Evolution of states and mesoscopic scaling for two-component birth-and-death dynamics in continuum

Martin Friesen, Oleksandr Kutoviy

Link to this article: http://mfat.imath.kiev.ua/article/?id=914

How to cite this article:

Martin Friesen and Oleksandr Kutoviy, *Evolution of states and mesoscopic scaling for two-component birth-and-death dynamics in continuum*, Methods Funct. Anal. Topology **22** (2016), no. 4, 346-374.

EVOLUTION OF STATES AND MESOSCOPIC SCALING FOR TWO-COMPONENT BIRTH-AND-DEATH DYNAMICS IN CONTINUUM

MARTIN FRIESEN AND OLEKSANDR KUTOVIY

ABSTRACT. Two coupled spatial birth-and-death Markov evolutions on $\mathbb{R}^d$ are obtained as unique weak solutions to the associated Fokker-Planck equations. Such solutions are constructed by its associated sequence of correlation functions satisfying the so-called Ruelle-bound. Using the general scheme of Vlasov scaling we are able to derive a system of non-linear, non-local mesoscopic equations describing the effective density of the particle system. The results are applied to several models of ecology and biology.

1. INTRODUCTION

In the last years we observe an increasing interest to interacting particle systems in the continuum in regard to particular models of ecology, biology and social sciences, cf. [4, 5, 6, 9, 33]. An important part of the general theory of interacting particle systems in the continuum is related to the construction of the associated dynamics and the study of their scalings. In particular, from the point of view of applications it is worthy to investigate the corresponding mesoscopic equations. Within this framework, each particle is described by its position $x \in \mathbb{R}^d$ and the collection of all particles by the configuration γ. Assuming that each two particles cannot occupy the same position and all particles are indistinguishable, we will choose the configuration space Γ to be the collection of all admissible particle configurations γ, i.e.

$$\Gamma = \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \text{ for all compacts } K \subset \mathbb{R}^d\}.$$

Here and subsequently $|A|$ denotes the number of elements in the set $A \subset \mathbb{R}^d$. It is assumed that the microscopic dynamics consists of two elementary events. Namely, the death of a particle ($\gamma \longmapsto \gamma \backslash \{x\}$) and the birth of a particle ($\gamma \longmapsto \gamma \cup \{x\}$). A detailed analysis of such birth-and-death dynamics can be found in [11], see also the references therein. However, many particular models from ecology and biology require that we have at least two different types of particles.

The aim of this work is to extend the known results for two-component interacting particle systems. Since two particles of different types cannot occupy the same position, a proper state space for the dynamics is naturally given by the configuration space

$$\Gamma^2 := \{(\gamma^+, \gamma^-) \in \Gamma \times \Gamma \mid \gamma^+ \cap \gamma^- = \emptyset\},$$

cf. [17]. The stochastic time-evolution is incorporated in the particular form of the associated Markov (pre-)generator L. The fact that particles can only die or create new particles leads to the special form of the corresponding generator $L = L^+ + L^-$, where

---

both operators $L^{\pm}$ are given on polynomially bounded cylinder functions F on Γ^2 by

$$(1.1) \qquad (L^-F)(\gamma) = \sum_{x \in \gamma^-} d^-(x, \gamma^+, \gamma^- \setminus \{x\})(F(\gamma^+, \gamma^- \setminus \{x\}) - F(\gamma))$$

$$+ \int_{\mathbb{R}^d} b^-(x, \gamma)(F(\gamma^+, \gamma^- \cup \{x\}) - F(\gamma)) \, dx$$

and

$$(1.2) \qquad (L^+F)(\gamma) = \sum_{x \in \gamma^+} d^+(x, \gamma^+ \setminus \{x\}, \gamma^-)(F(\gamma^+ \setminus \{x\}, \gamma^-) - F(\gamma))$$

$$+ \int_{\mathbb{R}^d} b^+(x, \gamma)(F(\gamma^+ \cup \{x\}, \gamma^-) - F(\gamma)) \, dx.$$

For simplicity of notation, we write $\gamma = (\gamma^+, \gamma^-)$, $\gamma^{\pm} \setminus x$ and $\gamma^{\pm} \cup x$ instead of $\gamma^{\pm} \cup \{x\}$ and $\gamma^{\pm} \setminus \{x\}$, respectively. The birth-and-death Markov process associated to the operator L, provided it exists, consists of two elementary events. The death intensities $d^{\pm}(x, \gamma) \geq 0$ determine the probability that the particle $x \in \gamma^{\pm}$ disappears from the configuration $\gamma^{\pm}$. The birth intensities $b^{\pm}(x, \gamma) \geq 0$ determine the probability for a new particle to appear at $x \in \mathbb{R}^d$. Multi-species birth-and-death dynamics in the continuum are especially important in ecological, biological and physical applications. For example, the famous Widom-Rowlinson approach to the phase transition in the continuum is based on a two-component classical gas model. This approach was extended to more general Potts-type systems (see [22]). Recently, the dynamical versions of these models were studied in [13]. The corresponding Markov statistical dynamics was constructed with the help of Glauber dynamics on Γ. The dynamical phase transition was shown there in the mesoscopic limit of this dynamics. In contrast to that work we are able to construct the corresponding dynamics for all $t \geq 0$. Different kinds of other models, additional comments and explanations for the modeling of tumor growth can be found in [9] and the references therein.

The Markov process associated with the (pre-)generator L may be obtained as the unique solution to certain stochastic differential equations, cf. [21]. Unfortunately the conditions given in the latter paper are too restrictive for some applications. A different, functional analytic, approach to the construction of the processes is related to the construction of solutions to the (backward) Kolmogorov equation on functions $F : \Gamma^2 \longrightarrow \mathbb{R}$

$$(1.3) \qquad \frac{\partial F_t}{\partial t} = LF_t, \quad F_t|_{t=0} = F_0, \quad t \geq 0.$$

Until now, however, there exist up to our knowledge no techniques which can be used to solve (1.3) in any space of continuous functions on Γ^2. The construction of an associated Markov process with general birth-and-death rates is, therefore, still unsolved. This difficulty comes from the necessity to control the number of particles in any bounded domain in $\mathbb{R}^d$. One therefore tries to construct an associated evolution of probability measures on Γ^2, that is the one-dimensional distributions of the Markov process. The evolution of one-dimensional distributions is an essential problem on its own right.

A probability measure μ on Γ^2 describes in this setting the distribution of particles located in $\mathbb{R}^d$. Functions $F : \Gamma^2 \longrightarrow \mathbb{R}$ are called observables and

$$\langle F, \mu \rangle := \int_{\Gamma^2} F(\gamma) \, d\mu(\gamma)$$

are considered as measurable quantities of the particle system. Above duality yields a weak formulation for the dual equation to (1.3), i.e.

$$(1.4) \qquad \frac{d}{dt} \langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \quad \mu_t|_{t=0} = \mu_0, \quad t \geq 0.$$

It is known as the (forward) Kolmogorov equation. In the physical literature (1.4) is also called Fokker-Planck equation. Because of the Markovian property of the operator L we expect that solutions to (1.4) can be constructed in the class of probability measures on Γ^2 and hence determine an evolution of states. Solutions to the Fokker-Planck equation determine the distributions of the associated Markov process, provided, of course, it exists. Hence $(\mu_t)_{t \geq 0}$ is referred to as the 'statistical description' of the birth-and-death process.

For many, from the point of view of applications, interesting interacting particle systems it seems impossible to solve (1.4) for any initial state μ_0. Note that the particular choice $\mu_0 = \delta_\gamma$, where $\gamma \in \Gamma^2$, is related with the construction of an associated Markov process on Γ^2. It is necessary to restrict ourselves to a certain set of admissible initial states. Here we see a crucial difference comparing with Markov stochastic processes framework where the evolution can be constructed for any initial data. One possible class of admissible initial states is given by the collection of all sub-Poissionian states, cf. [24] for the one-component case. Such states are described in terms of their associated correlation functions $(k_\mu^{(n,m)})_{n,m=0}^\infty$. For the convenience of the reader we repeat the definition of correlation functions and their properties in the next section. The one-component case with general location space X (here $X = \mathbb{R}^d$) was considered in the pioneering works [27, 29, 30] to mention some. The collection of correlation functions $(k_\mu^{(n,m)})_{n,m=0}^\infty$ is called sub-Poissonian if it satisfies the Ruelle bound

$$k_\mu^{(n,m)}(x_1, \ldots, x_n; y_1, \ldots, y_m) \leq A e^{\alpha n} e^{\beta m}, \quad n, m \in \mathbb{N}_0$$

for some constants $A > 0$ and $\alpha, \beta \in \mathbb{R}$. A sub-Poissonian state μ is therefore a probability measure on Γ^2 for which the associated correlation functions exist and are sub-Poissonian. The Cauchy problem (1.4) is then formally equivalent to a system of Banach space-valued differential equations

$$(1.5) \qquad \frac{\partial k_t^{(n,m)}}{\partial t} = (L^\Delta k_t)^{(n,m)}, \quad k_t^{(n,m)}|_{t=0} = k_0^{(n,m)}, \quad n, m \in \mathbb{N}_0,$$

where L^Δ can be seen as an infinite operator-valued matrix. We provide, under some reasonable conditions, existence and uniqueness of weak solutions to (1.5) and consequently derive existence and uniqueness of weak solutions to (1.4). Note that a solution $(k_t^{(n,m)})_{n,m=0}^\infty$ to (1.5), in general, does not need to be the correlation function of some state μ_t on Γ^2. For such property additional analysis is required which was only achieved for particular models (see e.g. [23, 24, 25]).

This work is organized as follows. Notations and preliminary results are introduced in the second section. The third section is devoted to the construction of solutions to the pre-dual equation to (1.5). To this end we first prove that the pre-dual equation is well-posed and show that solutions can be obtained by the action of an analytic semigroup of contractions, see Theorem 3.1. The adjoint semigroup is related to existence and uniqueness of solutions to (1.5). It is shown that such solutions are the correlation functions corresponding to an evolution of states, see Theorem 3.7 and Proposition 3.13. The fourth section extends the Vlasov scaling, cf. [11], to the case of two-component Markov evolutions, see Theorem 4.1–4.3. Examples are considered in the last section.

2. Preliminaries

Space of finite configuration. One-component case. Denote by Γ_0 the configuration space of all finite subsets of $\mathbb{R}^d$, i.e.

$$\Gamma_0 = \{\eta \subset \mathbb{R}^d \mid |\eta| < \infty\},$$

where $|\eta|$ denotes the number of particles in the set η. This space has a natural decomposition into n-particle spaces, $\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}$, where $\Gamma_0^{(n)} = \{\eta \subset \mathbb{R}^d \mid |\eta| = n\}$, $n \geq 1$ and in the case $n = 0$ we set $\Gamma_0^{(0)} = \{\emptyset\}$. Denote by $\widetilde{(\mathbb{R}^d)^n}$ the space of all sequences $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ with $x_i \neq x_j$ for $i \neq j$. $\Gamma_0^{(n)}$ can be identified with $\widetilde{(\mathbb{R}^d)^n}$ via the symmetrization map

$$\mathrm{sym}_n : \widetilde{(\mathbb{R}^d)^n} \longrightarrow \Gamma_0^{(n)}, \ (x_1, \ldots, x_n) \longmapsto \{x_1, \ldots, x_n\},$$

which defines a topology on $\Gamma_0^{(n)}$. Namely, a set $A \subset \Gamma_0^{(n)}$ is open if and only if $\mathrm{sym}_n^{-1}(A) \subset \widetilde{(\mathbb{R}^d)^n}$ is open. On Γ_0 we define the topology of disjoint unions, i.e. a set $A \subset \Gamma_0$ is open iff $A \cap \Gamma_0^{(n)}$ is open in $\Gamma_0^{(n)}$ for all $n \in \mathbb{N}$. Then Γ_0 is a locally compact Polish space. Let $\mathcal{B}(\Gamma_0)$ stand for the Borel-σ-algebra on Γ_0. Denote by dx the Lebesgue measure on $\mathbb{R}^d$ and by $d^{\otimes n}x$ the product measure on $(\mathbb{R}^d)^n$. Let $d^{(n)}x$ be the image measure $\Gamma_0^{(n)}$ w.r.t. sym_n. The Lebesgue-Poisson measure is defined by

$$\lambda = \delta_\emptyset + \sum_{n=1}^{\infty} \frac{1}{n!} d^{(n)}x.$$

Given a measurable function $G : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}$, then

(2.1)
$$\int_{\Gamma_0} \sum_{\xi \subset \eta} G(\xi, \eta\backslash\xi, \eta) \, d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\xi, \eta, \eta \cup \xi) \, d\lambda(\xi) d\lambda(\eta)$$

holds, provided one side of the equality is finite for $|G|$ (see e.g. [27]). For a given measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ the Lebesgue-Poisson exponential is defined by

$$e_\lambda(f; \eta) := \prod_{x \in \eta} f(x)$$

and satisfies the combinatorial formula

$$\sum_{\xi \subset \eta} e_\lambda(f; \xi) = e_\lambda(1 + f; \eta).$$

For computations we will use the identity

$$\int_{\Gamma_0} e_\lambda(f; \eta) \, d\lambda(\eta) = \exp\left(\int_{\mathbb{R}^d} f(x) \, dx\right),$$

whenever $f \in L^1(\mathbb{R}^d)$.

Space of finite configurations. Two-component case. In this part we want to provide a brief overview for the two-component configuration space Γ_0^2, see [15, 17] and the references therein. We suppose that two different particles cannot occupy the same location $x \in \mathbb{R}^d$ and therefore define the two-component state space by

$$\Gamma_0^2 = \{(\eta^+, \eta^-) \in \Gamma_0 \times \Gamma_0 \mid \eta^+ \cap \eta^- = \emptyset\}.$$

Here and in the following we simply write η instead of $(\eta^+, \eta^-) \in \Gamma_0^2$ if no confusion may arise. Set operations $\xi \subset \eta$, $\xi \cup \eta$ and $\eta\backslash\xi$ are defined component-wise, i.e. by $\xi^\pm \subset \eta^\pm$, etc. For $\eta \in \Gamma_0^2$ we let $|\eta| := |\eta^+| + |\eta^-|$. The space Γ_0^2 has the natural decomposition

$$\Gamma_0^2 = \bigsqcup_{n,m=0}^{\infty} \Gamma_0^{(n,m)},$$

where $\Gamma_0^{(n,m)} = \{(\eta^+, \eta^-) \subset \mathbb{R}^d \times \mathbb{R}^d \mid \eta^+ \cap \eta^- = \emptyset, \ |\eta^+| = n, \ |\eta^-| = m\}$. The topology on $\Gamma_0^{(n,m)}$ and Γ_0^2 is defined in the same way as for $\Gamma_0^{(n)}$ and Γ_0. It is not difficult to see that this topology is the same as the subspace topology of the product topology on

$\Gamma_0 \times \Gamma_0$. In particular Γ_0^2 is a Polish space. The product measure $\lambda \otimes \lambda$ on $\Gamma_0 \times \Gamma_0$ satisfies

$$(2.2) \qquad \lambda \otimes \lambda(\{(\eta^+, \eta^-) \in \Gamma_0 \times \Gamma_0 \mid \eta^+ \cap \eta^- \neq \emptyset\}) = 0,$$

see e.g. [16]. The Lebesgue-Poisson measure λ^2 on Γ_0^2 is defined as the restriction of $\lambda \otimes \lambda$ to Γ_0^2. Since no confusion may arise we use the same notation λ for the Lebesgue-Poisson measure λ^2 on Γ_0^2 and λ on Γ_0. One can show that integrals w.r.t. integrable functions $G : \Gamma_0^2 \longrightarrow \mathbb{R}$ can be also written as

$$(2.3) \qquad \int_{\Gamma_0^2} G(\eta)\, d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\eta^+, \eta^-)\, d\lambda(\eta^+) d\lambda(\eta^-).$$

By (2.1), (2.2) and, (2.3) we see that the two-component Lebesgue-Poisson measure satisfies

$$(2.4) \qquad \int_{\Gamma_0^2} \sum_{\xi \subset \eta} G(\xi, \eta \backslash \xi, \eta)\, d\lambda(\eta) = \int_{\Gamma_0^2} \int_{\Gamma_0^2} G(\xi, \eta, \eta \cup \xi)\, d\lambda(\xi) d\lambda(\eta)$$

provided one side of the equality is finite for $|G|$. A set $M \subset \Gamma_0^2$ is called bounded if there exist a compact $\Lambda \subset \mathbb{R}^d$ and $N \in \mathbb{N}_0$ such that

$$M \subset \{(\eta^+, \eta^-) \in \Gamma_0^2 \mid \eta^\pm \subset \Lambda,\ |\eta| \leq N\}.$$

A function G is said to have bounded support if it is supported on a bounded set. Denote by $B_{bs}(\Gamma_0^2)$ the space of all bounded, measurable functions having bounded support. We say that $H : \Gamma_0^2 \longrightarrow \mathbb{R}$ is locally integrable if it is integrable for any bounded set. This is the same as regarding that the integral $\int_{\Gamma_0^2} G(\eta)|H(\eta)|\, d\lambda(\eta)$ is finite for all non-negative functions $G \in B_{bs}(\Gamma_0^2)$.

Space of locally finite configurations. The one-component configuration space Γ consists of all locally finite subsets γ of $\mathbb{R}^d$ and it is equipped with the smallest topology for which $\gamma \longmapsto \sum_{x \in \gamma} f(x)$ is continuous for any continuous function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ having compact support. Then Γ is a Polish space on which we consider the associated Borel-σ-algebra. Given $\alpha \in \mathbb{R}$, the Poisson measure π_{e^α} on Γ is defined as the unique probability measure having the Laplace transform

$$\int_\Gamma e^{\sum_{x \in \gamma} f(x)}\, d\pi_{e^\alpha}(\gamma) = \exp\left(e^\alpha \int_{\mathbb{R}^d} (e^{f(x)} - 1)\, dx\right),$$

see [27]. For two-component systems we consider the product space $\Gamma \times \Gamma$ equipped with the product topology. For simplicity of notation we write $\gamma := (\gamma^+, \gamma^-)$. Likewise we use for $\eta \in \Gamma_0^2$ the notation $\eta \subset \gamma$ and $\gamma \backslash \eta$ by which we mean that $\eta^+ \subset \gamma^+, \eta^- \subset \gamma^-$ and $\gamma^+ \backslash \eta^+,\ \gamma^- \backslash \eta^-$.

By [16] it follows that the product measure $\pi_{e^\alpha} \otimes \pi_{e^\beta}$ on $\Gamma \times \Gamma$ with $\alpha, \beta \in \mathbb{R}$ is concentrated on

$$\Gamma^2 := \{(\gamma^+, \gamma^-) \in \Gamma \times \Gamma \mid \gamma^+ \cap \gamma^- = \emptyset\},$$

i.e. $\pi_{e^\alpha} \otimes \pi_{e^\beta}(\Gamma^2) = 1$. Here Γ^2 is the configuration space for Markov evolutions of particles with two different types (see [17]). It is equipped with the restriction of the product topology on $\Gamma \times \Gamma$. The additional restriction is due to the fact that any two particles cannot occupy the same position. The two-component Poisson measure is defined by $\pi_{e^\alpha, e^\beta} := \pi_{e^\alpha} \otimes \pi_{e^\beta}|_{\Gamma^2}$. Below we describe the class of measures for which equation (1.4) will be considered.

Let $\Lambda_+, \Lambda_- \subset \mathbb{R}^d$ be two compacts, define

$$(\Gamma \times \Gamma)_{\Lambda_+, \Lambda_-} = \{(\gamma^+, \gamma^-) \in \Gamma \times \Gamma \mid \gamma^\pm \subset \Lambda_\pm\},$$

and set projections $p_{\Lambda^+,\Lambda^-} : \Gamma \times \Gamma \longrightarrow (\Gamma \times \Gamma)_{\Lambda_+,\Lambda_-}, (\gamma^+, \gamma^-) \longmapsto (\gamma^+ \cap \Lambda_+, \gamma^- \cap \Lambda_-)$
A probability measure μ on $\Gamma \times \Gamma$ is said to be locally absolutely continuous w.r.t. the
Poisson measure if $\mu^{\Lambda_+,\Lambda_-} := \mu \circ p_{\Lambda_+,\Lambda_-}^{-1}$ is absolutely continuous w.r.t. $\pi_{e^\alpha} \otimes \pi_{e^\beta} \circ p_{\Lambda_+,\Lambda_-}^{-1}$.
Note that this definition is independent of the particular choice of α and β. Any such
measure μ satisfies $\mu(\Gamma^2) = 1$ (see [16]).

For $G \in B_{bs}(\Gamma_0^2)$ define the K-transform by

$$(2.5) \qquad (\mathbb{K}G)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta),$$

where $\eta \Subset \gamma$ means that the sum only runs over all finite subsets η of γ. Then $\mathbb{K}G$
is a polynomially bounded cylinder function, i.e. there exists a compact $\Lambda \subset \mathbb{R}^d$ and
constants $C > 0$, $N \in \mathbb{N}$ such that $(\mathbb{K}G)(\gamma^+, \gamma^-) = (\mathbb{K}G)(\gamma^+ \cap \Lambda, \gamma^- \cap \Lambda)$ and

$$|(\mathbb{K}G)(\gamma)| \le C(1 + |\gamma^+ \cap \Lambda| + |\gamma^- \cap \Lambda|)^N, \quad \gamma \in \Gamma^2$$

holds. The $\mathbb{K}$-transform $\mathbb{K} : B_{bs}(\Gamma_0^2) \longrightarrow \mathcal{FP}(\Gamma^2) := K(B_{bs}(\Gamma_0^2))$ is a positivity preserv-
ing isomorphism with inverse given by

$$(\mathbb{K}^{-1}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0^2.$$

Denote by $\mathbb{K}_0$ the restriction of $\mathbb{K}$ determined by evaluating $\mathbb{K}G$ only on Γ_0^2. Its inverse is
then denoted by $\mathbb{K}_0^{-1}$. Let μ be locally absolutely continuous w.r.t. the Poisson measure
and assume that it has finite local moments, i.e.

$$\int_{\Gamma^2} |\gamma^+ \cap \Lambda_+|^n |\gamma^- \cap \Lambda_-|^n \, d\mu(\gamma) < \infty$$

for all $n \in \mathbb{N}_0$ and all compacts $\Lambda_\pm \subset \mathbb{R}^d$. It can be shown that there exists a function
$k_\mu : \Gamma_0^2 \longrightarrow \mathbb{R}_+$ satisfying the relation

$$(2.6) \qquad \int_{\Gamma^2} \mathbb{K}G(\gamma) \, d\mu(\gamma) = \int_{\Gamma_0^2} G(\eta) k_\mu(\eta) \, d\lambda(\eta), \quad G \in B_{bs}(\Gamma_0^2),$$

see [16]. This function is the so-called correlation function associated with μ. In such
a case the $\mathbb{K}$-transform can be uniquely extended to a bounded linear operator $\mathbb{K} :$
$L^1(\Gamma_0^2, k_\mu d\lambda) \longrightarrow L^1(\Gamma^2, d\mu)$ such that $\|\mathbb{K}G\|_{L^1(\Gamma^2, d\mu)} \le \|G\|_{L^1(\Gamma_0^2, k_\mu d\lambda)}$ and (2.5) holds
for μ-a.a. $\gamma \in \Gamma^2$ (repeat e.g. the arguments given in [27]). Let $\mathcal{L}_{k_\mu} := L^1(\Gamma_0^2, k_\mu d\lambda)$, for
$k_\mu(\eta) := e^{\alpha|\eta^+|} e^{\beta|\eta^-|}$ we also write $\mathcal{L}_{k_\mu} \equiv \mathcal{L}_{\alpha,\beta}$. A function $G \in \mathcal{L}_{\alpha,\beta}$ is called positive
definite if $\mathbb{K}G \ge 0$. Denote by $\mathcal{L}_{\alpha,\beta}^+$ the cone of all positive definite functions.

The duality $\langle G, k \rangle := \int_{\Gamma_0^2} G(\eta) k(\eta) \, d\lambda(\eta)$ is used for the identification $\mathcal{L}_{\alpha,\beta}^* \cong \mathcal{K}_{\alpha,\beta}$.
Here $\mathcal{K}_{\alpha,\beta}$ stands for the Banach space of all equivalence classes of functions $k : \Gamma_0^2 \longrightarrow \mathbb{R}$
equipped with norm

$$\|k\|_{\mathcal{K}_{\alpha,\beta}} = \underset{\eta \in \Gamma_0^2}{\text{ess sup}} \ |k(\eta)| e^{-\alpha|\eta^+|} e^{-\beta|\eta^-|}.$$

A function $k \in \mathcal{K}_{\alpha,\beta}$ is said to be positive definite if $\langle G, k \rangle \ge 0$ holds for all $G \in$
$B_{bs}^+(\Gamma_0^2) := B_{bs}(\Gamma_0^2) \cap \mathcal{L}_{\alpha,\beta}^+$. Let $\mathcal{K}_{\alpha,\beta}^+$ be the space of all positive definite functions
in $\mathcal{K}_{\alpha,\beta}$. Denote by $\mathcal{P}_{\alpha,\beta}$ the collection of all probability measures having finite local
moments and being locally absolutely continuous w.r.t. the Poisson measure such that
for each $\mu \in \mathcal{P}_{\alpha,\beta}$ its correlation function k_μ belongs to $\mathcal{K}_{\alpha,\beta}$.

Theorem 2.1. [27, 29, 30] *The following assertions are satisfied.*

(1) *Let $\mu \in \mathcal{P}_{\alpha,\beta}$ with correlation function k_μ. Then $k_\mu(\emptyset) = 1$ and k_μ is positive
definite.*

(2) *Conversely, let $k \in \mathcal{K}_{\alpha,\beta}^+$ and assume that $k(\emptyset) = 1$ holds. Then there exists a unique $\mu \in \mathcal{P}_{\alpha,\beta}$ with k as its correlation function.*

Let $\mu \in \mathcal{P}_{\alpha,\beta}$, then by Fubini's theorem

$$\int_{\Gamma^2} \mathbb{K}G(\gamma)\, d\mu(\gamma) = \int_\Gamma \int_\Gamma \mathbb{K}G(\gamma^+, \gamma^-)\, d\mu(\gamma^+, \gamma^-)$$

holds for all $G \in \mathcal{L}_{\alpha,\beta}$.

3. Construction of dynamics

Let $L = L^- + L^+$ be given by (1.1) and (1.2). We suppose that the intensities satisfy the condition given below.

(A) For all $x \in \mathbb{R}^d$

(3.1) $\mathbb{R}^d \times \Gamma^2 \ni (x, \gamma) \longmapsto d^+(x, \gamma^+\backslash x, \gamma^-), d^-(x, \gamma^+, \gamma^-\backslash x), b^+(x, \gamma), b^-(x, \gamma) \in [0, \infty]$

are measurable w.r.t the product Borel-σ-algebras on $\mathbb{R}^d \times \Gamma^2$ and $\mathcal{B}(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Moreover, their restrictions to $\mathbb{R}^d \times \Gamma_0^2$ take values in $[0, \infty)$ and, for any compact $\Lambda \subset \mathbb{R}^d$ and bounded set $M \subset \Gamma_0^2$

(3.2) $\qquad \int_\Lambda \int_M (d^+(x, \eta) + d^-(x, \eta) + b^+(x, \eta) + b^-(x, \eta))\, d\lambda(\eta)dx < \infty.$

3.1. Quasi-observables.

Introduce the cumulative death intensity by

$$M(\eta) := \sum_{x \in \eta^-} d^-(x, \eta^+, \eta^-\backslash x) + \sum_{x \in \eta^+} d^+(x, \eta^+\backslash x, \eta^-)$$

and set $c(L, \alpha, \beta; \eta) = c(\eta) = c(\alpha, \beta; \eta)$ by

(3.3) $\quad c(L, \alpha, \beta; \eta) := \sum_{x \in \eta^-} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1} d^-(x, \cdot \cup \eta^+, \cdot \cup \eta^-\backslash x)|(\xi)\, d\lambda(\xi)$

$$+ \sum_{x \in \eta^+} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1} d^+(x, \cdot \cup \eta^+\backslash x, \cdot \cup \eta^-)|(\xi)\, d\lambda(\xi)$$

$$+ e^{-\beta} \sum_{x \in \eta^-} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1} b^-(x, \cdot \cup \eta^+, \cdot \cup \eta^-\backslash x)|(\xi)\, d\lambda(\xi)$$

$$+ e^{-\alpha} \sum_{x \in \eta^+} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1} b^+(x, \cdot \cup \eta^+\backslash x, \cdot \cup \eta^-)|(\xi)\, d\lambda(\xi).$$

Define a linear mapping on $B_{bs}(\Gamma_0^2)$ by $\widehat{L} := \mathbb{K}_0^{-1} L \mathbb{K}_0$. Using the methods proposed in [11, 17] we can compute $\widehat{L}$. It has the form $\widehat{L} = A + B$. The latter expressions are given by $(AG)(\eta) = -M(\eta)G(\eta)$ and by

$$(BG)(\eta) = -\sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi^-} (\mathbb{K}_0^{-1} d^-(x, \cdot \cup \xi^+, \cdot \cup \xi^-\backslash x))(\eta\backslash\xi)$$

$$-\sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi^+} (\mathbb{K}_0^{-1} d^+(x, \cdot \cup \xi^+\backslash x, \cdot \cup \xi^-))(\eta\backslash\xi)$$

$$+\sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+, \xi^- \cup x)(\mathbb{K}_0^{-1} b^-(x, \cdot \cup \xi^+, \cdot \cup \xi^-))(\eta\backslash\xi)\, dx$$

$$+\sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-)(\mathbb{K}_0^{-1} b^+(x, \cdot \cup \xi^+, \cdot \cup \xi^-))(\eta\backslash\xi)\, dx.$$

Below we consider the linear mapping $\widehat{L}$ in one fixed Banach space and construct a semigroup associated to the Cauchy problem for quasi-observables

$$(3.4) \qquad \frac{\partial G_t}{\partial t} = \widehat{L}G_t, \quad G_t|_{t=0} = G_0 \in \mathcal{L}_{\alpha,\beta}.$$

Note that solutions to (1.3) are formally related to (3.4) by $F_t = \mathbb{K}G_t$. Denote by $\mathbb{1}^*$ the function given by $\mathbb{1}^*(\eta) := 0^{|\eta|} = \begin{cases} 1, & |\eta| = 0 \\ 0, & \text{otherwise} \end{cases}$.

Theorem 3.1. *Suppose that (A) is satisfied and assume that there exists $\alpha, \beta \in \mathbb{R}$ and a constant $a = a(\alpha, \beta) \in (0, 2)$ such that*

$$(3.5) \qquad c(\alpha, \beta; \eta) \le a(\alpha, \beta)M(\eta), \quad \eta \in \Gamma_0^2$$

holds. Then the following assertions are true:

(a) *The closure of $(\widehat{L}, B_{bs}(\Gamma_0^2))$ is the generator of an analytic semigroup $(\widehat{T}_{\alpha,\beta}(t))_{t \ge 0}$ of contractions on $\mathcal{L}_{\alpha,\beta}$ such that $\widehat{T}_{\alpha,\beta}(t)\mathbb{1}^* = \mathbb{1}^*$ holds. Moreover, the closure is given by*

$$D_{\alpha,\beta}(\widehat{L}) = \{G \in \mathcal{L}_{\alpha,\beta} \mid M \cdot G \in \mathcal{L}_{\alpha,\beta}\}$$

with $\widehat{L} = A + B$ defined as above.

(b) *Suppose that (3.5) holds for all $\alpha \in (\alpha_*, \alpha^*)$ and $\beta \in (\beta_*, \beta^*)$ with $\alpha_* < \alpha^*$ and $\beta_* < \beta^*$, and possibly different constants $a = a(\alpha, \beta)$. Then, for any $\alpha' < \alpha$ and $\beta' < \beta$ the space $\mathcal{L}_{\alpha,\beta}$ is invariant for $\widehat{T}_{\alpha',\beta'}(t)$ and $\widehat{T}_{\alpha,\beta}(t) = \widehat{T}_{\alpha',\beta'}(t)|_{\mathcal{L}_{\alpha,\beta}}$ holds.*

Proof. (a) Set $D_{\alpha,\beta}(A) := \{G \in \mathcal{L}_{\alpha,\beta} \mid M \cdot G \in \mathcal{L}_{\alpha,\beta}\}$. Then, since $M \ge 0$, the operator $(A, D_{\alpha,\beta}(A))$ is the generator of an analytic (of angle $\frac{\pi}{2}$), positive C_0-semigroup $(e^{-tM})_{t \ge 0}$ on $\mathcal{L}_{\alpha,\beta}$, see [7]. Let B' be defined for any $G \in B_{bs}(\Gamma_0^2)$ by

$$(B'G)(\eta^+, \eta^-) := \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi^-} |\mathbb{K}_0^{-1} d^-(x, \cdot \cup \xi^+, \cdot \cup \xi^- \backslash x)|(\eta \backslash \xi)$$

$$+ \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi^+} |\mathbb{K}_0^{-1} d^+(x, \cdot \cup \xi^+ \backslash x, \cdot \cup \xi^-)|(\eta \backslash \xi)$$

$$+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+, \xi^- \cup x)|\mathbb{K}_0^{-1} b^-(x, \cdot \cup \xi^+, \cdot \cup \xi^-)|(\eta \backslash \xi) \, dx$$

$$+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-)|\mathbb{K}_0^{-1} b^+(x, \cdot \cup \xi^+, \cdot \cup \xi^-)|(\eta \backslash \xi) \, dx.$$

Fix $r \in (0, 1)$, cf. (3.5), such that $a(\alpha, \beta) < 1 + r < 2$. For each $0 \le G \in D_{\alpha,\beta}(A)$, see (2.4), we obtain by (3.5)

$$\int_{\Gamma_0^2} B'G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|} \, d\lambda(\eta) = \int_{\Gamma_0^2} (c(\alpha, \beta; \eta) - M(\eta))G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|} \, d\lambda(\eta)$$

$$\le r \int_{\Gamma_0^2} M(\eta)G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|} \, d\lambda(\eta)$$

and hence $\int_{\Gamma_0^2} \left(A + \frac{1}{r}B'\right)G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|} \, d\lambda(\eta) \le 0$ holds. Therefore by [34, Theorem 2.2] the operator $(A + B', D_{\alpha,\beta}(A))$ is the generator of a strongly continuous semigroup $(U(s))_{s \ge 0}$ of contractions which preserves positivity. By [1, Theorem 1.1, Theorem 1.2] the operator $(A + B, D_{\alpha,\beta}(A))$ is the generator of an analytic semigroup $(\widehat{T}(s))_{s \ge 0}$ such that $|\widehat{T}(s)G| \le U(s)|G|$ holds for all $G \in \mathcal{L}_{\alpha,\beta}$. Since $U(s)$ is a contraction operator, so is $\widehat{T}(s)$. It remains to show that the closure of $(\widehat{L}, B_{bs}(\Gamma_0^2))$ is given by $(A + B, D_{\alpha,\beta}(A))$. To this end it suffices to show that $B_{bs}(\Gamma_0^2)$ is a core for $A + B$.

Let $G \in D_{\alpha,\beta}(A)$, $A_n \subset \Gamma_0^2$ an increasing sequence of bounded sets with $\bigcup_{n \geq 1} A_n = \Gamma_0^2$ and let $G_n(\eta) := \mathbb{1}_{A_n}(\eta)\mathbb{1}_{|G| \leq n}(\eta)G(\eta)$. Then $|G_n| \leq |G|$ and $G_n \longrightarrow G$ almost everywhere. Hence, $M \cdot G_n \longrightarrow M \cdot G$ and by dominated convergence also $BG_n \longrightarrow BG$, as $n \to \infty$ a.e. As a consequence $\widehat{L}G_n \longrightarrow \widehat{L}G$, as $n \to \infty$ almost everywhere. By

$$|\widehat{L}G_n| \leq M|G_n| + B'|G_n| \leq (M + B')|G| \in \mathcal{L}_{\alpha,\beta}$$

and dominated convergence we obtain $\widehat{L}G_n \longrightarrow \widehat{L}G$ in $\mathcal{L}_{\alpha,\beta}$. Therefore $B_{bs}(\Gamma_0^2) \subset D_{\alpha,\beta}(\widehat{L})$ is dense in the graph norm, i.e. the closure of $(\widehat{L}, B_{bs}(\Gamma_0^2))$ is given by $(A + B, D_{\alpha,\beta}(A))$.

(b) Let $\alpha' < \alpha$ and $\beta' < \beta$ such that (3.5) also holds for (α', β'). Denote by $(\widehat{T}_{\alpha',\beta'}(s))_{s \geq 0}$ the corresponding semigroup on $\mathcal{L}_{\alpha',\beta'}$ with generator $(\widehat{L}, D_{\alpha',\beta'}(\widehat{L}))$ on the domain

$$D_{\alpha',\beta'}(\widehat{L}) := \{G \in \mathcal{L}_{\alpha',\beta'} \mid M \cdot G \in \mathcal{L}_{\alpha',\beta'}\}.$$

We have to show that $\mathcal{L}_{\alpha,\beta}$ is invariant for $\widehat{T}_{\alpha',\beta'}(s)$ and

$$(3.6) \qquad \widehat{T}(s)G = \widehat{T}_{\alpha',\beta'}(s)G, \quad G \in \mathcal{L}_{\alpha,\beta}, \quad s \geq 0.$$

To this end we define a linear isomorphism

$$S : \mathcal{L}_{\alpha,\beta} \longrightarrow \mathcal{L}_{\alpha',\beta'}, \quad (SG)(\eta) = e^{(\alpha-\alpha')|\eta^+|}e^{(\beta-\beta')|\eta^-|}G(\eta)$$

with inverse S^{-1} given by $(S^{-1}G)(\eta) = e^{-(\alpha-\alpha')|\eta^+|}e^{-(\beta-\beta')|\eta|}G(\eta)$. Define on $\mathcal{L}_{\alpha',\beta'}$ a new operator by $\widehat{L}_1 := S\widehat{L}S^{-1}$ equipped with the domain

$$D_{\alpha',\beta'}(\widehat{L}_1) = \{G \in \mathcal{L}_{\alpha',\beta'} \mid S^{-1}G \in D_{\alpha,\beta}(\widehat{L})\} = \{G \in \mathcal{L}_{\alpha',\beta'} \mid MS^{-1}G \in \mathcal{L}_{\alpha,\beta}\}.$$

Since $\|MS^{-1}G\|_{\mathcal{L}_{\alpha,\beta}} = \|MG\|_{\mathcal{L}_{\alpha',\beta'}}$ we obtain $D_{\alpha',\beta'}(\widehat{L}_1) = D_{\alpha',\beta'}(\widehat{L})$. Let us show that $(\widehat{L}_1, D_{\alpha',\beta'}(\widehat{L}_1))$ is the generator of a C_0-semigroup on $\mathcal{L}_{\alpha',\beta'}$. The definition of S and S^{-1} implies $\widehat{L}_1 = A + B_1$ where A is the same as for $\widehat{L}$ and B_1 is given by

$$(B_1 G)(\eta) =$$
$$-\sum_{\xi \subsetneq \eta} G(\xi)e^{(\alpha-\alpha')|\eta^+ \setminus \xi^+|}e^{(\beta-\beta')|\eta^- \setminus \xi^-|} \sum_{x \in \xi^-} (\mathbb{K}_0^{-1}d^-(x, \cdot \cup \xi^+, \cdot \cup \xi^- \setminus x))(\eta \setminus \xi)$$
$$-\sum_{\xi \subsetneq \eta} G(\xi)e^{(\alpha-\alpha')|\eta^+ \setminus \xi^+|}e^{(\beta-\beta')|\eta^- \setminus \xi^-|} \sum_{x \in \xi^+} (\mathbb{K}_0^{-1}d^+(x, \cdot \cup \xi^+ \setminus x, \cdot \cup \xi^-))(\eta \setminus \xi)$$
$$+ e^{-(\beta-\beta')}\sum_{\xi \subset \eta}\int_{\mathbb{R}^d} G(\xi^+, \xi^- \cup x)e^{(\alpha-\alpha')|\eta^+ \setminus \xi^+|}e^{(\beta-\beta')|\eta^- \setminus \xi^-|}(\mathbb{K}_0^{-1}b^-(x, \cdot \cup \xi))(\eta \setminus \xi)\, dx$$
$$+ e^{-(\alpha-\alpha')}\sum_{\xi \subset \eta}\int_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-)e^{(\alpha-\alpha')|\eta^+ \setminus \xi^+|}e^{(\beta-\beta')|\eta^- \setminus \xi^-|}(\mathbb{K}_0^{-1}b^+(x, \cdot \cup \xi))(\eta \setminus \xi)\, dx.$$

Define analogously to B' the positive operator B_1' such that $|B_1 G| \leq B_1'|G|$, then for any non-negative function $G \in D_{\alpha',\beta'}(\widehat{L}_1)$ we obtain

$$\int_{\Gamma_0^2} B_1'G(\eta)e^{\alpha'|\eta^+|}e^{\beta'|\eta^-|}\, d\lambda(\eta) = \int_{\Gamma_0^2} (c(\alpha,\beta;\eta) - M(\eta))G(\eta)e^{\alpha'|\eta^+|}e^{\beta'|\eta^-|}\, d\lambda(\eta).$$

The same arguments as for the construction of $\widehat{T}_{\alpha,\beta}(t)$ show that $(A + B_1', D_{\alpha',\beta'}(\widehat{L}_1))$ is the generator of a sub-stochastic semigroup and hence $(\widehat{L}_1, D_{\alpha',\beta'}(\widehat{L}_1))$ is the generator of a C_0-semigroup. Now [31, Chapter 4, Theorem 5.5, Theorem 5.8] implies that $\mathcal{L}_{\alpha,\beta}$ is invariant for $\widehat{T}_{\alpha',\beta'}(t)$ and the restriction to $\mathcal{L}_{\alpha,\beta}$ is a C_0-semigroup given by $\widetilde{T}_{\alpha,\beta}(t) :=$

$\widehat{T}_{\alpha',\beta'}(t)|_{\mathcal{L}_{\alpha,\beta}}$. The generator of $\widetilde{T}_{\alpha,\beta}(t)$ is given by the part of $(\widehat{L}, D_{\alpha',\beta'}(\widehat{L}))$ in $\mathcal{L}_{\alpha,\beta}$, that is by

$$D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}} := \{G \in D_{\alpha',\beta'}(\widehat{L}) \cap \mathcal{L}_{\alpha,\beta} \mid \widehat{L}G \in \mathcal{L}_{\alpha,\beta}\}$$
$$= \{G \in \mathcal{L}_{\alpha,\beta} \mid M \cdot G \in \mathcal{L}_{\alpha',\beta'}, \ \widehat{L}G \in \mathcal{L}_{\alpha,\beta}\}.$$

Condition (3.5) therefore implies $D_{\alpha,\beta}(\widehat{L}) \subset D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}}$ and hence $(\widehat{L}, D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}})$ is an extension of $(\widehat{L}, D_{\alpha,\beta}(\widehat{L}))$. Denote by $R(\lambda; \widehat{L})$ the resolvent for $(\widehat{L}, D_{\alpha,\beta}(\widehat{L}))$ and by $\widetilde{R}(\lambda; \widehat{L})$ the resolvent for $(\widehat{L}, D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}})$. For sufficiently large $\lambda > 0$ it follows that $R(\lambda, \widehat{L})G \in D_{\alpha,\beta}(\widehat{L}) \subset D_{\alpha',\beta'}(\widehat{L})|_{\mathcal{L}_{\alpha,\beta}}$ for any $G \in \mathcal{L}_{\alpha,\beta}$ and thus

$$\widetilde{R}(\lambda; \widehat{L})G - R(\lambda; \widehat{L})G = \widetilde{R}(\lambda; \widehat{L})((\lambda - \widehat{L}) - (\lambda - \widehat{L}))R(\lambda; \widehat{L})G = 0,$$

where we have used that for elements in $D_{\alpha,\beta}(\widehat{L})$ the action of the generators is given by the formulas for $\widehat{L} = A + B$ and hence coincide. $\qquad\square$

For one-component models, i.e. $b^- = 0 = d^-$, a similar construction was already done in [11]. The main assumption was that each term in $c(\alpha, \beta; \eta)$ is bounded by $\frac{3}{2}M(\eta)$ and it was not clear whether $\widehat{T}_{\alpha,\beta}(t)$ is a contraction operator for $t \geq 0$. The next example shows that the constant 2 in (3.5) is optimal.

Example 3.2. *Take $d^- = 1$, $b^- = z > 0$ constant and $b^+ = d^+ = 0$, then condition (3.5) can be restated to $z < e^\beta$ and $\alpha \in \mathbb{R}$ is arbitrary. The evolution equation (3.4) is in this case exactly solvable and hence has for every initial condition G_0 the solution $(G_t)_{t \geq 0}$ given by*

$$G_t(\eta) = e^{-t|\eta^-|} \int_{\Gamma_0} G(\eta^+, \eta^- \cup \xi^-) e_\lambda \left(z(1 - e^{-t}); \xi^-\right) d\lambda(\xi^-), \quad \eta \in \Gamma_0^2,$$

see [8] for the one-component case. If condition (3.5) is satisfied, then $G_t \in \mathcal{L}_{\alpha,\beta}$. Suppose that $a(\alpha, \beta) > 2$, i.e. $z > e^\beta$ and let $t_0 > 0$ such that $(1 - e^{-t})z > e^\beta$ for all $t \geq t_0$ and hence

$$z(1 - e^{-t})(e^{-\beta} + e^{-t}) \geq z(1 - e^{-t})e^{-\beta} > 1.$$

Take $0 \leq G \in \mathcal{L}_{\alpha,\beta}$ such that $G \notin \mathcal{L}_{\alpha,\beta'}$ for any $\beta' > \beta$. The unique solution G_t is then positive and satisfies

$$\|G_t\|_{\mathcal{L}_{\alpha,\beta}} = \int_{\Gamma_0^2} \int_{\Gamma_0} e^{-t|\eta^-|} G(\eta^+, \eta^- \cup \xi^-) e_\lambda(z(1 - e^{-t}); \xi^-) e^{\alpha|\eta^+|} e^{\beta|\eta^-|} d\lambda(\xi^-) d\lambda(\eta)$$

$$= \int_{\Gamma_0^2} G(\eta^+, \xi^-) \left((e^{-\beta} + e^{-t})z(1 - e^{-t})\right)^{|\xi^-|} e^{\alpha|\eta^+|} e^{\beta|\xi^-|} d\lambda(\eta^+, \xi^-) = \infty.$$

Below we show that $\widehat{T}_{\alpha,\beta}(t)$ depends continuously (in a certain sense) on the birth-and-death rates. Let $d_n^\pm, d^\pm$ and $b_n^\pm, b^\pm$ be such that condition (A) is satisfied. Let

$$c_n(\alpha, \beta; \eta) = \sum_{x \in \eta^-} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1}(d^- - d_n^-)(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi)\, d\lambda(\xi)$$

$$+ \sum_{x \in \eta^+} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1}(d^+ - d_n^+)(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi)\, d\lambda(\xi)$$

$$+ e^{-\beta} \sum_{x \in \eta^-} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1}(b^- - b_n^-)(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi)\, d\lambda(\xi)$$

$$+ e^{-\alpha} \sum_{x \in \eta^+} \int_{\Gamma_0^2} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} |\mathbb{K}_0^{-1}(b^+ - b_n^+)(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi)\, d\lambda(\xi)$$

and $M_n(\eta) = \sum_{x \in \eta^-} d_n^-(x, \eta^+, \eta^- \backslash x) + \sum_{x \in \eta^+} d_n^+(x, \eta^+ \backslash x, \eta^-)$.

Theorem 3.3. *Suppose that the following conditions are satisfied.*

(1) *There exist $\alpha, \beta \in \mathbb{R}$ and a constant $a = a(\alpha, \beta) \in (0, 2)$ (independent of n) such that*

$$c_n(\alpha, \beta; \eta) \le a(\alpha, \beta)M_n(\eta), \quad \eta \in \Gamma_0^2, \quad n \ge 1$$

holds.

(2) *There exist constants $C > 0$, $N \in \mathbb{N}$ and $\tau \ge 0$ such that*

$$d_n^-(x, \eta) + d_n^+(x, \eta) \le C(1 + |\eta|)^N e^{\tau|\eta|}, \quad \eta \in \Gamma_0^2, \quad x \in \mathbb{R}^d$$

holds.

(3) *$c_n(\alpha, \beta; \eta) \longrightarrow 0$, $n \to \infty$ holds for all $\eta \in \Gamma_0^2$.*

Then (3.5) is satisfied. Let $\widehat{T}_{\alpha,\beta}(t)$ and $\widehat{T}_{\alpha,\beta}^n(t)$ be the associated semigroups on $\mathcal{L}_{\alpha,\beta}$. Then $\widehat{T}_{\alpha,\beta}^n(t)G \longrightarrow \widehat{T}_{\alpha,\beta}(t)G$, as $n \to \infty$ in $\mathcal{L}_{\alpha,\beta}$ uniformly on compacts for $t \ge 0$ and all $G \in \mathcal{L}_{\alpha,\beta}$.

Proof. Denote by $c(L, \alpha, \beta; \eta)$ the function defined in (3.3). Let $c(L_n, \alpha, \beta; \eta)$ be given as in (3.3) with $d^\pm, b^\pm$ replaced by $d_n^\pm, b_n^\pm$. It is not difficult to see that

$$|c(L_n, \alpha, \beta; \eta) - c(L, \alpha, \beta; \eta)| \le c_n(\alpha, \beta; \eta)$$

and $|M_n(\eta) - M(\eta)| \le c_n(\alpha, \beta; \eta)$ holds. Thus, for any $\eta \in \Gamma_0^2$

$$c(L, \alpha, \beta; \eta) = \lim_{n \to \infty} c(L_n, \alpha, \beta; \eta) \le a(\alpha, \beta) \lim_{n \to \infty} M_n(\eta) = a(\alpha, \beta)M(\eta)$$

and hence (3.5) holds. Let $\widehat{T}_{\alpha,\beta}(t)$ and $\widehat{T}_{\alpha,\beta}^n(t)$ be the associated semigroups on $\mathcal{L}_{\alpha,\beta}$. By Trotter-Kato approximation and since $B_{bs}(\Gamma_0^2)$ is a core for the generators of both semigroups, it suffices to show $\widehat{L}_n G \longrightarrow \widehat{L}G$ for any $G \in B_{bs}(\Gamma_0^2)$. To this end one can show that

$$\|\widehat{L}_n G - \widehat{L}G\|_{\mathcal{L}_{\alpha,\beta}} \le \int_{\Gamma_0^2} c_n(\alpha, \beta; \eta)|G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\, d\lambda(\eta).$$

Note that the integrand tends for a.a. η to zero. Since $c_n(\alpha, \beta; \eta) \le a(\alpha, \beta)(M(\eta) + M_n(\eta))$ and by $M(\eta) = \lim_{n \to \infty} M_n(\eta) \le C|\eta|^{N+1}e^{\tau|\eta|}$ it follows that $c_n(\alpha, \beta; \eta) \le 2a(\alpha, \beta)C|\eta|^{N+1}e^{\tau|\eta|}$. Dominated convergence implies $\widehat{L}_n G \longrightarrow \widehat{L}G$, as $n \to \infty$ for all $G \in B_{bs}(\Gamma_0^2)$. $\qquad\qquad\square$

3.2. **Correlation functions.** Suppose that condition (A) and (3.5) are fulfilled. Denote by $\widehat{T}_{\alpha,\beta}(t)^*$ the adjoint semigroup on $\mathcal{K}_{\alpha,\beta}$ and by $(\widehat{L}^*, D_{\alpha,\beta}(\widehat{L}^*))$ the adjoint operator to $(\widehat{L}, D_{\alpha,\beta}(\widehat{L}))$, i.e. $\langle \widehat{L}G, k \rangle = \langle G, \widehat{L}^*k \rangle$ for $G \in D_{\alpha,\beta}(\widehat{L})$ and $k \in D_{\alpha,\beta}(\widehat{L}^*)$.

Remark 3.4. *Let $\alpha' < \alpha$ and $\beta' < \beta$ be such that condition (3.5) holds for α', β' and α, β. Let $(\widehat{T}_{\alpha',\beta'}(s))_{s \ge 0}$ be the analytic semigroup constructed in Theorem 3.1. Then by (3.6) for any $G \in \mathcal{L}_{\alpha,\beta} \subset \mathcal{L}_{\alpha',\beta'}$ and $k \in \mathcal{K}_{\alpha',\beta'} \subset \mathcal{K}_{\alpha,\beta}$ we obtain*

$$\langle G, \widehat{T}_{\alpha',\beta'}(t)^*k \rangle = \langle \widehat{T}_{\alpha',\beta'}(t)G, k \rangle = \langle \widehat{T}_{\alpha,\beta}(t)G, k \rangle = \langle G, \widehat{T}_{\alpha,\beta}(t)^*k \rangle$$

and hence $\widehat{T}_{\alpha,\beta}(t)^*k = \widehat{T}_{\alpha',\beta'}(t)^*k$ holds.

We consider the linear operator L^Δ

$$
\begin{aligned}
(L^\Delta k)(\eta) = &- \sum_{x \in \eta^-} \int_{\Gamma_0^2} k(\eta \cup \xi)(\mathbb{K}_0^{-1} d^-(x, \cdot \cup \eta^+, \cdot \cup \eta^- \setminus x))(\xi) \, d\lambda(\xi) \\
&- \sum_{x \in \eta^+} \int_{\Gamma_0^2} k(\eta \cup \xi)(\mathbb{K}_0^{-1} d^+(x, \cdot \cup \eta^+ \setminus x, \cdot \cup \eta^-))(\xi) \, d\lambda(\xi) \\
&+ \sum_{x \in \eta^-} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+, \eta^- \cup \xi^- \setminus x)(\mathbb{K}_0^{-1} b^-(x, \cdot \cup \eta^+, \cdot \cup \eta^- \setminus x))(\xi) \, d\lambda(\xi) \\
&+ \sum_{x \in \eta^+} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+ \setminus x, \eta^- \cup \xi^-)(\mathbb{K}_0^{-1} b^+(x, \cdot \cup \eta^+ \setminus x, \cdot \cup \eta^-))(\xi) \, d\lambda(\xi)
\end{aligned}
$$

on the maximal domain

$$
D_{\alpha,\beta}(L^\Delta) = \{k \in \mathcal{K}_{\alpha,\beta} \mid L^\Delta k \in \mathcal{K}_{\alpha,\beta}\}.
$$

Introduce the following additional conditions.

(B) There exist constants $C > 0$, $\tau \geq 0$ and $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$ and $\eta \in \Gamma_0^2$

$$
(3.7) \qquad d^+(x, \eta) + d^-(x, \eta) + b^+(x, \eta) + b^-(x, \eta) \leq C(1 + |\eta|)^N e^{\tau|\eta|}
$$

holds.

(C) Let α, β and $a(\alpha, \beta)$ be as in (3.5). There exist constants $\alpha', \beta' \in \mathbb{R}$ with $\alpha' + \tau < \alpha$, $\beta' + \tau < \beta$ and another constant $a(\alpha', \beta') > 0$ such that

$$
c(\alpha', \beta'; \eta) \leq a(\alpha', \beta') M(\eta), \quad \eta \in \Gamma_0^2
$$

holds.

Lemma 3.5. *Suppose that (3.5) and conditions (A)–(C) are satisfied. Then*

$$
\widehat{L}^* = L^\Delta \in L(\mathcal{K}_{\alpha',\beta'}, \mathcal{K}_{\alpha,\beta}).
$$

In particular $\mathcal{K}_{\alpha',\beta'} \subset D_{\alpha,\beta}(L^\Delta)$ holds.

Proof. Observe that for $k \in \mathcal{K}_{\alpha',\beta'}$

$$
|L^\Delta k(\eta)| \leq \|k\|_{\mathcal{K}_{\alpha',\beta'}} a(\alpha', \beta') C |\eta|^{N+1} e^{(\alpha'+\tau)|\eta^+|} e^{(\beta'+\tau)|\eta^-|}.
$$

Then

$$
\begin{aligned}
|\eta|^{N+1} e^{-(\alpha-\alpha'-\tau)|\eta^+|} e^{-(\beta-\beta'-\tau)|\eta^-|} &\leq |\eta|^{N+1} e^{-\min\{\alpha-\alpha'-\tau, \beta-\beta'-\tau\}|\eta|} \\
&\leq \frac{(N+1)^{N+1} e^{-(N+1)}}{(\min\{\alpha-\alpha'-\tau, \beta-\beta'-\tau\})^{N+1}}
\end{aligned}
$$

implies

$$
|L^\Delta k(\eta)| \leq \frac{C a(\alpha', \beta')(N+1)^{N+1} e^{-(N+1)}}{(\min\{\alpha-\alpha'-\tau, \beta-\beta'-\tau\})^{N+1}} e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \|k\|_{\mathcal{K}_{\alpha',\beta'}},
$$

i.e. L^Δ is bounded from $\mathcal{K}_{\alpha',\beta'}$ to $\mathcal{K}_{\alpha,\beta}$. Let us prove the second assertion. Define a positive linear mapping

$$(\mathcal{C}k)(\eta) = \sum_{x\in\eta^-}\int_{\Gamma_0^2} k(\eta\cup\xi)|\mathbb{K}_0^{-1}d^-(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi)\,d\lambda(\xi)$$

$$+ \sum_{x\in\eta^+}\int_{\Gamma_0^2} k(\eta\cup\xi)|\mathbb{K}_0^{-1}d^+(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi)\,d\lambda(\xi)$$

$$+ \sum_{x\in\eta^-}\int_{\Gamma_0^2} k(\eta^+\cup\xi^+,\eta^-\cup\xi^-\backslash x)|\mathbb{K}_0^{-1}b^-(x,\cdot\cup\eta^+,\cdot\cup\eta^-\backslash x)|(\xi)\,d\lambda(\xi)$$

$$+ \sum_{x\in\eta^+}\int_{\Gamma_0^2} k(\eta^+\cup\xi^+\backslash x,\eta^-\cup\xi^-)|\mathbb{K}_0^{-1}b^+(x,\cdot\cup\eta^+\backslash x,\cdot\cup\eta^-)|(\xi)\,d\lambda(\xi).$$

Then, for $G\in B_{bs}(\Gamma_0^2)$ and $k\in\mathcal{K}_{\alpha,\beta}$ we have

$$|k(\eta)|\cdot(M(\eta)|G(\eta)| + B'|G|(\eta)) \le \|k\|_{\mathcal{K}_{\alpha,\beta}}e^{\alpha|\eta^+|}e^{\beta|\eta^-|}(M(\eta)|G(\eta)| + B'|G|(\eta))$$

and

$$|G(\eta)|\cdot\mathcal{C}|k|(\eta) \le c(\alpha,\beta;\eta)\|k\|_{\mathcal{K}_{\alpha,\beta}}|G(\eta)|e^{\alpha|\eta^+|}e^{\beta|\eta^-|}$$

which shows that $|k|\cdot(A+B')|G|$ and $|G(\eta)|\cdot\mathcal{C}|k|$ are both integrable. Hence we can apply (2.4) and obtain by a simple computation

$$\int_{\Gamma_0^2}(\widehat{L}G)(\eta)k(\eta)\,d\lambda(\eta) = \int_{\Gamma_0^2}G(\eta)(L^\Delta k)(\eta)\,d\lambda(\eta).$$

Let $k\in D_{\alpha,\beta}(L^\Delta)$, then above equality implies $L^\Delta\subset\widehat{L}^*$. Conversely take $k\in D_{\alpha,\beta}(\widehat{L}^*)$. Then

$$\int_{\Gamma_0^2}G(\eta)(\widehat{L}^*k)(\eta)\,d\lambda(\eta) = \int_{\Gamma_0^2}(\widehat{L}G)(\eta)k(\eta)\,d\lambda(\eta) = \int_{\Gamma_0^2}G(\eta)(L^\Delta k)(\eta)\,d\lambda(\eta)$$

implies $L^\Delta k = \widehat{L}^*k\in\mathcal{K}_{\alpha,\beta}$ and $D_{\alpha,\beta}(\widehat{L}^*)\subset D_{\alpha,\beta}(L^\Delta)$. $\qquad\square$

Since $\mathcal{L}_{\alpha,\beta}$ is not reflexive, $\widehat{T}_{\alpha,\beta}(t)^*$ does not need to be strongly continuous. However, it is continuous w.r.t. the topology $\sigma((\mathcal{L}_{\alpha,\beta})^*,\mathcal{K}_{\alpha,\beta}) = \sigma(\mathcal{K}_{\alpha,\beta},\mathcal{L}_{\alpha,\beta})$. Here $\sigma(\mathcal{K}_{\alpha,\beta},\mathcal{L}_{\alpha,\beta})$ is the smallest topology such that for each $k\in\mathcal{K}_{\alpha,\beta}$, $\mathcal{L}_{\alpha,\beta}\ni G\longmapsto\langle G,k\rangle$ is continuous. It is well-known that $\widehat{T}_{\alpha,\beta}(t)^*$ is strongly continuous on the proper subspace $\mathcal{K}_{\alpha,\beta}^\odot = \overline{D_{\alpha,\beta}(L^\Delta)}$ and its restriction $\widehat{T}_{\alpha,\beta}(t)^\odot := \widehat{T}_{\alpha,\beta}(t)^*|_{\mathcal{K}_{\alpha,\beta}^\odot}$ is a C_0-semigroup with generator $\widehat{L}^\odot k = L^\Delta k$,

$$D_{\alpha,\beta}(\widehat{L}^\odot) = \{k\in D_{\alpha,\beta}(L^\Delta)\mid L^\Delta k\in\mathcal{K}_{\alpha,\beta}^\odot\},$$

see e.g. [7]. Hence, for each $k_0\in D_{\alpha,\beta}(\widehat{L}^\odot)$ there exists a unique classical solution to (1.5) on the Banach space $\mathcal{K}_{\alpha,\beta}^\odot$, which is given by $\widehat{T}_{\alpha,\beta}(t)^*k_0 =: k_t$ (that is k_t is continuously differentiable w.r.t. the norm in $\mathcal{K}_{\alpha,\beta}$ and its derivative satisfies (1.5)).

Remark 3.6. *Note that $D_{\alpha,\beta}(\widehat{L}^\odot)$ depends on the generator. Using the theory of sundual semigroups it is possible to find certain invariant subspaces which are only characterized by the constant τ in condition (B), and prove that the restriction of $\widehat{T}_{\alpha,\beta}(t)^*$ onto such spaces is a strongly continuous semigroup. This can be used to prove existence and uniqueness of classical solutions to (1.5) on this subspaces (see [11, 12]).*

Below we consider another approach, which can be used to show weak uniqueness of solutions to (1.5). Let $\mathcal{C}(\mathcal{K}_{\alpha,\beta},\mathcal{L}_{\alpha,\beta}) =: \mathcal{C}$ be the topology of uniform convergence on

compact subsets of $\mathcal{L}_{\alpha,\beta}$. A basis of neighbourhoods around 0 is given by sets of the form

$$\{k \in \mathcal{K}_{\alpha,\beta} \mid \sup_{G \in K} |\langle G, k\rangle| < \varepsilon\},$$

with $\varepsilon > 0$ and a compact $K \subset \mathcal{L}_{\alpha,\beta}$, see [28, 35, 36] and the references therein. The semigroup $(\widehat{T}_{\alpha,\beta}(t)^*)_{t \geq 0}$ becomes continuous w.r.t. $\mathcal{C}$ and its generator w.r.t. $\mathcal{C}$ is exactly the adjoint operator $(L^{\Delta}, D_{\alpha,\beta}(L^{\Delta}))$, cf. [36, Theorem 1.4]. The next theorem provides existence, uniqueness and regularity of solutions to the Cauchy problem

$$(3.8) \qquad \frac{d}{dt}\langle G, k_t\rangle = \langle \widehat{L}G, k_t\rangle, \quad k_t|_{t=0} = k_0, \quad G \in B_{bs}(\Gamma_0^2).$$

Theorem 3.7. *Suppose that (3.5) and (A) are satisfied. Then for any $k_0 \in \mathcal{K}_{\alpha,\beta}$ the equation (3.8) has a unique solution given by $k_t = \widehat{T}_{\alpha,\beta}(t)^* k_0$. This means that k_t is continuous w.r.t. the topology $\mathcal{C}$ and satisfies*

$$(3.9) \qquad \langle G, k_t\rangle = \langle G, k_0\rangle + \int_0^t \langle \widehat{L}G, k_s\rangle\, ds, \quad G \in B_{bs}(\Gamma_0^2).$$

Assume that conditions (B) and (C) are fulfilled and let α', β' be the corresponding constants. Then the following assertions are true:

(1) *If $k_0 \in \mathcal{K}_{\alpha',\beta'}$, then k_t is continuous w.r.t. to the norm in $\mathcal{K}_{\alpha,\beta}$.*
(2) *If $k_0 \in \mathcal{K}_{\alpha',\beta'}$ and $\alpha' + 2\tau < \alpha$, $\beta' + 2\tau < \beta$, then k_t is also continuously differentiable w.r.t. to the norm in $\mathcal{K}_{\alpha,\beta}$ and the unique classical solution to (1.5).*

Proof. Existence and uniqueness for the Cauchy problem (3.9) follows from [36, Theorem 2.1] and Theorem 3.1 if one replaces $B_{bs}(\Gamma_0^2)$ by $D_{\alpha,\beta}(\widehat{L})$ in (3.9). But since $B_{bs}(\Gamma_0^2)$ is a core, uniqueness is proved. Moreover, since $\widehat{T}_{\alpha,\beta}(t)^*$ is continuous w.r.t. $\sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$, $t \longmapsto \langle \widehat{L}G, k_t\rangle$ is continuous and hence (3.9) implies (3.8).

1. If $k_0 \in \mathcal{K}_{\alpha',\beta'}$, then by Lemma 3.5 $L^{\Delta}k_0 \in \mathcal{K}_{\alpha,\beta}$ and hence $k_0 \in D_{\alpha,\beta}(L^{\Delta}) \subset \mathcal{K}_{\alpha,\beta}^{\odot}$ which implies the assertion.

2. Suppose that $\alpha' + 2\tau < \alpha$, $\beta' + 2\tau < \beta$ and let $\alpha'' \in (\alpha', \alpha)$, $\beta'' \in (\beta', \beta)$ be such that $\alpha' + \tau < \alpha''$, $\alpha'' + \tau < \alpha$ and $\beta' + \tau < \beta''$, $\beta'' + \tau < \beta$. By Lemma 3.5 the operator L^{Δ} is bounded as $\mathcal{K}_{\alpha',\beta'} \to \mathcal{K}_{\alpha'',\beta''}$ and $\mathcal{K}_{\alpha'',\beta''} \to \mathcal{K}_{\alpha,\beta}$. Therefore $k_0 \in D_{\alpha,\beta}(L^{\Delta})$ and $L^{\Delta}k_0 \in \mathcal{K}_{\alpha'',\beta''} \subset D_{\alpha,\beta}(L^{\Delta})$. Thus $k_0 \in D_{\alpha,\beta}(\widehat{L}^{\odot})$ implies that k_t is continuously differentiable w.r.t. the norm in $\mathcal{K}_{\alpha,\beta}$ and it is a classical solution to (1.5). $\qquad \square$

3.3. **Positive definiteness.** Suppose that (A) and (3.5) are satisfied. We start with the definition of solutions to the Fokker-Planck equation (1.4).

Definition 3.8. *A family of Borel probability measures $(\mu_t)_{t \geq 0} \subset \mathcal{P}_{\alpha,\beta}$ is said to be a weak solution to (1.4) if for any $F \in \mathcal{FP}(\Gamma^2)$, $t \longmapsto \langle LF, \mu_t\rangle$ is locally integrable and satisfies*

$$(3.10) \qquad \langle F, \mu_t\rangle = \langle F, \mu_0\rangle + \int_0^t \langle LF, \mu_s\rangle\, ds, \quad t \geq 0.$$

Note that $\langle F, \mu_t\rangle$ is well-defined since F is a polynomially bounded cylinder function and μ_t has finite local moment. Uniqueness is stated in the next theorem.

Theorem 3.9. *(Uniqueness) Suppose that (A) and (3.5) are fulfilled. Then equation (1.4) has at most one weak solution $(\mu_t)_{t \geq 0} \subset \mathcal{P}_{\alpha,\beta}$ such that its correlation functions $(k_t)_{t \geq 0}$ satisfy*

$$\sup_{t \in [0,T]} \|k_t\|_{\mathcal{K}_{\alpha,\beta}} < \infty, \quad \forall T > 0.$$

Proof. Let $(\mu_t)_{t\geq 0} \subset \mathcal{P}_{\alpha,\beta}$ be a solution to (1.4), and denote by $(k_t)_{t\geq 0} \subset \mathcal{K}_{\alpha,\beta}$ the associated correlation functions. Let $F \in \mathcal{FP}(\Gamma^2)$ and $G \in B_{bs}(\Gamma_0^2) \subset D_{\alpha,\beta}(\widehat{L})$ such that $F = \mathbb{K}G$. Then by $k_t(\eta) \leq \|k_t\|_{\mathcal{K}_{\alpha,\beta}} e^{\alpha|\eta^+|} e^{\beta|\eta^-|}$ it follows that $G, \widehat{L}G \in \mathcal{L}_{\alpha,\beta} \subset \mathcal{L}_{k_t}$. Since $\mathbb{K} : \mathcal{L}_{k_t} \longrightarrow L^1(\Gamma^2, d\mu_t)$ is continuous it follows that $F = \mathbb{K}G, L\mathbb{K}G = \mathbb{K}\widehat{L}G$ belong to $L^1(\Gamma^2, d\mu_t)$ for any $t \geq 0$. Moreover,

$$\langle LF, \mu_t \rangle = \langle \mathbb{K}\widehat{L}G, \mu_t \rangle = \langle \widehat{L}G, k_t \rangle$$

shows that $t \longmapsto \langle \widehat{L}G, k_t \rangle$ is locally integrable and hence

$$\langle G, k_t \rangle = \langle G, k_0 \rangle + \int_0^t \langle \widehat{L}G, k_s \rangle \, ds, \quad t \geq 0, \quad G \in B_{bs}(\Gamma_0^2)$$

holds. Thus k_t is continuous w.r.t. $\sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$ and since k_t is norm-bounded on $[0,T]$ [36, Lemma 1.10] implies that k_t is also continuous w.r.t. the topology $\mathcal{C}$. As a consequence $(k_t)_{t\geq 0}$ is a weak solution to (3.9) and hence it is given by $k_t = \widehat{T}_{\alpha,\beta}(t)^* k_0$. $\square$

Remark 3.10. *Let $k_0 \in \mathcal{K}_{\alpha,\beta}$ be positive definite and suppose that $k_t := \widehat{T}_{\alpha,\beta}(t)^* k_0 \in \mathcal{K}_{\alpha,\beta}$ is positive definite. Then $(k_t)_{t\geq 0}$ is a weak solution to (3.9) and for each $t \geq 0$ there exists a unique $\mu_t \in \mathcal{P}_{\alpha,\beta}$ having correlation function k_t. By $\langle G, k_t \rangle = \langle F, \mu_t \rangle$ and $\langle \widehat{L}G, k_t \rangle = \langle LF, \mu_t \rangle$ it follows that $(\mu_t)_{t\geq 0}$ is a weak solution to (1.4).*

Above considerations show that for existence of weak solutions to (1.4), it suffices to show that $\widehat{T}_{\alpha,\beta}(t)^*$ preserves the cone of positive definite functions. To this and we approximate $k_t = \widehat{T}_{\alpha,\beta}(t)^* k_0$ by an auxiliary evolution $\widehat{T}_{\alpha,\beta}^\delta(t)^* k_0$ and prove that $\widehat{T}_{\alpha,\beta}^\delta(t)^* k_0$ is positive definite. Such approximation scheme was used for particular models in [14, 23].

Let $(R_\delta)_{\delta>0}$ be a sequence of continuous integrable functions with $0 < R_\delta \leq 1$ and $R_\delta(x) \nearrow 1$ as $\delta \to 0$ for all $x \in \mathbb{R}^d$. In the following we simply say that $(R_\delta)_{\delta>0}$ is a localization sequence. Define new birth intensities by $b_\delta^+(x,\eta) := R_\delta(x) b^+(x,\eta)$ and $b_\delta^-(x,\eta) := R_\delta(x) b^-(x,\eta)$ for all $x \in \mathbb{R}^d$ and $\eta \in \Gamma_0^2$. Then for any $\eta \in \Gamma_0^2$ and $\delta > 0$

$$(3.11) \qquad \int_{\mathbb{R}^d} \left(b_\delta^+(x,\eta) + b_\delta^-(x,\eta) \right) \, dx < \infty$$

holds. Denote by L_δ the operator L with b^+, b^- replaced by b_δ^+, b_δ^-. We will consider the latter operator on a proper set of functions $F : \Gamma_0^2 \longrightarrow \mathbb{R}$.

Let $D_\delta(\eta) = M(\eta) + \int_{\mathbb{R}^d} b_\delta^+(x,\eta) \, dx + \int_{\mathbb{R}^d} b_\delta^-(x,\eta) \, dx$ and define

$$Q_\delta \rho(\eta) = \int_{\mathbb{R}^d} d^-(x,\eta) \rho(\eta^+, \eta^- \cup x) \, dx + \int_{\mathbb{R}^d} d^+(x,\eta) \rho(\eta^+ \cup x, \eta^-) \, dx$$

$$+ \sum_{x \in \eta^-} b_\delta^-(x, \eta^+, \eta^- \backslash x) \rho(\eta^+, \eta^- \backslash x) + \sum_{x \in \eta^+} b_\delta^+(x, \eta^+ \backslash x, \eta^-) \rho(\eta^+ \backslash x, \eta^-).$$

The linear operator $\mathcal{I}_\delta = -D_\delta + Q_\delta$ is considered on the domain

$$D(\mathcal{I}_\delta) = \{ \rho \in L^1(\Gamma_0^2, d\lambda) \mid D_\delta \rho \in L^1(\Gamma_0^2, d\lambda) \},$$

where D_δ acts as a multiplication operator on $L^1(\Gamma_0^2, d\lambda)$. Note that an analogue of D_δ was already introduced in [32]. Since Q_δ preserves positivity and satisfies

$$\int_{\Gamma_0^2} Q_\delta \rho(\eta) \, d\lambda(\eta) = \int_{\Gamma_0^2} D_\delta(\eta) \rho(\eta) \, d\lambda(\eta), \quad 0 \leq \rho \in D(\mathcal{I}_\delta),$$

it follows that $(\mathcal{I}_\delta, D(\mathcal{I}_\delta))$ has an extension $(\mathcal{G}_\delta, D(\mathcal{G}_\delta))$ which is the generator of a sub-stochastic semigroup $(S_\delta(t))_{t\geq 0}$ on $L^1(\Gamma_0^2, d\lambda)$, cf. [34, Theorem 2.2]. Here sub-stochastic means that $S_\delta(t)$ is a strongly continuous semigroup of contractions which preserves

positivity. Moreover, this semigroup is minimal in the sense that given another sub-stochastic semigroup $U_\delta(t)$ such that its generator is an extension of $(\mathcal{I}_\delta, D(\mathcal{I}_\delta))$, then $S_\delta(t) \leq U_\delta(t)$. Note that, in general, $\|S_\delta(t)\rho_0\|_{L^1(\Gamma_0^2, d\lambda)} < \|\rho_0\|_{L^1(\Gamma_0^2, d\lambda)}$ may happen for some $t > 0$ (see e.g. [3] and the references therein). It is related to $\overline{\mathcal{I}}_\delta = \mathcal{G}_\delta$ and is the main ingredient for the next condition.

(D) There exists a localization sequence $(R_\delta)_{\delta>0}$ such that $\overline{\mathcal{I}}_\delta = \mathcal{G}_\delta$, i.e. for each $\rho_0 \in D(\mathcal{G}_\delta)$ there exists a unique classical solution to

$$(3.12) \qquad \frac{\partial \rho_t^\delta}{\partial t} = \mathcal{G}_\delta \rho_t^\delta, \quad \rho_t^\delta|_{t=0} = \rho_0.$$

Note that condition (D) implies that the semigroup $S_\delta(t)$ is stochastic. For technical reasons we will need the adjoint semigroup on $L^\infty(\Gamma_0^2, d\lambda)$. Let $(\mathcal{J}_\delta, D(\mathcal{J}_\delta))$ be the adjoint operator to $(\mathcal{I}_\delta, D(\mathcal{I}_\delta))$ on $L^\infty(\Gamma_0^2, d\lambda)$. The next lemma follows by standard arguments and (2.4).

Lemma 3.11. *For any $F \in D(\mathcal{J}_\delta)$ it holds that $\mathcal{J}_\delta F = L_\delta F$.*

Remark 3.12. *Condition (D) is a non-explosion condition. It is fulfilled, provided one can find a proper Lyapunov functional. Sufficient conditions and related results how such condition can be checked are given in [18, 20]. A general (and rather easy) criterion can be found in [34]. A general approach to study condition (D) in an abstract setting is given in [3] (see also the references therein).*

The next statement provides existence and uniqueness of solutions to (1.4).

Proposition 3.13. *(Existence) Suppose that (A)–(D) and (3.5) are fulfilled. Then $\widehat{T}_{\alpha,\beta}(t)^* \mathcal{K}_{\alpha',\beta'}^+ \subset \mathcal{K}_{\alpha,\beta}^+$. In particular for any $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ there exists exactly one weak solution $(\mu_t)_{t\geq 0} \subset \mathcal{P}_{\alpha,\beta}$ to (1.4). The correlation functions are given by $k_{\mu_t} = \widehat{T}_{\alpha,\beta}(t)^* k_{\mu_0}$. If conditions (B) and (C) hold for all $\tau > 0$, then $\widehat{T}_{\alpha,\beta}(t)^* \mathcal{K}_{\alpha,\beta}^+ \subset \mathcal{K}_{\alpha,\beta}^+$ holds.*

Existence of an associated Markov function is stated in the next corollary, cf. [24].

Corollary 3.14. *Suppose that (A)–(D) hold for any $\tau > 0$ and assume that (3.5) holds. Then for any $\mu \in \mathcal{P}_{\alpha,\beta}$ there exists a Markov function $(X_t^\mu)_{t\geq 0}$ on the configuration space Γ^2 with the initial distribution μ associated with the generator L.*

The rest of this section is devoted to the proof of Proposition 3.13.

Lemma 3.15. *For any $\delta > 0$ Theorem 3.1 and Theorem 3.7 hold with L replaced by L_δ. Let $\widehat{T}_{\alpha,\beta}^\delta(t)$ be the semigroup on $\mathcal{L}_{\alpha,\beta}$. Then, for any $G \in \mathcal{L}_{\alpha,\beta}$*

$$\widehat{T}_{\alpha,\beta}^\delta(t)G \longrightarrow \widehat{T}_{\alpha,\beta}(t)G, \quad \delta \to 0$$

holds uniformly on compacts in $t \geq 0$.

Proof. The first claim follows by $R_\delta(x) \leq 1$ and an repetition of the arguments given in the proofs of Theorem 3.1 and Theorem 3.7. In particular, such repetition shows that $\widehat{L}_\delta := \mathbb{K}_0^{-1} L_\delta \mathbb{K}_0$ is given by $\widehat{L}_\delta = A + B_\delta$, where A is given as before and B_δ is obtained from B by multiplication of the terms for the birth by $R_\delta(x)$. In particular, it is the generator of an analytic semigroup $\widehat{T}_{\alpha,\beta}^\delta(t)$ of contractions on $\mathcal{L}_{\alpha,\beta}$. Note that this generator is considered on the same domain $D_{\alpha,\beta}(\widehat{L})$ as $\widehat{L}$. It is not difficult to see that $\widehat{L}_\delta G \longrightarrow \widehat{L}G$, as $\delta \to 0$ for all $G \in D_{\alpha,\beta}(\widehat{L})$. In view of Trotter-Kato approximation the assertion is proved. $\qquad\square$

Let $\mathcal{B}_{\alpha,\beta}$ be the Banach space of all equivalence classes of functions G with norm

$$\||G\||_{\mathcal{B}_{\alpha,\beta}} = \int_{\Gamma_0^2} |G(\eta)| e_\lambda(R_\delta; \eta^+) e_\lambda(R_\delta; \eta^-) e^{\alpha|\eta^+|} e^{\beta|\eta^-|} \, d\lambda(\eta).$$

Its dual Banach space $\mathcal{B}^*_{\alpha,\beta}$ can be identified with the Banach space of all equivalence classes of functions k with norm

$$|||k|||_{\mathcal{B}^*_{\alpha,\beta}} = \operatorname{ess\,sup}_{\eta \in \Gamma^2_0} \frac{|k(\eta)|}{e_\lambda(R_\delta; \eta^+) e_\lambda(R_\delta; \eta^-) e^{\alpha|\eta^+|} e^{\beta|\eta^-|}}.$$

Here and in the following we let $\delta > 0$ be arbitrary but fixed. To omit cumbersome notation will not explicitly state the dependence of $\mathcal{B}_{\alpha,\beta}, \mathcal{B}^*_{\alpha,\beta}$ on δ.

Remark 3.16. *Note that such spaces have the following two important properties.*

1. *The space $\mathcal{B}_{\alpha,\beta}$ is large in the sense that*

$$\bigcup_{\alpha'',\beta'' \in \mathbb{R}} \mathcal{K}_{\alpha'',\beta''} \subset \mathcal{B}_{\alpha,\beta}.$$

2. *The space $\mathcal{B}^*_{\alpha,\beta}$ is small in the sense that*

$$\mathcal{B}^*_{\alpha,\beta} \subset \bigcap_{\alpha'',\beta'' \in \mathbb{R}} \mathcal{L}_{\alpha'',\beta''}.$$

The same arguments as for the proof of Theorem 3.1 and Theorem 3.7 show that we can replace $\mathcal{L}_{\alpha,\beta}, \mathcal{K}_{\alpha,\beta}$ also by $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{B}^*_{\alpha,\beta}$. Note that it requires to introduce another function $c_\delta(\alpha,\beta;\eta)$ which is an analogue of $c(\alpha,\beta;\eta)$. A simple computation shows that it is given by $c(\alpha,\beta;\eta)$ with $e^{\alpha|\xi^+|} e^{\beta|\xi^-|}$ replaced by $e_\lambda(R_\delta; \xi^+) e_\lambda(R_\delta; \xi^-) e^{\alpha|\xi^+|} e^{\beta|\xi^-|}$. At this point it is necessary to use the additional factor R_δ in $b^\pm_\delta$. Now $c_\delta(\alpha,\beta;\eta) \leq c(\alpha,\beta;\eta)$ shows that the same arguments as for the proof of Theorem 3.1 and Theorem 3.7 can be applied. Denote by $U_\delta(t)$ and $U_\delta(t)^*$ the corresponding semigroups on $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{B}^*_{\alpha,\beta}$, respectively. Let $(\widehat{L}_\delta, D^{\mathcal{B}}_{\alpha,\beta}(\widehat{L}))$ be the generator of $U_\delta(t)$. The proofs of Theorem 3.1 and 3.7 show that

$$D^{\mathcal{B}}_{\alpha,\beta}(\widehat{L}) = \{G \in \mathcal{B}_{\alpha,\beta} \mid M \cdot G \in \mathcal{B}_{\alpha,\beta}\}.$$

Thus the Cauchy problem

$$(3.13) \qquad \frac{d}{dt}\langle G, u^\delta_t \rangle = \langle \widehat{L}_\delta G, u^\delta_t \rangle, \quad u^\delta_t|_{t=0} = u_0, \quad \forall G \in B_{bs}(\Gamma^2_0)$$

has for every $u_0 \in \mathcal{B}^*_{\alpha,\beta}$ a unique weak solution in $\mathcal{B}^*_{\alpha,\beta}$ given by $U_\delta(t)^* u_0$.

Lemma 3.17. *Let $k_0 \in \mathcal{B}^*_{\alpha,\beta}$, then $\widehat{T}^\delta_{\alpha,\beta}(t)^* k_0 = U_\delta(t)^* k_0$ holds.*

Proof. First observe that $\mathcal{B}^*_{\alpha,\beta} \subset \mathcal{K}_{\alpha,\beta}$ continuously and hence $k_0 \in \mathcal{K}_{\alpha,\beta}$. In particular $u^\delta_t := U_\delta(t)^* k_0$ and $k^\delta_t := \widehat{T}^\delta_{\alpha,\beta}(t)^* k_0$ are well-defined. Moreover, since $\mathcal{L}_{\alpha,\beta}$ is continuously embedded into $\mathcal{B}_{\alpha,\beta}$ we obtain $D_{\alpha,\beta}(\widehat{L}) \subset D^{\mathcal{B}}_{\alpha,\beta}(\widehat{L})$, i.e. $(\widehat{L}_\delta, D^{\mathcal{B}}_{\alpha,\beta}(\widehat{L}))$ is an extension of $(\widehat{L}_\delta, D_{\alpha,\beta}(\widehat{L}))$. Therefore $(u^\delta_t)_{t \geq 0}$ is also a weak solution to (3.9) and thus by uniqueness $u^\delta_t = k^\delta_t$, $t \geq 0$. $\qquad \square$

Lemma 3.18. *Let $k_0 \in \mathcal{B}^*_{\alpha',\beta'}$ be positive definite. Denote by $u^\delta_t \in \mathcal{B}^*_{\alpha,\beta}$ the unique weak solution to (3.13), then u^δ_t is positive definite for any $t \geq 0$.*

Proof. Define for any $u \in \mathcal{B}^*_{\alpha,\beta}$ a linear operator $\mathcal{H}u(\eta) := \int_{\Gamma^2_0}(-1)^{|\xi|} u(\eta \cup \xi)\, d\lambda(\xi)$. Then $\mathcal{H}u$ is well-defined and satisfies for any $C_+, C_- > 0$

$$\int_{\Gamma^2_0} |\mathcal{H}u(\eta)| C_+^{|\eta^+|} C_-^{|\eta^-|}\, d\lambda(\eta)$$

$$\leq |||u|||_{\mathcal{B}^*_{\alpha,\beta}} \int_{\Gamma^2_0} (1+C_+)^{|\eta^+|} (1+C_-)^{|\eta^-|} e^{\alpha|\eta^+|} e^{\beta|\eta^-|} e_\lambda(R_\delta; \eta^+) e_\lambda(R_\delta; \eta^-)\, d\lambda(\eta),$$

i.e. $\mathcal{H} : \mathcal{B}^*_{\alpha,\beta} \longrightarrow \mathcal{L}_{\log(C_+),\log(C_-)}$ is continuous. Let $G \in \mathcal{B}_{\alpha,\beta}$ be arbitrary, then for any $u \in \mathcal{B}^*_{\alpha,\beta}$ we get by Fubini's theorem and (2.4) that

$$(3.14) \qquad\qquad \langle \mathbb{K}_0 G, \mathcal{H}u \rangle = \langle G, u \rangle$$

holds. We can apply Fubini's theorem and (2.4) since

$$\int_{\Gamma_0^2} \int_{\Gamma_0^2} \int_{\Gamma_0^2} |G(\xi)||u(\eta \cup \xi \cup \zeta)| \, d\lambda(\zeta) d\lambda(\xi) d\lambda(\eta)$$

$$\leq \|u\|_{\mathcal{B}^*_{\alpha,\beta}} e^{2e^\alpha \langle R_\delta \rangle} e^{2e^\beta \langle R_\delta \rangle} \int_{\Gamma_0^2} |G(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|} e_\lambda(R_\delta; \xi^+) e_\lambda(R_\delta; \xi^-) \, d\lambda(\xi)$$

is satisfied, where $\langle R_\delta \rangle := \int_{\mathbb{R}^d} R_\delta(x) \, dx$. For the same u and $G \in D^{\mathcal{B}}_{\alpha,\beta}(\widehat{L})$ we obtain by (3.14) and $\mathbb{K}_0 \widehat{L}_\delta G = L_\delta \mathbb{K}_0 G$

$$(3.15) \qquad\qquad \langle \widehat{L}_\delta G, u \rangle = \langle \mathbb{K}_0 \widehat{L}_\delta G, \mathcal{H}u \rangle = \langle L_\delta \mathbb{K}_0 G, \mathcal{H}u \rangle.$$

Now let $U_\delta(t)^* k_0 = u_t^\delta \in \mathcal{B}^*_{\alpha,\beta}$, then

$$\langle G, u_t^\delta \rangle = \langle G, u_0 \rangle + \int_0^t \langle \widehat{L}_\delta G, u_s^\delta \rangle \, ds, \quad G \in D^{\mathcal{B}}_{\alpha,\beta}(\widehat{L}).$$

Observe that condition (B) implies $\mathcal{K}_{\log(2),\log(2)} \subset D^{\mathcal{B}}_{\alpha,\beta}(\widehat{L})$. Hence by (3.14) and (3.15) it follows for $R_t^\delta := \mathcal{H}u_t^\delta \in L^1(\Gamma_0^2, d\lambda)$, $t \geq 0$ that

$$\langle \mathbb{K}_0 G, R_t^\delta \rangle = \langle \mathbb{K}_0 G, R_0 \rangle + \int_0^t \langle L_\delta \mathbb{K}_0 G, R_s^\delta \rangle \, ds, \quad G \in \mathcal{K}_{\log(2),\log(2)}$$

holds. For any $F \in D(\mathcal{J}_\delta) \subset L^\infty(\Gamma_0, d\lambda)$ we get $|\mathbb{K}_0^{-1} F(\eta)| \leq \|F\|_{L^\infty} 2^{|\eta|}$ and hence $D(\mathcal{J}_\delta) \subset \mathbb{K}_0 \mathcal{K}_{\log(2),\log(2)}$. Thus we can find $G \in \mathcal{K}_{\log(2),\log(2)}$ such that $\mathbb{K}_0 G = F \in D(\mathcal{J}_\delta)$. Lemma 3.11 therefore implies

$$\langle F, R_t^\delta \rangle = \langle F, R_0 \rangle + \int_0^t \langle \mathcal{J}_\delta F, R_s^\delta \rangle \, ds, \quad F \in D(\mathcal{J}_\delta).$$

Since $k_0 \in \mathcal{B}^*_{\alpha',\beta'}$ we get by Theorem 3.7.1 that u_t^δ is continuous in $t \geq 0$ w.r.t. the norm in $\mathcal{B}^*_{\alpha,\beta}$. Because $\mathcal{H} : \mathcal{B}^*_{\alpha,\beta} \longrightarrow L^1(\Gamma_0^2, d\lambda)$ is continuous, $R_t^\delta = \mathcal{H}u_t^\delta$ is continuous w.r.t. $t \geq 0$ on $L^1(\Gamma_0^2, d\lambda)$. Hence $(R_t^\delta)_{t \geq 0}$ is a weak solution to (3.12). The main result from [2] therefore implies $R_t^\delta = S_\delta(t) R_0 \geq 0$. Finally, for any $G \in B^+_{bs}(\Gamma_0^2)$ we get

$$\langle G, u_t^\delta \rangle = \langle \mathbb{K}_0 G, R_t^\delta \rangle \geq 0, \quad t \geq 0. \qquad\qquad \square$$

We are now prepared to complete the proof of positive definiteness.

Proof. (Proposition 3.13) Let $\mu_0 \in \mathcal{P}_{\alpha',\beta'}$ with correlation function $k_0 \in \mathcal{K}_{\alpha',\beta'}$. Define

$$k_{0,\delta}(\eta) := k_0(\eta) e_\lambda(R_\delta; \eta^+) e_\lambda(R_\delta; \eta^-), \quad \delta > 0, \quad \eta \in \Gamma_0^2,$$

then $k_{0,\delta} \in \mathcal{B}^*_{\alpha',\beta'}$ and it is positive definite, cf. [8, 15]. By Lemma 3.17 we get $\widehat{T}^\delta_{\alpha,\beta}(t)^* k_{0,\delta} = U_\delta(t)^* k_{0,\delta} \in \mathcal{B}^*_{\alpha,\beta}$ and by Lemma 3.18 the latter expression is positive definite. Let $G \in B^+_{bs}(\Gamma_0^2)$. Then it suffices to show that

$$\langle G, \widehat{T}^\delta_{\alpha,\beta}(t)^* k_{0,\delta} \rangle \longrightarrow \langle G, \widehat{T}_{\alpha,\beta}(t)^* k_0 \rangle, \quad \delta \to 0.$$

To this end observe that

$$\langle G, \widehat{T}^\delta_{\alpha,\beta}(t)^* k_{0,\delta} \rangle = \langle \widehat{T}^\delta_{\alpha,\beta}(t)^* G - \widehat{T}_{\alpha,\beta}(t) G, k_{0,\delta} \rangle + \langle \widehat{T}_{\alpha,\beta}(t) G, k_{0,\delta} \rangle.$$

The first term can be estimated by

$$\|\widehat{T}^\delta_{\alpha,\beta}(t) G - \widehat{T}_{\alpha,\beta}(t) G\|_{\mathcal{L}_{\alpha,\beta}} \|k_0\|_{\mathcal{K}_{\alpha,\beta}}$$

and hence tends by Lemma 3.15 to zero. The second term tends by dominated convergence to $\langle \widehat{T}_{\alpha,\beta}(t)G, k_0 \rangle = \langle G, \widehat{T}_{\alpha,\beta}(t)^* k_0 \rangle$, which implies that $\widehat{T}_{\alpha,\beta}(t)^* k_0$ is positive definite.

If conditions (B) and (C) hold for all $\tau > 0$, then $k_{0,\delta}(\eta) := e^{-\delta|\eta|} k_0(\eta)$ belongs to $\mathcal{K}_{\alpha-\delta,\beta-\delta}$ for any $\delta > 0$. Consequently, above considerations imply that $\widehat{T}_{\alpha,\beta}(t)^* k_{0,\delta} \in \mathcal{K}_{\alpha,\beta}$ is positive definite. Taking the limit $\delta \to 0$ yields the assertion. $\square$

Remark 3.19. *Suppose instead of (B) the following to be satisfied: There exist $C > 0$, $N \in \mathbb{N}$ and $\nu_b \geq 0$, $\nu_1, \nu_2 \geq 0$ such that for all $x \in \mathbb{R}^d$ and $\eta \in \Gamma_0^2$:*

$$b^+(x,\eta) + b^-(x,\eta) \leq C(1+|\eta|)^N e^{\nu_b|\eta|},$$
$$d^+(x,\eta) \leq C(1+|\eta|)^N e^{\nu_1|\eta|},$$
$$d^-(x,\eta) \leq C(1+|\eta|)^N e^{\nu_2|\eta|}.$$

Then for any positive definite $k_0 \in \mathcal{K}_{\alpha',\beta'}$ the evolution $\widehat{T}_{\alpha,\beta}(t)^* k_0$ is positive definite, provided (C) holds for $\alpha' + \nu_1 < \alpha$, $\beta' + \nu_2 < \beta$.

4. Vlasov scaling

General description of Vlasov scaling. Let us briefly explain the Vlasov scaling in the two-component case. A motivation and additional explanations can be found in [10]. Let L be a Markov (pre-)generator on Γ^2, the aim is to find a scaling L_n such that the following scheme holds. Let $T_n^\Delta(t) = e^{tL_n^\Delta}$ be the (heuristic) representation of the scaled evolution of correlation functions, see (1.5). The particular choice of $L \to L_n$ should preserve the order of singularity, that is the limit

$$(4.1) \qquad\qquad n^{-|\eta|} T_n^\Delta(t) n^{|\eta|} k \longrightarrow T_V^\Delta(t)k, \quad n \to 0$$

should exist and the evolution $T_V^\Delta(t)$ should preserve Lebesgue-Poisson exponentials, i.e. if $r_0(\eta) = e_\lambda(\rho_0^-, \eta^-) e_\lambda(\rho_0^+; \eta^+)$, then $T_V^\Delta(t) r_0(\eta) = e_\lambda(\rho_t^-, \eta^-) e_\lambda(\rho_t^+; \eta^+)$. In such a case ρ_t^-, ρ_t^+ satisfy the system of non-linear integro-differential equations

$$(4.2) \qquad\qquad \frac{\partial \rho_t^-}{\partial t} = v_-(\rho_t^-, \rho_t^+), \quad \frac{\partial \rho_t^+}{\partial t} = v_+(\rho_t^-, \rho_t^+).$$

The functionals v_-, v_+ can be computed explicitly for a large class of models and (4.2) is the system of kinetic equations for the densities ρ_t^-, ρ_t^+ for the particle system. Instead of investigating the limit (4.1), we define renormalized operators $L_{n,\mathrm{ren}}^\Delta := n^{-|\eta|} L_n^\Delta n^{|\eta|}$ and study the behaviour of the semigroups $T_{n,\mathrm{ren}}^\Delta(t)$ when $n \to \infty$. In such a case one can compute a limiting operator

$$(4.3) \qquad\qquad L_{n,\mathrm{ren}}^\Delta \longrightarrow L_V^\Delta$$

and show that L_V^Δ is associated to a semigroup $T_V^\Delta(t)$. The limit (4.1) is then obtained by showing the convergence

$$(4.4) \qquad\qquad T_{n,\mathrm{ren}}^\Delta(t) \longrightarrow T_V^\Delta(t)$$

in a proper sense.

Scaling of two-component model. Consider the scaled intensities $d_n^+, d_n^-, b_n^+, b_n^- \geq 0$ and suppose they satisfy condition (A) for any $n \in \mathbb{N}$. Let $L_n = L_n^- + L_n^+$ where

$$L_n^- F(\gamma) = \sum_{x \in \gamma^-} d_n^-(x, \gamma^+, \gamma^- \backslash x)(F(\gamma^+, \gamma^- \backslash x) - F(\gamma^+, \gamma^-))$$

$$+ n \int_{\mathbb{R}^d} b_n^-(x, \gamma^+, \gamma^-)(F(\gamma^+, \gamma^- \cup x) - F(\gamma^+, \gamma^-))\, dx$$

and

$$L_n^+(t)F(\gamma) = \sum_{x \in \gamma^+} d_n^+(x, \gamma^+ \backslash x, \gamma^-)(F(\gamma^+ \backslash x, \gamma^-) - F(\gamma^+, \gamma^-))$$
$$+ n \int_{\mathbb{R}^d} b_n^+(x, \gamma^+, \gamma^-)(F(\gamma^+ \cup x, \gamma^-) - F(\gamma^+, \gamma^-)) \, dx.$$

Introduce

$$c_n(\alpha, \beta; \eta) := + \sum_{x \in \eta^-} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d_n^-(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \, d\lambda(\xi)$$

$$\sum_{x \in \eta^+} \int_{\Gamma_0^2} |\mathbb{K}_0^{-1} d_n^+(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \, d\lambda(\xi)$$

$$+ e^{-\beta} \sum_{x \in \eta^-} \int_{\Gamma_0} |\mathbb{K}_0^{-1} b_n^-(x, \cdot \cup \eta^+, \cdot \cup \eta^- \backslash x)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \, d\lambda(\xi)$$

$$+ e^{-\alpha} \sum_{x \in \eta^+} \int_{\Gamma_0} |\mathbb{K}_0^{-1} b_n^+(x, \cdot \cup \eta^+ \backslash x, \cdot \cup \eta^-)|(\xi) n^{|\xi|} e^{\alpha|\xi^+|} e^{\beta|\xi^-|} \, d\lambda(\xi)$$

and $M_n(\eta) := \sum_{x \in \eta^-} d_n^-(x, \eta^+, \eta^- \backslash x) + \sum_{x \in \eta^+} d_n^+(x, \eta^+ \backslash x, \eta^-)$. We will suppose the following conditions to be satisfied:

(V1) There exists $a(\alpha, \beta) \in (0, 2)$ such that for all $\eta \in \Gamma_0^2$ and $n \in \mathbb{N}$

$$c_n(\alpha, \beta; \eta) \le a(\alpha, \beta) M_n(\eta)$$

is satisfied.

(V2) For all $\xi \in \Gamma_0^2$ and $x \in \mathbb{R}^d$ the following limits exist in $\mathcal{L}_{\alpha, \beta}$ and are independent of ξ

$$\lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^-(x, \cdot \cup \xi)) = \lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^-(x, \cdot)) =: D_x^{V,-},$$
$$\lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^+(x, \cdot \cup \xi)) = \lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} d_n^+(x, \cdot)) =: D_x^{V,+},$$
$$\lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^-(x, \cdot \cup \xi)) = \lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^-(x, \cdot)) =: B_x^{V,-},$$
$$\lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^+(x, \cdot \cup \xi)) = \lim_{n \to \infty} n^{|\cdot|}(\mathbb{K}_0^{-1} b_n^+(x, \cdot)) =: B_x^{V,+}.$$

(V3) Let $M_V(\eta) := \sum_{x \in \eta^+} D_x^+(\emptyset) + \sum_{x \in \eta^-} D_x^-(\emptyset)$, then there exists $\sigma > 0$ such that either

$$M_n(\eta) \le \sigma M_V(\eta), \quad \eta \in \Gamma_0^2, \quad n \in \mathbb{N}$$

or

$$M_n(\eta) \le \sigma M_V(\eta), \quad \eta \in \Gamma_0^2, \quad n \in \mathbb{N}$$

are satisfied.

Note that here and below the constants α, β are fixed. For simplicity of notation, we omit, whenever it is possible, the dependence on these constants. Define $\widehat{L}_n := \mathbb{K}_0^{-1} L_n \mathbb{K}_0$ and the renormalized operators $\widehat{L}_{n,\text{ren}} := R_n \widehat{L}_n R_{n^{-1}}$, where $R_\kappa G(\eta) = \kappa^{|\eta|} G(\eta)$. Then for any $G \in B_{bs}(\Gamma_0^2)$ and $k \in \mathcal{K}_{\alpha, \beta}$ the relation $\langle \widehat{L}_{n,\text{ren}} G, k \rangle = \langle G, L_{n,\text{ren}}^\Delta k \rangle$ holds.

Statements. The next statement provides the existence and uniqueness of an evolution of quasi-observables and correlation functions for any fixed $n \in \mathbb{N}$.

Theorem 4.1. *Suppose that condition (V1) is satisfied. Then for any fixed $n \in \mathbb{N}$ the following assertions are true:*

(1) *The closure of $(\widehat{L}_{n,\mathrm{ren}}, B_{bs}(\Gamma_0^2))$ is given by $(\widehat{L}_{n,\mathrm{ren}}, D(\widehat{L}_{n,\mathrm{ren}}))$, where*

$$D(\widehat{L}_{n,\mathrm{ren}}) = \{G \in \mathcal{L}_{\alpha,\beta} \mid M_n \cdot G \in \mathcal{L}_{\alpha,\beta}\}.$$

It is the generator of an analytic semigroup $(\widehat{T}_{n,\mathrm{ren}}(s))_{s \geq 0}$ of contractions on $\mathcal{L}_{\alpha,\beta}$.

(2) *Let $\widehat{T}_{n,\mathrm{ren}}(t)^*$ be the adjoint semigroup with $(L_{n,\mathrm{ren}}^{\Delta}, D(L_{n,\mathrm{ren}}^{\Delta}))$ considered on the (maximal) domain*

$$D(L_{n,\mathrm{ren}}^{\Delta}) = \{k \in \mathcal{K}_{\alpha,\beta} \mid L_{n,\mathrm{ren}}^{\Delta} k \in \mathcal{K}_{\alpha,\beta}\}.$$

For any $n \in \mathbb{N}$ and $k_0 \in \mathcal{K}_{\alpha,\beta}$, there exists a unique weak solution to

$$\frac{\partial}{\partial t}\langle G, k_{t,n}\rangle = \langle \widehat{L}_{n,\mathrm{ren}}G, k_{t,n}\rangle, \quad k_{t,n}|_{t=0} = k_0, \quad G \in B_{bs}(\Gamma_0^2)$$

given by $k_{t,n} = \widehat{T}_{n,\mathrm{ren}}(t)^* k_0$.

The case $n = 1$ is covered by the results obtained in Theorem 3.7. Following the arguments there, it is not difficult to adopt the proofs to this case. In the next step we construct the limiting dynamics when $n \to \infty$. Condition (V2) suggests to consider the limit $\widehat{L}_{n,\mathrm{ren}}G \longrightarrow \widehat{L}_V G$, as $n \to \infty$. The operator $\widehat{L}_V := A_V + B_V$ is given by $A_V G(\eta) = -M_V(\eta)G(\eta)$, where

$$M_V(\eta) = \sum_{x \in \eta^+} D_x^{V,+}(\emptyset) + \sum_{x \in \eta^-} D_x^{V,-}(\emptyset),$$

$$B_V G(\eta) = - \sum_{\substack{\xi^+ \subsetneq \eta^+ \\ \xi^- \subsetneq \eta^-}} G(\xi) \sum_{x \in \xi^+} D_x^{V,+}(\eta\backslash\xi) - \sum_{\substack{\xi^+ \subsetneq \eta^+ \\ \xi^- \subsetneq \eta^-}} G(\xi) \sum_{x \in \xi^-} D_x^{V,-}(\eta\backslash\xi)$$

$$+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+ \cup x, \xi^-) B_x^{V,+}(\eta\backslash\xi)\,dx + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi^+, \xi^- \cup x) B_x^{V,-}(\eta\backslash\xi)\,dx.$$

Let $D(\widehat{L}_V) := \{G \in \mathcal{L}_{\alpha,\beta} \mid M_V \cdot G \in \mathcal{L}_{\alpha,\beta}\}$, define

$$c_V(\alpha,\beta;\eta) :=$$

$$+ \sum_{x \in \eta^+} \int_{\Gamma_0^2} |D_x^{V,+}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|}\,d\lambda(\xi) + e^{-\alpha} \sum_{x \in \eta^+} \int_{\Gamma_0^2} |B_x^{V,+}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|}\,d\lambda(\xi)$$

$$+ \sum_{x \in \eta^-} \int_{\Gamma_0^2} |D_x^{V,-}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|}\,d\lambda(\xi) + e^{-\beta} \sum_{x \in \eta^-} \int_{\Gamma_0^2} |B_x^{V,-}(\xi)| e^{\alpha|\xi^+|} e^{\beta|\xi^-|}\,d\lambda(\xi)$$

and finally

$$(L_V^{\Delta} k)(\eta) :=$$

$$- \sum_{x \in \eta^+} \int_{\Gamma_0^2} k(\eta \cup \xi) D_x^{V,+}(\xi)\,d\lambda(\xi) + \sum_{x \in \eta^+} \int_{\Gamma_0^2} k(\eta^+\backslash x \cup \xi^+, \eta^+ \cup \xi^+) B_x^{V,+}(\xi)\,d\lambda(\xi)$$

$$- \sum_{x \in \eta^-} \int_{\Gamma_0^2} k(\eta \cup \xi) D_x^{V,-}(\xi)\,d\lambda(\xi) + \sum_{x \in \eta^-} \int_{\Gamma_0^2} k(\eta^+ \cup \xi^+, \eta^-\backslash x \cup \xi^-) B_x^{V,-}(\xi)\,d\lambda(\xi).$$

Theorem 4.2. *Assume that conditions (V1), (V2) are satisfied. Then the following assertions are true:*

(1) *The operator $(\widehat{L}_V, D(\widehat{L}_V))$ is the generator of an analytic semigroup $(\widehat{T}^V(t))_{t \geq 0}$ of contractions on $\mathcal{L}_{\alpha,\beta}$.*

(2) *Let* $(\widehat{T}^V(t)^*)_{t\geq 0}$ *be the adjoint semigroup on* $\mathcal{K}_{\alpha,\beta}$, *then for any* $r_0 \in \mathcal{K}_{\alpha,\beta}$ *there exists a unique solution* $r_t = \widehat{T}^V(t)^*r_0$ *to the Cauchy problem*

$$(4.5) \qquad \frac{\partial}{\partial t}\langle G, r_t\rangle = \langle \widehat{L}_V G, r_t\rangle, \quad r_t|_{t=0} = r_0, \quad G \in B_{bs}(\Gamma_0^2).$$

(3) *Let* $r_0(\eta) = \prod_{x\in\eta^+}\rho_0^+(x)\prod_{x\in\eta^-}\rho_0^-(x)$ *and* $\rho_0^+, \rho_0^- \in L^\infty(\mathbb{R}^d)$ *with* $\|\rho_0^+\|_{L^\infty} \leq e^\alpha$, $\|\rho_0^-\|_{L^\infty} \leq e^\beta$. *Assume that* (ρ_t^+, ρ_t^-) *is a classical solution to*

$$\frac{\partial \rho_t^-}{\partial t}(x) = -\int_{\Gamma_0^2} e_\lambda(\rho_t^+;\xi^+)e_\lambda(\rho_t^-;\xi^-)D_x^{V,-}(\xi)\,d\lambda(\xi)\rho_t^-(x)$$

$$+ \int_{\Gamma_0^2} e_\lambda(\rho_t^+;\xi^+)e_\lambda(\rho_t^-;\xi^-)B_x^{V,-}(\xi)\,d\lambda(\xi),$$

$$\frac{\partial \rho_t^+}{\partial t}(x) = -\int_{\Gamma_0^2} e_\lambda(\rho_t^+;\xi^+)e_\lambda(\rho_t^-;\xi^-)D_x^{V,+}(\xi)\,d\lambda(\xi)\rho_t^+(x)$$

$$+ \int_{\Gamma_0^2} e_\lambda(\rho_t^+;\xi^+)e_\lambda(\rho_t^-;\xi^-)B_x^{V,+}(\xi)\,d\lambda(\xi)$$

with initial conditions $\rho_t^+|_{t=0} = \rho_0^+$, $\rho_t^-|_{t=0} = \rho_0^-$ *and*

$$\|\rho_t^+\|_{L^\infty} \leq e^\alpha \quad \|\rho_t^-\|_{L^\infty} \leq e^\beta.$$

Then $r_t(\eta) := \prod_{x\in\eta^+}\rho_t^+(x)\prod_{x\in\eta^-}\rho_t^-(x)$ *is a weak solution to* (4.5) *in* $\mathcal{K}_{\alpha,\beta}$.

Proof. By conditions (V1) and (V2) it follows that $c_V(\alpha,\beta;\eta) \leq a(\alpha,\beta)M_V(\eta)$ holds. Define a positive operator B_V' on $D(\widehat{L}_V)$ by

$$B_V'G(\eta) = \sum_{\substack{\xi^+\subsetneq\eta^+ \\ \xi^-\subsetneq\eta^-}} G(\xi)\sum_{x\in\xi^+}|D_x^{V,+}(\eta\backslash\xi)| + \sum_{\substack{\xi^+\subsetneq\eta^+ \\ \xi^-\subsetneq\eta^-}} G(\xi)\sum_{x\in\xi^-}|D_x^{V,-}(\eta\backslash\xi)|$$

$$+ \sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi^+\cup x, \xi^-)|B_x^{V,+}(\eta\backslash\xi)|\,dx$$

$$+ \sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi^+, \xi^-\cup x)|B_x^{V,-}(\eta\backslash\xi)|\,dx.$$

Then it is not difficult to see that for any $0 \leq G \in D(\widehat{L}_V)$

$$\int_{\Gamma_0^2} B_V'G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\,d\lambda(\eta) \leq (a(\alpha,\beta)-1)\int_{\Gamma_0^2} M_V(\eta)G(\eta)e^{\alpha|\eta^+|}e^{\beta|\eta^-|}\,d\lambda(\eta)$$

is fulfilled. The same arguments as in the proof of Theorem 4.1 yield existence, analyticity and the contraction property of the semigroup $\widehat{T}^V(t)$. For the last assertion we only show that r_t is continuous w.r.t. $\mathcal{C}$. The other assertions are simple computations, see e.g. [10]. First observe that by $|r_t(\eta)| \leq e^{\alpha|\eta^+|}e^{\beta|\eta^-|}$ the function r_t is norm-bounded and hence it suffices to show that it is continuous w.r.t. $\sigma(\mathcal{K}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta})$. But this function is continuous in $t \geq 0$ for any η and hence the assertion follows by dominated convergence. $\qquad\square$

Theorem 4.3. *Suppose that conditions (V1)–(V3) are fulfilled. Then* $\widehat{T}_{n,\mathrm{ren}}(t) \longrightarrow \widehat{T}^V(t)$ *holds strongly in* $\mathcal{L}_{\alpha,\beta}$ *and uniformly on compacts in* $t \geq 0$.

Proof. We are going to apply [11, Lemma 4.3] and Trotter-Kato approximation. Fix $\lambda > 0$ and denote by $R(\lambda; A_n)$ and $R(\lambda, A_V)$ the resolvent for A_n and A_V, respectively. Then it follows that $\|R(\lambda; A_n)\|_{L(\mathcal{L}_{\alpha,\beta})}, \|R(\lambda; A_V)\|_{L(\mathcal{L}_{\alpha,\beta})} \leq \frac{1}{\lambda}$, $\|B_n R(\lambda; A_n)G\|_{\mathcal{L}_{\alpha,\beta}} \leq (a(\alpha,\beta)-1)\|G\|_{\mathcal{L}_{\alpha,\beta}}$ and likewise $\|B_V R(\lambda; A_V)G\|_{\mathcal{L}_{\alpha,\beta}} \leq (a(\alpha,\beta)-1)\|G\|_{\mathcal{L}_{\alpha,\beta}}$. Since

$M_n \longrightarrow M_V$ as $n \to \infty$, it is easy to show by dominated convergence that $R(\lambda; A_n) \longrightarrow R(\lambda; A_V)$ holds strongly in $\mathcal{L}_{\alpha,\beta}$ as $n \to \infty$. Hence it remains to show the convergence

$$(4.6) \qquad\qquad B_n R(\lambda; A_n)G \longrightarrow B_V R(\lambda; A_V)G, \quad n \to \infty.$$

This can be proved, similarly to [11], by dominated convergence. $\qquad\qquad\square$

Remark 4.4. *The proof shows that condition (V3) can be replaced by*

$$d_n^-(x,\eta) + d_n^+(x,\eta) \le C(1+|\eta|)^N e^{\tau|\eta|}, \quad x \in \mathbb{R}^d, \quad \eta \in \Gamma_0^2, \quad n \in \mathbb{N}$$

for some constants $C > 0$, $N \in \mathbb{N}$ and $\tau \ge 0$.

5. Examples

We introduce four models describing the stochastic behaviour of particle systems on Γ^2. Interactions of particles, of the same and also of different type, are modelled by the relative energy function

$$E_\varphi(x, \gamma^\pm) := \sum_{y \in \gamma^\pm} \varphi(x-y), \quad x \in \mathbb{R}^d, \quad \gamma^\pm \in \Gamma,$$

where φ is a symmetric, non-negative and integrable function. Associated to above relative energy is the following functional

$$(5.1) \qquad\qquad C(\varphi) := \int_{\mathbb{R}^d} |e^{-\varphi(x)} - 1| \, dx.$$

Since φ is non-negative, we obtain $C(\varphi) \le \int_{\mathbb{R}^d} \varphi(x) \, dx =: \langle \varphi \rangle$. Note that below we will give no proofs of the statements. Conditions (A)–(C) are rather standard and can be checked by similar methods to [11, 12]. The last condition can be checked similarly to [18, 20]. Note that a general criterion for condition (D) can be also found in [34].

5.1. **Two-interacting BDLP-model.** Suppose that the death intensities are given by

$$d^-(x, \gamma^+, \gamma^-\backslash x) = m^- + \sum_{y \in \gamma^-\backslash x} a^-(x-y),$$

$$d^+(x, \gamma^+\backslash x, \gamma^-) = m^+ + \sum_{y \in \gamma^+\backslash x} b^-(x-y) + \sum_{y \in \gamma^-} \varphi^-(x-y).$$

This means that the particles in $\gamma^\pm$ have a random lifetime determined by the parameters $m^+, m^- > 0$. The additional terms describe the competition of particles for resources. Namely, each particle $x \in \gamma^-$ may die due to the competition for resources with another particle $y \in \gamma^-\backslash x$ from the same type. The rate of this event is determined by $a^-(x-y)$. Likewise each particle $x \in \gamma^+$ may be die due to the interaction with another particle by $y \in \gamma^+\backslash x$ with the rate $b^-(x-y)$. Moreover, this particle $x \in \gamma^+$ may also die due to the competition for resources with another particle $y \in \gamma^-$ of different type. The rate of this event is described by the interaction potential $\varphi^-(x-y)$.

The birth intensities are assumed to be given by

$$b^-(x, \gamma) = \sum_{y \in \gamma^-} a^+(x-y) + z,$$

$$b^+(x, \gamma) = \sum_{y \in \gamma^+} b^+(x-y) + \sum_{y \in \gamma^-} \varphi^+(x-y).$$

The terms corresponding to a^+ and b^+ describe the free branching of particles $\gamma^\pm$ independently of each other. Additionally, each particle $y \in \gamma^-$ may create a new particle at position $x \in \mathbb{R}^d$ of opposite type. The spatial distribution of the new particle is determined by the function φ^+. By definition, such events occur independently of each

other. The term including $z \geq 0$ describes the additional creation of particles by an outer source. Suppose that $a^{\pm}, b^{\pm}, \varphi^{\pm}$ are non-negative, symmetric and integrable.

Theorem 5.1. *Suppose that $a^{\pm}, b^{\pm}, \varphi^{\pm}$ are bounded and there exist constants $b_1, b_2 \geq 0$ and $\vartheta_1, \vartheta_2, \vartheta_3 > 0$ such that*

$$\sum_{x \in \eta^+} \sum_{y \in \eta^+ \setminus x} b^+(x-y) \leq \vartheta_1 \sum_{x \in \eta^+} \sum_{y \in \eta^+ \setminus x} b^-(x-y) + b_1 |\eta^+|,$$

$$\sum_{x \in \eta^-} \sum_{y \in \eta^- \setminus x} a^+(x-y) \leq \vartheta_2 \sum_{x \in \eta^-} \sum_{y \in \eta^- \setminus x} a^-(x-y) + b_2 |\eta^-|,$$

and $\varphi^+ \leq \vartheta_3 \varphi^-$ hold. Moreover, assume that the parameters satisfy the relations $\vartheta_1, \vartheta_3 < e^{\alpha}, \vartheta_2 < e^{\beta}$,

$$m^+ > e^{\alpha} \langle b^- \rangle + e^{\beta} \langle \varphi^- \rangle + e^{-\alpha} b_1 + \langle b^+ \rangle + \langle \varphi^+ \rangle,$$

$$m^- > e^{\beta} \langle a^- \rangle + e^{-\beta} (b_2 + z) + \langle a^+ \rangle.$$

Then conditions (A)–(D) hold for $\tau = 0$ and (3.5) is fulfilled.

Remark 5.2. *Note that the one-component case has been considered in [11, 23].*

Suppose that the conditions given above are fulfilled. Then (V1)–(V3) are satisfied and after Vlasov scaling we arrive at the kinetic equations

$$\frac{\partial \rho_t^-}{\partial t}(x) = -m^- \rho_t^-(x) - \rho_t^-(x)(a^- * \rho_t^-)(x) + (a^+ * \rho_t^-)(x) + z,$$

$$\frac{\partial \rho_t^+}{\partial t}(x) = -\left(m^+ + (\varphi^- * \rho_t^-)(x)\right) \rho_t^+(x) - \rho_t^+(x)(b^- * \rho_t^+)(x)$$
$$+ (b^+ * \rho_t^+)(x) + (\varphi^+ * \rho_t^-)(x).$$

Here $(f * g)(x) := \int_{\mathbb{R}^d} f(x-y) g(y) \, dy$ denotes the usual convolution of functions on $\mathbb{R}^d$.

5.2. **Two interacting Glauber-models.** Suppose that the death intensities are given by

$$d^-(x, \gamma^+, \gamma^- \setminus x) = \exp\left(-s E_{\psi^+}(x, \gamma^+)\right),$$
$$d^+(x, \gamma^+ \setminus x, \gamma^-) = \exp\left(-s E_{\psi^-}(x, \gamma^-)\right),$$

where $s \in [0, \frac{1}{2}]$ and ψ^+, ψ^- are symmetric, non-negative and integrable. The birth intensities are assumed to be of the form

$$b^-(x, \gamma) = z^- \exp\left(-(1-s) E_{\psi^+}(x, \gamma^+)\right) \exp\left(-E_{\phi^-}(x, \gamma^-)\right),$$
$$b^+(x, \gamma) = z^+ \exp\left(-(1-s) E_{\psi^-}(x, \gamma^-)\right) \exp\left(-E_{\phi^+}(x, \gamma^+)\right),$$

where $z^-, z^+ > 0$ and ϕ^-, ϕ^+ are assumed to be non-negative, symmetric and integrable. This model describes the time-evolution of two interacting types of particles. In contrast to the previous model, the branching of this particles is replaced by an additional source. The next theorem provides an evolution of states.

Theorem 5.3. *Let $\phi^+, \phi^-, \psi^+, \psi^-$ be symmetric, non-negative and integrable and assume that the paramters satisfy the relations*

(5.2)
$$e^{e^{\alpha} C(s\psi^+)} + e^{-\beta} z^- e^{e^{\alpha} C((1-s)\psi^+)} e^{e^{\beta} C(\phi^-)} < 2,$$

(5.3)
$$e^{e^{\beta} C(s\psi^-)} + e^{-\alpha} z^+ e^{e^{\beta} C((1-s)\psi^-)} e^{e^{\alpha} C(\phi^+)} < 2.$$

Then conditions (A)–(D) are satisfied for $\tau = 0$ and (3.5) holds.

In the case $s = 0$ conditions (5.2) and (5.3) simplify to

$$(5.4) \qquad\qquad z^- e^{e^\alpha C(\psi^+)} e^{e^\beta C(\phi^-)} < e^\beta,$$

$$(5.5) \qquad\qquad z^+ e^{e^\beta C(\psi^-)} e^{e^\alpha C(\phi^+)} < e^\alpha.$$

Of particular interest is the special case $\phi^+ = 0 = \phi^-$, also known as the Widom-Rowlinson model. The non-equilibrium dynamics for this model has recently been analysed in [13], but without conditions (5.4) and (5.5) only existence of a local evolution of correlation functions could have been shown. Conditions (5.4) and (5.5) are satisfied for $e^{-\alpha} = C(\psi^+)$ and $e^{-\beta} = C(\psi^-)$ if

$$z^- < \frac{1}{eC(\psi^-)} \quad \text{and} \quad z^+ < \frac{1}{eC(\psi^+)}$$

are satisfied. An extension of such model with density dependent mutation rates (i.e. particles can change their types) has been recently considered in [19].

Vlasov scaling. For simplicity we consider the case $s = 0$, hence the death intensities need not to be scaled, i.e. are given by

$$d^-(x, \gamma^+, \gamma^- \backslash x) = 1 = d^+(x, \gamma^+ \backslash x, \gamma^-).$$

The scaled birth intensities are given by

$$b_n^-(x, \gamma) = z^- \exp\left(-\frac{1}{n} E_{\psi^+}(x, \gamma^+)\right) \exp\left(-\frac{1}{n} E_{\phi^-}(x, \gamma^-)\right),$$

$$b_n^+(x, \gamma) = z^+ \exp\left(-\frac{1}{n} E_{\psi^-}(x, \gamma^-)\right) \exp\left(-\frac{1}{n} E_{\phi^+}(x, \gamma^+)\right).$$

All previous results can be applied which yields the mesoscopic equations, cf. (4.2)

$$(5.6) \qquad \frac{\partial \rho_t^-}{\partial t}(x) = -\rho_t^-(x) + z^- e^{-(\phi^- * \rho_t^-)(x)} e^{-(\psi^+ * \rho_t^+)(x)},$$

$$(5.7) \qquad \frac{\partial \rho_t^+}{\partial t}(x) = -\rho_t^+(x) + z^+ e^{-(\phi^+ * \rho_t^+)(x)} e^{-(\psi^- * \rho_t^-)(x)}.$$

5.3. **BDLP-model in Glauber environment.** Let us consider death intensities given by

$$d^-(x, \gamma^+, \gamma^- \backslash x) = 1,$$
$$d^+(x, \gamma^+ \backslash x, \gamma^-) = m^+ + \sum_{y \in \gamma^+ \backslash x} a^-(x - y) + \sum_{y \in \gamma^-} \phi(x - y),$$

where $m^+ > 0$ and $0 \le a^-, \phi \in L^1(\mathbb{R}^d)$ are symmetric. The birth intensities are assumed to be of the form

$$b^-(x, \gamma) = z^- \exp\left(-E_\psi(x, \gamma^-)\right),$$
$$b^+(x, \gamma) = \sum_{y \in \gamma^+} a^+(x - y) + \sum_{y \in \gamma^-} b^+(x - y),$$

where $z^- > 0$ and $0 \le \psi, a^+, b^+ \in L^1(\mathbb{R}^d)$ are symmetric.

Theorem 5.4. *Suppose that $a^\pm, b^+, \phi$ are bounded and there exist $\theta \in (0, e^\alpha)$ and $b \ge 0$ such that*

$$(5.8) \qquad \sum_{x \in \eta^+} \sum_{y \in \eta^+ \backslash x} a^+(x - y) \le \theta \sum_{x \in \eta^+} \sum_{y \in \eta^+ \backslash x} a^-(x - y) + b|\eta^+|$$

is satisfied. Moreover, assume that for some $\vartheta \in (0, e^{\alpha})$ *and*

$$(5.9) \qquad\qquad \vartheta\phi \geq b^{+},$$

$$(5.10) \qquad\qquad e^{\beta} > z^{-} \exp\left(e^{\beta} C(\psi)\right),$$

$$(5.11) \qquad\qquad m^{+} > e^{\alpha}\langle a^{-}\rangle + e^{\beta}\langle\phi\rangle + \langle a^{+}\rangle + \langle b^{+}\rangle + e^{-\alpha}b$$

hold. Then conditions (A)–(D) are satisfied with $\tau = 0$ *and (3.5) holds.*

Vlasov scaling. Suppose that $a^{\pm}, b^{+}, \phi, \psi$ are bounded and (5.8)–(5.11) with $z^{-}e^{e^{\beta}\langle\psi\rangle} < e^{\beta}$ instead of (5.10) hold. Scaling of the potentials by $\frac{1}{n}$ yields for the death

$$d^{-}(x, \gamma^{+}, \gamma^{-}\backslash x) = 1,$$

$$d_{n}^{+}(x, \gamma^{+}\backslash x, \gamma^{-}) = m^{+} + \frac{1}{n}\sum_{y\in\gamma^{+}\backslash x} a^{-}(x-y) + \frac{1}{n}\sum_{y\in\gamma^{-}}\phi(x-y).$$

For the birth we obtain

$$b_{n}^{-}(x, \gamma) = z^{-}\exp\left(-\frac{1}{n}E_{\psi}(x, \gamma^{-})\right),$$

$$b_{n}^{+}(x, \gamma) = \frac{1}{n}\sum_{y\in\gamma^{+}} a^{+}(x-y) + \frac{1}{n}\sum_{y\in\gamma^{-}} b^{+}(x-y).$$

Taking $n \to \infty$ yields $D_{x}^{V,-}(\eta) = 0^{|\eta|}$ and

$$D_{x}^{V,+}(\eta) = 0^{|\eta|}m^{+} + 0^{|\xi^{-}|}\mathbb{I}_{\Gamma^{(1)}}(\xi^{+})\sum_{y\in\xi^{+}} a^{-}(x-y) + 0^{|\xi^{+}|}\mathbb{I}_{\Gamma^{(1)}}(\xi^{-})\sum_{y\in\xi^{-}}\phi(x-y).$$

We obtain for the birth intensities

$$B_{x}^{V,-}(\eta) = z^{-}e_{\lambda}\left(-\psi(x-\cdot); \xi^{-}\right)0^{|\xi^{+}|},$$

$$B_{x}^{V,+}(\eta) = 0^{|\xi^{-}|}\mathbb{I}_{\Gamma^{(1)}}(\xi^{+})\sum_{y\in\xi^{+}} a^{+}(x-y) + 0^{|\xi^{+}|}\mathbb{I}_{\Gamma^{(1)}}(\xi^{-})\sum_{y\in\xi^{-}} b^{+}(x-y).$$

The kinetic equation is therefore given by

$$\frac{\partial\rho_{t}^{-}}{\partial t}(x) = -\rho_{t}^{-}(x) + z^{-}e^{-(\psi*\rho_{t}^{-})(x)},$$

$$\frac{\partial\rho_{t}^{+}}{\partial t}(x) = -\left(m^{+} + (\phi*\rho_{t}^{-})(x)\right)\rho_{t}^{+}(x) - \rho_{t}^{+}(x)(a^{-}*\rho_{t}^{+})(x)$$

$$+ (a^{+}*\rho_{t}^{+})(x) + (b^{+}*\rho_{t}^{-})(x).$$

5.4. Density dependent branching in Glauber environment. Suppose that the death intensities are given by

$$d^{-}(x, \gamma^{+}, \gamma^{-}\backslash x) = 1,$$

$$d^{+}(x, \gamma^{+}\backslash x, \gamma^{-}) = m^{+}\exp\left(E_{\phi^{+}}(x, \gamma^{+}\backslash x)\right),$$

where $m^{+} > 0$. The birth intensities are given by

$$b^{-}(x, \gamma) = z^{-}\exp\left(-E_{\phi^{-}}(x, \gamma^{-})\right),$$

$$b^{+}(x, \gamma) = \sum_{y\in\gamma^{+}}\exp\left(-E_{\psi^{-}}(y, \gamma^{-})\right)a^{+}(x-y)$$

with $z^{-} > 0$ and $a^{+}, \phi^{-}, \phi^{+}, \psi^{-}$ symmetric, non-negative and integrable.

Theorem 5.5. *Suppose that* $0 \neq \phi^+, a^+$ *are bounded, there exist constants* $\kappa > 0$ *and* $b \geq 0$ *such that for all* $\eta^+ \in \Gamma_0$

$$(5.12) \qquad \sum_{x \in \eta^+} \sum_{y \in \eta^+ \setminus x} a^+(x-y) \leq \vartheta \sum_{x \in \eta^+} \sum_{y \in \eta^+ \setminus x} \phi^+(x-y) + b|\eta^+|$$

and the parameters satisfy the relations

$$e^\beta > z^- \exp\left(e^\beta C(\phi^-)\right),$$

$$2 > e^{e^\alpha C(-\phi^+)} + \frac{\max\{\langle a^+ \rangle + b e^{-\alpha}, \vartheta e^{-\alpha}\}}{m^+} e^{e^\beta C(\psi^-)}.$$

Then conditions (A)–(D) hold with $\tau = \|\phi^+\|_\infty$ *and (3.5) is satisfied.*

Vlasov scaling. Scaling all potentials by $\frac{1}{n}$ gives $d^-(x, \gamma^+, \gamma^- \setminus x) = 1$,

$$d^+(x, \gamma^+ \setminus x, \gamma^-) = m^+ \exp\left(\frac{1}{n} E_{\phi^+}(x, \gamma^+ \setminus x)\right)$$

and for the birth intensities

$$b^-(x, \gamma) = z^- \exp\left(-\frac{1}{n} E_{\phi^-}(x, \gamma^-)\right),$$

$$b^+(x, \gamma) = \frac{1}{n} \sum_{y \in \gamma^+} \exp\left(-\frac{1}{n} E_{\psi^-}(y, \gamma^-)\right) a^+(x-y).$$

Suppose that $0 \neq \phi^+, a^+, \phi^-, \psi^-, a^+$ are bounded, (5.12) holds and the parameters satisfy the stronger relations

$$e^\beta > z^- \exp\left(e^\beta \langle \phi^- \rangle\right),$$

$$2 > e^{e^\alpha \langle \phi^+ \rangle} + \frac{\max\{\langle a^+ \rangle + b e^{-\alpha}, \vartheta e^{-\alpha}\}}{m^+} e^{e^\beta \langle \psi^- \rangle}.$$

Then conditions (V1)–(V3) are satisfied. This yields the kinetic equations

$$\frac{\partial \rho_t^-}{\partial t}(x) = -\rho_t^-(x) + z^- e^{-(\phi^- * \rho_t^-)(x)},$$

$$\frac{\partial \rho_t^+}{\partial t}(x) = -m^+ \rho_t^+(x) e^{(\phi^+ * \rho_t^+)(x)} + (a^+ * \rho_t^+)(x) e^{-(\psi^- * \rho_t^-)(x)}.$$

References

1. W. Arendt and A. Rhandi, *Perturbation of positive semigroups*, Arch. Math. (Basel) **56** (1991), no. 2, 107–119.
2. J. M. Ball, *Strongly continuous semigroups, weak solutions, and the variation of constants formula*, Proc. Amer. Math. Soc. **63** (1977), no. 2, 370–373.
3. J. Banasiak and L. Arlotti, *Perturbations of positive semigroups with applications*, Springer Monographs in Mathematics, Springer-Verlag, London, 2006.
4. B. Bolker, S. Cornell, D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and O. Ovaskainen, *A general mathematical framework for the analysis of spatio-temporal point processes*, Theoretical Ecology **7** (2014), no. 1, 101–113.
5. B. Bolker and S. W. Pacala, *Spatial moment equations for plant competition: Understanding spatial strategies and the advantages of short dispersal*, The American Naturalist **153** (1999), no. 6, 575–602.
6. U. Dieckmann and R. Law, *Relaxation projections and the method of moments*, The geometry of ecological interactions: simplifying spatial complexity, Cambridge University Press, Cambridge, 2005, pp. 412–455.

7. K. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.

8. D. Finkelshtein, *Functional evolutions for homogeneous stationary death-immigration spatial dynamics*, Methods Funct. Anal. Topology **17** (2011), no. 4, 300–318.

9. D. Finkelshtein, M. Friesen, H. Hatzikirou, Y. Kondratiev, T. Krüger, and O. Kutoviy, *Stochastic models of tumour development and related mesoscopic equations*, Inter. Stud. Comp. Sys. **7** (2015), 5–85.

10. D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, *Vlasov scaling for stochastic dynamics of continuous systems*, J. Stat. Phys. **141** (2010), no. 1, 158–178.

11. D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, *Semigroup approach to birth-and-death stochastic dynamics in continuum*, J. Funct. Anal. **262** (2012), no. 3, 1274–1308.

12. D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, *Statistical dynamics of continuous systems: perturbative and approximative approaches*, Arab. J. Math. **4** (2015), no. 4, 255–300.

13. D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and M. J. Oliveira, *Dynamical Widom-Rowlinson model and its mesoscopic limit*, J. Stat. Phys. **158** (2015), no. 1, 57–86.

14. D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and E. Zhizhina, *An approximative approach for construction of the Glauber dynamics in continuum*, Math. Nachr. **285** (2012), no. 2-3, 223–235.

15. D. L. Finkel'shtein, *On convolutions on configuration spaces. I. Spaces of finite configurations*, Ukrainian Math. J. **64** (2013), no. 11, 1752–1775.

16. D. L. Finkelshtein, *Measures on two-component configuration spaces*, Cond. Matt. Phys. **12** (2015), no. 1, 5–18.

17. D. L. Finkelshtein, Y. G. Kondratiev, and M. J. Oliveira, *Markov evolutions and hierarchical equations in the continuum. II: Multicomponent systems*, Rep. Math. Phys. **71** (2013), no. 1, 123–148.

18. M. Friesen, *Non-autonomous interacting particle systems in continuum*, Methods Funct. Anal. Topology **22** (2016), no. 3, 220–244.

19. M. Friesen, *Non-equilibrium dynamics for a widom-rowlinson type model with mutations*, (2016), `arXiv:1609.01929`.

20. M. Friesen and Y. Kondratiev, *Weak-coupling limits in ergodic environments*, (2016), in preperation.

21. N. L. Garcia and T. G. Kurtz, *Spatial birth and death processes as solutions of stochastic equations*, ALEA Lat. Am. J. Probab. Math. Stat. **1** (2006), 281–303.

22. H.-O. Georgii and O. Häggström, *Phase transition in continuum Potts models*, Comm. Math. Phys. **181** (1996), no. 2, 507–528.

23. Y. Kondratiev and Y. Kozitsky, *The evolution of states in a spatial population model*, J. Dyn. Diff. Equat. **28** (2016), no. 1, 1–39.

24. Y. Kondratiev, O. Kutoviy, and R. Minlos, *On non-equilibrium stochastic dynamics for interacting particle systems in continuum*, J. Funct. Anal. **255** (2008), no. 1, 200–227.

25. Y. Kondratiev, O. Kutoviy, and S. Pirogov, *Correlation functions and invariant measures in continuous contact model*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **11** (2008), no. 2, 231–258.

26. Y. Kondratiev and A. Skorokhod, *On contact processes in continuum*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9** (2006), no. 2, 187–198.

27. Y. G. Kondratiev and T. Kuna, *Harmonic analysis on configuration space. I. General theory*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5** (2002), no. 2, 201–233.

28. L. D. Lemle, *Existence and uniqueness for C_0-semigroups on the dual of a Banach space*, Carpathian J. Math. **26** (2010), no. 1, 67–76.

29. A. Lenard, *Correlation functions and the uniqueness of the state in classical statistical mechanics*, Comm. Math. Phys. **30** (1973), 35–44.

30. A. Lenard, *States of classical statistical mechanical systems of infinitely many particles. II. Characterization of correlation measures*, Arch. Rational Mech. Anal. **59** (1975), no. 3, 241–256.

31. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

32. C. Preston, *Spatial birth-and-death processes*, Bull. Inst. Internat. Statist. **46** (1975), no. 2, 371–391.

33. D. Steinsaltz, S. N. Evans, and K. W. Wachter, *A generalized model of mutation-selection balance with applications to aging*, Adv. in Appl. Math. **35** (2005), no. 1, 16–33.

34. H. R. Thieme and J. Voigt, *Stochastic semigroups: their construction by perturbation and approximation*, Positivity IV—theory and applications, Tech. Univ. Dresden, Dresden, 2006, pp. 135–146.

35. L. Wu and Y. Zhang, *Existence and uniqueness of C_0-semigroup in L^∞: a new topological approach*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 8, 699–704.
36. L. Wu and Y. Zhang, *A new topological approach to the L^∞-uniqueness of operators and the L^1-uniqueness of Fokker-Planck equations*, J. Funct. Anal. **241** (2006), no. 2, 557–610.

DEPARTMENT OF MATHEMATICS, BIELEFELD UNIVERSITY, GERMANY
E-mail address: `mfriesen@math.uni-bielefeld.de`

DEPARTMENT OF MATHEMATICS, BIELEFELD UNIVERSITY, GERMANY
E-mail address: `kutoviy@math.uni-bielefeld.de`