R(A)$ of A is closed, i.e., $\overline{R(A)} = R(A)$.*

Proof.

"Only if" part. Suppose that 0 is an *isolated point* of $\sigma(A)$.

Considering that

$$\sigma(A) \setminus \{0\} = \sigma(A) \setminus \{\lambda \in \mathbb{C} \,|\, |\lambda| < \gamma\}$$

with some $\gamma > 0$, to the *bounded* Borel measurable function

$$F(\lambda) := \begin{cases} 0 & \text{for } \lambda \in \mathbb{C} \text{ with } |\lambda| < \gamma, \\ \dfrac{1}{\lambda} & \text{for } \lambda \in \mathbb{C} \text{ with } |\lambda| \geq \gamma \end{cases}$$

by the properties of the *operational calculus* ([8, Theorem XVIII.2.11]), there corresponds a *bounded* scalar type spectral operator

$$F(A) = \int_{\mathbb{C}} F(\lambda)\, dE_A(\lambda)$$

and, for each $f \in X$,

$$\begin{aligned} E_A(\sigma(A) \setminus \{0\})f &= E_A\left(\sigma(A) \setminus \{\lambda \in \mathbb{C} \,|\, |\lambda| < \gamma\}\right) f \\ &= \int_{\{\lambda \in \mathbb{C} \,|\, |\lambda| \geq \gamma\}} 1\, dE_A(\lambda) f = \int_{\mathbb{C}} \lambda F(\lambda)\, dE_A(\lambda) f = AF(A)f \in R(A). \end{aligned}$$

Since, by Theorem 3.1, $E_A(\sigma(A) \setminus \{0\})$ is the projection onto $\overline{R(A)}$ along $\ker A$ [17], we infer that $\overline{R(A)} = R(A)$.

"If" part. Suppose that $\overline{R(A)} = R(A)$, which, considering $\sigma_r(A) = \emptyset$, implies that $0 \in \sigma_p(A)$, i.e., $\ker A \neq \{0\}$.

Then, by Theorem 3.1, the direct sum decomposition

$$X = \ker A \oplus R(A), \tag{3.3}$$

where $\ker A = E_A(\{0\})X$ and $R(A) = E_A(\sigma(A) \setminus \{0\})X$, holds, and hence, A can be treated as the matrix operator

$$\begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix},$$

in $\ker A \oplus R(A)$, where $A_1 : D(A) \cap R(A) \to R(A)$ is the restriction of A to $R(A)$. Such a consideration makes apparent the fact that

$$\sigma(A) = \{0\} \cup \sigma(A_1).$$

Since $\ker A \cap R(A) = \{0\}$, the *closed linear operator* $A_1 : D(A) \cap R(A) \to R(A)$ is *bijective* and has an *inverse* defined on $R(A)$, which, in respect that $(R(A), \|\cdot\|)$ is a Banach space, by the *Closed Graph Theorem* (see, e.g., [6]), is *bounded*.

Hence, 0 is a *regular point* of A_1. Considering the fact that the *resolvent set* of a closed operator is *open* in $\mathbb{C}$ (see, e.g., [6]), we infer that, there is a neighborhood of 0 not containing points of $\sigma(A_1)$, i.e., other points of $\sigma(A)$, which makes 0 to be an *isolated point* of $\sigma(A)$. □

Remark 3.1. Observe that, the fact that λ_0 is an *isolated point* of the spectrum $\sigma(A)$ of a scalar type spectral operator A, necessarily implies that $\lambda_0 \in \sigma_p(A)$. Indeed, the spectrum being the *support* of the operator's spectral measure $E_A(\cdot)$, we immediately infer that

$$E_A(\{\lambda_0\}) \neq 0,$$

which makes λ_0 to be an *eigenvalue* of A with the *eigenspace* $E_A(\{\lambda_0\})X$ [5, 8]. The converse, however, is not true.

Example 3.3. In l_2, for the *self-adjoint* operator

$$l_2 \ni (x_1, x_2, \dots) \mapsto A(x_1, x_2, \dots) = (0, x_2, x_3/2, x_4/3, \dots) \in l_2,$$

the eigenvalue 0 is not an isolated point of $\sigma(A) = \sigma_p(A) = \{0, 1, 1/2, 1/3, \dots\}$.

Corollary 3.2. *If, for a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$, 0 is a regular point or an isolated point of the spectrum $\sigma(A)$, direct sum decomposition (3.3) holds and the operator $A + E_A(\{0\})$ has a bounded inverse defined on X, i.e., $0 \in \rho\,(A + E_A(\{0\}))$ ($\rho(\cdot)$ is the resolvent set of an operator).*

Proof. The validity of decomposition (3.3) follows immediately from Theorem 3.2.

Since, by Theorem 3.1, the projection $E_A(\{0\})$ is onto $\ker A$ along $R(A)$, the rest follows from a more general statement concerning the existence of a bounded inverse defined on X of $A + P$ with a closed linear operator A, for which decomposition (3.3) holds, and P is the projection onto $\ker A$ along $R(A)$ ([13, 12, 3], cf. also [15, 16, 14]). Such operators are naturally called *reducibly invertible*. □

Thus, a scalar type spectral operator A, for which 0 is a regular point or an isolated point of spectrum, is *reducibly invertible*.

4. Final remarks

As Examples 3.1 and 3.2 demonstrate, Corollary 3.1 and Proposition 3.1 are ready tests for disqualifying an operator from being scalar type spectral.

Theorem 3.2 relates a peculiar topological property of the spectrum of a scalar type spectral operator to a rather natural topological property of its range.

Observe also that decompositions (3.2) and (3.3) are essential in the context of the asymptotic behavior of *weak/mild solutions* of the associated abstract evolution equation

$$y'(t) = Ay(t), \quad t \geq 0,$$

[2, 9, 11, 15, 16, 14].

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