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FREDHOLM THEORY CONNECTED WITH A DOUGLIS-NIRENBERG SYSTEM OF DIFFERENTIAL EQUATIONS OVER $\mathbb{R}^n$

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ABSTRACT. We consider a spectral problem over $\mathbb{R}^n$ for a Douglis-Nirenberg system of differential operators under limited smoothness assumptions and under the assumption of parameter-ellipticity in a closed sector $\mathcal{L}$ in the complex plane with vertex at the origin. We pose the problem in an L_p Sobolev-Bessel potential space setting, $1 < p < \infty$, and denote by A_p the operator induced in this setting by the spectral problem. We then derive results pertaining to the Fredholm theory for A_p for values of the spectral parameter λ lying in $\mathcal{L}$ as well as results pertaining to the invariance of the Fredholm domain of A_p with p.

1. INTRODUCTION

The Fredholm properties of elliptic pseudodifferential operators, as well as systems of such operators, defined over $\mathbb{R}^n$ has been the subject of investigation by various authors over the past few decades. We refer for example to [10], [15], [17], [19], and to the references listed therein for further details. Of particular interest to us are the works dealing with Douglis-Nirenberg systems (cf. [10], [17], [18], [19], and [22]) as well as those dealing with parameter-elliptic operators (cf. [17]), since, as far as we are aware from an inspection of the literature, there are no works dealing with parameter-elliptic Douglis-Nirenberg systems over $\mathbb{R}^n$ in the classical L_p Sobolev-Bessel potential space setting, $1 < p < \infty$, when the diagonal operators are not all of the same order.

In light of what has just been said, let us mention at this point the paper [13] of Kozhevnikov wherein a Douglis-Nirenberg system of pseudodifferential operators acting over a compact manifold without boundary is considered. By posing the problem in a classical setting (as mentioned above) and by introducing the so-called Kohzevnikov conditions, the author was able to establish a priori estimates for solutions as well as various spectral results. Problems similar to those considered in [13] were also investigated by Denk, Mennicken, and Volevich [9], and by introducing conditions, which by the use of the Newton polygon they show to be equivalent to those of Kozhevnikov, they also establish a priori estimates for solutions as well as various spectral results.

Motivated by what has been said above, the object of this paper is to derive information concerning the Fredholm properties of the operator A_p induced in a L_p Sobolev-Bessel potential space setting, $1 < p < \infty$, by a spectral problem over $\mathbb{R}^n$ for a Douglis-Nirenberg system of differential operators under limited smoothness assumptions as well as under the assumption of parameter-ellipticity in a closed sector $\mathcal{L}$ of the complex plane with vertex at the origin. And fundamental to our work will be the uniform version of the Kozhevnikov conditions given in Definition 2.3 below which will enable us to derive information pertaining to that part of the Fredholm domain of A_p lying in $\mathcal{L}$.

Turning to the problem under consideration here, let $N \in \mathbb{N}$ with $N > 1$ and let $\{s_j\}_1^N$ and $\{t_j\}_1^N$ denote sequences of integers satisfying $s_1 \geq s_2 \geq \cdots \geq s_N \geq 0$, $t_1 \geq t_2 \geq \cdots \geq t_N \geq 0$, and put $m_j = s_j + t_j$ for $j = 1, \ldots, N$. We suppose that $m_1 = m_2 = \cdots = m_{k_1} > m_{k_1+1} = \cdots = m_{k_{d-1}} > m_{k_{d-1}+1} = \cdots = m_{k_d} > 0$, where $k_d = N$, and let $\tilde{I}_r$ denote the $(k_r - k_{r-1}) \times (k_r - k_{r-1})$ identity matrix for $r = 1, \ldots, d$, where $k_0 = 0$. We will also use the notation I_ℓ to denote the $\ell \times \ell$ unit matrix for $\ell \in \mathbb{N}$. Then we shall be concerned here with the spectral problem

(1.1) $$A(x, D)u(x) - \lambda u(x) = f(x) \quad \text{in} \quad \mathbb{R}^n,$$

where $u(x) = (u_1(x), \ldots, u_N(x))^T$, and $f(x) = (f_1(x), \ldots, f_N(x))^T$ are $N \times 1$ matrix functions defined in $\mathbb{R}^n$, ^{T} denotes transpose, $A(x, D)$ is an $N \times N$ matrix operator whose entries $A_{jk}(x, D)$ are linear differential operators defined on $\mathbb{R}^n$ of order not exceeding $s_j + t_k$ and defined to be 0 if $s_j + t_k < 0$. Our assumptions concerning the spectral problem (1.1) will be made precise in Section 2.

In Section 2 we make precise the concept of parameter-ellipticity for the spectral problem (1.1) and the conditions under which the problem is dealt with. In Section 3 we pose the spectral problem in an L_p Sobolev-Bessel potential space setting, $1 < p < \infty$, and obtain estimates for solutions for various values of the spectral parameter. Finally in Section 4 we we fix our attention upon the operator A_p induced in the L_p Sobolev-Bessel potential space setting just cited by the spectral problem (1.1), and using results from Section 3, we derive information concerning the Fredholm properties of A_p for various values of the spectral parameter lying in $\mathcal{L}$.

2. Preliminaries

In this section we are going to introduce some terminology, definitions, and assumptions concerning the spectral problem (1.1), which we require for our work.

Accordingly, we let $x = (x_1, \ldots, x_n) = (x', x_n)$ denote a generic point in $\mathbb{R}^n$ and use the notation $D_j = -i\partial/\partial x_j$, $D = (D_1, \ldots, D_n)$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = D'^{\alpha'} D_n^{\alpha_n}$, and $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$ for $\xi = (\xi_1, \ldots, \xi_n) = (\xi', \xi_n) \in \mathbb{R}^n$, where $\alpha = (\alpha_1, \ldots, \alpha_n) = (\alpha', \alpha_n)$ is a multi-index whose length $\sum_{j=1}^n \alpha_j$ is denoted by $|\alpha|$. Differentiation with respect to another variable, say $y \in \mathbb{R}^n$, instead of x will be indicated by replacing D and D^α by D_y and D_y^α, respectively. For $1 < p < \infty$, $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and G an open set in $\mathbb{R}^n$, we let $W_p^s(G)$ denote the Sobolev space of order s related to $L_p(G)$ and denote the norm in this space by $\| \cdot \|_{s,p,G}$, where $\|u\|_{s,p,G} = \left(\sum_{|\alpha| \leq s} \int_G |D^\alpha u(x)|^p \, dx \right)^{1/p}$ for $u \in W_p^s(G)$. In addition we shall use norms depending upon a parameter $\lambda \in \mathbb{C} \backslash \{0\}$, namely for $1 \leq j \leq N$, we let

$$|||u|||_{s,p,G}^{(j)} = \|u\|_{s,p,G} + |\lambda|^{s/m_j} \|u\|_{0,p,G} \quad \text{for} \quad u \in W_p^s(G).$$

In the sequel we shall at times deal with the Bessel-potential space $H_p^s(G)$ for $0 \geq s \in \mathbb{Z}$ and equipped with either its ordinary norm $\| \cdot \|_{s,p,G}$ or its parameter dependent norm $||| \cdot |||_{s,p,G}^{(j)}, 1 \leq j \leq N$. Here, for $u \in H_p^s(G)$, $\|u\|_{s,p,G} = \| F^{-1} \langle \xi \rangle^s F u \|_{0,p,\mathbb{R}^n}$ (resp. $|||u|||_{s,p,G}^{(j)} = \| F^{-1} \langle \xi, \lambda \rangle_j^s F u \|_{0,p,\mathbb{R}^n}$) if $G = \mathbb{R}^n$, while $\|u\|_{s,p,G} = \inf \| v \|_{s,p,\mathbb{R}^n}$ (resp. $|||u|||_{s,p,G}^{(j)} = \inf |||v|||_{s,p,\mathbb{R}^n}^{(j)}$) otherwise, where the infimum is taken over all $v \in H_p^s(\mathbb{R}^n)$ for which $u = v_{|_G}$, F denotes the Fourier transformation in $\mathbb{R}^n (x \to \xi)$, $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, $\lambda \in \mathbb{C} \backslash \{0\}$, and $\langle \xi, \lambda \rangle_j = \left(|\xi|^2 + |\lambda|^{\frac{2}{m_j}} \right)^{\frac{1}{2}}$ (see [11, Section 1], [21, p. 177]). Analogous definitions also hold for $s > 0$. However when $s \geq 0$, then $W_p^s(G)$ and $H_p^s(G)$ coincide algebraically and their norms, both ordinary and parameter dependent, are equivalent. Hence in the sequel, when dealing with the space $H_p^s(G)$ for $s \geq 0$, we shall suppose that

it is equipped with either the ordinary or parameter dependent norm of $W_p^s(G)$. Lastly, let $\mathbb{R}_+ = \{\, t \in \mathbb{R} | t > 0 \,\}$, $\mathbb{R}_- = \{\, t \in \mathbb{R} \,|\, t < 0 \,\}$.

Next for $\ell \in \mathbb{N}_0$, we will use the usual notation $C^\ell(\mathbb{R}^n)$ to denote the vector space consisting of all those functions ϕ which, together with their partial derivatives of order up to ℓ, are continuous on $\mathbb{R}^n$. In addition we let $C^\ell(\overline{\mathbb{R}^n})$ denote the subspace of $C^\ell(\mathbb{R}^n)$ consisting of all those functions $\phi \in C^\ell(\mathbb{R}^n)$ for which ϕ as well as its partial derivative of order up to ℓ are bounded and uniformly continuous on $\mathbb{R}^n$. Lastly for $\ell \in \mathbb{N}$ we let $C^{\ell,0}(\overline{\mathbb{R}^n})$ denote the subspace of $C^\ell(\overline{\mathbb{R}^n})$ for which $|D^\alpha \phi(x)| \to 0$ as $|x| \to \infty$ for $1 \le |\alpha| \le \ell$, while for $\ell = 0$ we let $C^{0,0}(\overline{\mathbb{R}^n})$ denote the subspace of $C^0(\overline{\mathbb{R}^n})$ consisting of those functions ϕ for which $\omega_\phi(x) \to 0$ as $|x| \to \infty$, where $\omega_\phi(x) = \sup |\phi(x) - \phi(y)|$ for $x, y \in \mathbb{R}^n$, and where for each x the supremum is taken over those values of y for which $|x - y| \le 1$. Note that for $\ell \ge 1, C^{\ell,0}(\overline{\mathbb{R}^n}) \subset C^{0,0}(\overline{\mathbb{R}^n})$.

Turning now to the spectral problem (1.1), (1.2), let us write

$$A_{jk}(x, D) = \sum_{|\alpha| \le s_j + t_k} a_\alpha^{jk}(x) D^\alpha \quad \text{for} \quad x \in \mathbb{R}^n \quad \text{and} \quad 1 \le j, k \le N.$$

Then as pointed out in [3] there is no loss of generality in making the following assumptions.

Assumption 2.1. It will henceforth be supposed that $t_j > 0$ and $s_j \ge 0$ for $j = 1, \ldots, N$.

Definition 2.2. We say that the spectral problem (1.1) is minimally smooth if for each pair j, k, $a_\alpha^{jk} \in C^{s_j}(\overline{\mathbb{R}^n})$ for $|\alpha| \le s_j + t_k$ if $s_j > 0$, while if $s_j = 0$, then $a_\alpha^{jk} \in L_\infty(\mathbb{R}^n)$ for $|\alpha| < t_k$ and $a_\alpha^{jk} \in C^0(\overline{\mathbb{R}^n})$ for $|\alpha| = t_k$.

For $x, \xi \in \mathbb{R}^n$ let

$$\mathring{A}(x, \xi) = \left(\mathring{A}_{jk}(x, \xi) \right)_{j,k=1}^N,$$

where $\mathring{A}_{jk}(x, \xi)$ consists of those terms in $A_{jk}(x, \xi)$ which are just of order $s_j + t_k$. Then in the sequel we shall also require the following notation. For $x, \xi \in \mathbb{R}^n$ let

$$\mathcal{A}_{11}^{(r)}(x, \xi) = \left(\mathring{A}_{jk}(x, \xi) \right)_{j,k=1}^{k_r} \quad \text{for} \quad 1 \le r \le d.$$

In addition we let $\tilde{I}_{r,0} = \operatorname{diag}(0 \cdot \tilde{I}_1, \ldots, 0 \cdot \tilde{I}_{r-1}, \tilde{I}_r)$ for $r = 2, \ldots, d$ and $\tilde{I}_{1,0} = \tilde{I}_1$.

Definition 2.3. Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin. Then the spectral problem (1.1) will be called parameter-elliptic in $\mathcal{L}$ if it is minimally smooth and for each $r, 1 \le r \le d$,

$$(2.1) \qquad\qquad \left| \det \left(\mathcal{A}_{11}^{(r)}(x, \xi) - \lambda \tilde{I}_{r,0} \right) \right| \ge \kappa_r |\xi|^{N_{r-1}}$$

for $x, \xi \in \mathbb{R}^n$, and $\lambda \in \mathcal{L}$ with $\langle \xi, \lambda \rangle_{k_r} = 1$, where the κ_r denote positive constants and $N_r = \sum_{j=1}^{k_r} m_j$ for $r \ge 1$ and $N_0 = 0$.

Remark 2.4. It follows from the arguments of [5, Proposition 2.2] that when (2.1) holds, then N_r is even if $r = 1$ and if $r > 1$ and $n > 2$.

Definition 2.5. We say that the spectral problem (1.1) is weakly smooth if it is minimally smooth and in addition $a_\alpha^{jk} \in C^{0,0}(\overline{\mathbb{R}^n})$ if $|\alpha| \le t_k$ and to $C^{|\alpha|-t_k,0}(\overline{R^n})$ otherwise, $1 \le j, k \le N$.

3. Some estimates

In this section we are going to establish some a priori estimates for solutions of the spectral problem (1.1), which will be used in the sequel.

Accordingly, let us introduce some further notation. For G an open set in $\mathbb{R}^n$ and $\tau = t$ or s we let $W_p^{(\tau)}(G) = \prod_{k=1}^N W_p^{\tau_k}(G)$ and equip $W_p^{(\tau)}(G)$ with either its ordinary norm, $\| u \|_{(\tau),p,G} = \sum_{k=1}^N \| u_k \|_{\tau_k,p,G}$ or its parameter dependent norm $\||u\||_{(\tau),p,G} = \sum_{k=1}^N \||u_k\||_{\tau_k,p,G}^{(k)}$ for $u = (u_1,\dots,u_N)^T \in W_p^{(\tau)}(G)$. The subspace $\prod_{k=1}^N \mathring{W}_p^{\tau_k}(G)$ is denoted by $\mathring{W}_p^{(\tau)}(G)$. In addition we let $H_p^{(-s)}(G) = \prod_{k=1}^N H_p^{-s_k}(G)$ and equip this space with either its ordinary norm $\| \cdot \|_{(-s),p,G}$ or its parameter dependent norm $\||\cdot\||_{(-s),p,G}$, which are defined in an analogous manner to the way they were defined for $W_p^{(\tau)}(G)$.

Proposition 3.1. *Suppose that the spectral problem* (1.1) *is minimally smooth. Suppose also that* $\lambda \in \mathbb{C}$ *with* $|\lambda| \geq \lambda^\sharp$ *for some* $\lambda^\sharp \in \mathbb{R}_+$, *that* $u \in W_p^{(t)}(\mathbb{R}^n)$, *and that* f *is defined by* (1.1). *Then* $f \in H_p^{(-s)}(\mathbb{R}^n)$ *and* $\||f\||_{(-s),p,\mathbb{R}^n} \leq C\||u\||_{(t),p,\mathbb{R}^n}$, *where the constant* C *does not depend upon* u *and* λ.

Proof. Let $\langle \cdot,\cdot \rangle_{\mathbb{R}^n}$ denote the pairing between $H_p^{(-s)}(\mathbb{R}^n)$ and its adjoint space $W_{p'}^{(s)}(\mathbb{R}^n)$, where $p' = p/(p-1)$ and both spaces are equipped with their parameter dependent norms (see [11, Section 1] and [21, Theorem 2.6.1, p. 198]). Then for $u \in W_p^{(t)}(\mathbb{R}^n)$ and $\zeta = (\zeta_1,\dots,\zeta_N)^T \in C_0^\infty(\mathbb{R}^n)^N$,

$$\langle (A(x,D)u - \lambda\,u)\,,\zeta\rangle_{\mathbb{R}^n} = \sum_{j=1}^N \left(\sum_{k=1}^N A_{jk}(x,D)u_k - \lambda\,\delta_{jk}u_k \right)(\overline{\zeta_j}),$$

where δ_{jk} denotes the Kronecker delta, and for each j

$$\left(\sum_{k=1}^N A_{jk}(x,D)u_k - \lambda\,\delta_{jk}u_k \right)(\overline{\zeta_j})$$

denotes the value of the distribution $\sum_{k=1}^N A_{jk}(x,D)u_k - \lambda\,\delta_{jk}u_k$ at $\overline{\zeta_j}$.

Let us fix our attention upon a particular pair j,k. Then

$$(A_{jk}(x,D)u_k - \lambda\,\delta_{jk}u_k)(\overline{\zeta_j}) = \sum_{\substack{t_k \leq |\alpha| \leq s_j + t_k \\ |\beta| = t_k}} \left(D^\beta u_k, D^{\alpha-\beta}\overline{a_\alpha^{jk}(x)}\zeta_j \right)_{\mathbb{R}^n}$$

$$+ \left(\sum_{|\alpha| < t_k} a_\alpha^{jk}(x)D^\alpha u_k, \zeta_j \right)_{\mathbb{R}^n} - \lambda\,\delta_{jk}\,(u_k,\zeta_j)_{\mathbb{R}^n},$$

where $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the pairing between $L_p(\mathbb{R}^n)$ and its adjoint space $L_{p'}(\mathbb{R}^n)$. Hence

$$\left| (A_{jk}(x,D)u_k - \lambda\,\delta_{jk}u_k)(\overline{\zeta_j}) \right| \leq C\Big(\| u_k \|_{t_k,p,\mathbb{R}^n} \| \zeta_j \|_{s_j,p',\mathbb{R}^n}$$

$$+ |\lambda|^{t_k/m_k}|\lambda|^{s_j/m_j} \| u_k \|_{0,p,\mathbb{R}^n} \| \zeta_j \|_{0,p',\mathbb{R}^n} \Big)$$

$$\leq C\||u_k\||_{t_k,p,\mathbb{R}^n}^{(k)}\||\zeta_j\||_{s_j,p',\mathbb{R}^n}^{(j)},$$

where the constant C does not depend upon u_k, ζ_j, and λ. It now follows from the foregoing results that

$$\left| \langle (A(x,D) - \lambda\,I_N)\,u,\zeta\rangle_\Omega \right| \leq C\||u\||_{(t),p,\mathbb{R}^n}\||\zeta\||_{(s),p',\mathbb{R}^n},$$

and the assertion of the proposition follows immediately from this last result. □

We now turn to the main results of this section.

Proposition 3.2. *Suppose that the spectral problem* (1.1) *is parameter-elliptic in* $\mathcal{L}$. *Suppose also that* $u \in W_p^{(t)}(\mathbb{R}^n)$ *and that* f *is defined by* (1.1). *Then there exists the constant* $\lambda' = \lambda'(p) > 0$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \geq \lambda'$, *the a priori estimate*

$$(3.1) \qquad\qquad |||u|||_{(t),p,\mathbb{R}^n} \leq C|||f|||_{(-s),p,\mathbb{R}^n}$$

holds, where the constant C *does not depend upon* u *and* λ.

Proposition 3.3. *Suppose that the spectral problem* (1.1) *is parameter-elliptic in* $\mathcal{L}$. *Then there exists the constant* $\lambda^0 = \lambda^0(p) > 0$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \geq \lambda^0$, *the spectral problem has a unique solution* $u \in W_p^{(t)}(\mathbb{R}^n)$ *for every* $f \in H_p^{(-s)}(\mathbb{R}^n)$, *and the a priori estimate*

$$|||u|||_{(t),p,\mathbb{R}^n} \leq C|||f|||_{(-s),p,\mathbb{R}^n}$$

holds, where the constant C *does not depend upon* f *and* λ.

The proofs of Propositions 3.2 and 3.3 can be achieved by modifying the arguments given in the proofs of Theorems 4.1 and 5.1 of [5]. Furthermore, since the proof of Proposition 3.3 is somewhat similar to that of Proposition 3.2, we will restrict ourselves to the proof of this latter proposition.

Before beginning the proof, let us now present some results which we require below.

Proposition 3.4. *Suppose that the spectral problem* (1.1) *is parameter-elliptic in* $\mathcal{L}$ *and that* $f \in H_p^{(-s)}(\mathbb{R}^n)$. *Suppose also that* $x^0 \in \mathbb{R}^n$. *Then there exits the constant* $\lambda_1 = \lambda_1(p) > 0$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \geq \lambda_1$, *the equation*

$$\mathring{A}(x^0, D)u(x) - \lambda\, u(x) = f(x) \quad for \quad x \in \mathbb{R}^n$$

has a unique solution $u \in W_p^{(t)}(\mathbb{R}^n)$ *and the a priori estimate*

$$|||u|||_{(t),p,\mathbb{R}^n} \leq C_1|||f|||_{(-s),p,\mathbb{R}^n},$$

holds, where the constant C_1 *does not depend upon* x^0, f, *and* λ.

Proof. It follows from Definition 2.3 and [9] that there exists the constant $\lambda_0 > 0$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0, \xi \in \mathbb{R}^n$, and α a multi-index whose entries are either 0 or 1,

$$\left| \det \left(\mathring{A}(x^0, \xi) - \lambda\, I_N \right) \right| \geq C_0 \prod_{j=1}^{N} \langle \xi, \lambda \rangle_j^{m_j} \quad \text{and}$$

$$\left| \xi^\alpha D_\xi^\alpha \tilde{a}_{jk}(x^0, \xi, \lambda) \right| \leq C_0' \langle \xi, \lambda \rangle_k^{-s_k} \langle \xi, \lambda \rangle_j^{-t_j},$$

where the constants C_0 and C_0' do not depend upon $\alpha, x^0, \xi, \lambda, j, k$, and where we have written $\left(\mathring{A}(x^0, \xi) - \lambda\, I_N \right)^{-1} = \left(\tilde{a}_{jk}(x^0, \xi, \lambda) \right)_{j,k=1}^{N}$. The assertions of the proposition now follow from the same arguments as those used in the proof of [8, Proposition 3.2]. $\qquad\square$

Proposition 3.5. *Suppose that the spectral problem* (1.1) *is parameter-elliptic in* $\mathcal{L}$. *Then there exist the constants* $r_0 = r_0(p)$ *and* $\lambda_2 = \lambda_2(p)$ *in* $\mathbb{R}_+$ *such that for each* $x^0 \in \mathbb{R}^n$ *one can find a neighbourhood* V *of this point with* $\operatorname{diam} V \leq r_0$ *for which the a priori estimate*

$$|||u|||_{(t),p,\mathbb{R}^n} \leq C|||A(x,D)u - \lambda\, u|||_{(-s),p,\mathbb{R}^n}$$

holds for each $u \in W_p^{(t)}(\mathbb{R}^n)$ *and* $\lambda \in \mathcal{L}$ *for which* $\operatorname{supp} u \subset V$ *and* $|\lambda| \geq \lambda_2$, *where the constant* C *does not depend upon* u *and* λ, *and where* diam *denotes diameter and* supp *denotes support.*

Proof. To begin with let us choose r_0 so that for $x, y \in \mathbb{R}^n$ with $|x - y| \leq r_0$ we have for each triple (j, k, α) with $1 \leq j, k \leq N$ and $|\alpha| = s_j + t_k$, $|a_\alpha^{jk}(x) - a_\alpha^{jk}(y)| \leq 1/8C_1\delta_1$, where C_1 denotes the constant of Proposition 3.4, $\delta_1 = N^2 \max \left\{ \frac{(s_j + t_k)!}{s_j!} n_{jk} \right\}_{j,k=1}^N$ and n_{jk} denotes the number of distinct multi-indices α for which $|\alpha| = s_j + t_k$.

Next let $x^0 \in \mathbb{R}^n$ and let $V \subset \mathbb{R}^n$ be a neighbourhood of x^0 with diam $V \leq r_0$. Also let $u \in W_p^{(t)}(\mathbb{R}^n)$ such that supp $u \subset V$. Then bearing in mind the proof of Proposition 3.1 as well as Proposition 3.4, we see that

$$|||u|||_{(t),p,V} \leq C_1 |||\left(\mathring{A}(x^0, D) - \lambda I_N\right) u|||_{(-s),p,\mathbb{R}^n}$$

for λ and C_1 satisfying the conditions cited in Proposition 3.4. Hence it follows from arguments similar to those used in the proof of [8, Proposition 4.1] that there exists the constant $\lambda_2 = \lambda_2(p) > 0$ such that

$$|||u|||_{(t),p,V} \leq C_1' |||\left(A(x, D) - \lambda I_N\right) u|||_{(-s),p,\mathbb{R}^n} \quad \text{for} \quad \lambda \in \mathcal{L} \quad \text{with} \quad |\lambda| \geq \lambda_2,$$

where the constant C_1' has the same properties as the constant C_1. On the other hand, if we refer to the proof of Proposition 3.1 for notation and argue as in that proof, then we obtain for $\zeta \in C_0^\infty(\mathbb{R}^n)^N$ and for values of λ just cited,

$$\langle \left(A(x, D) - \lambda I_N\right) u, \zeta \rangle_{\mathbb{R}^n} = \langle \left(A(x, D) - \lambda, I_N\right) u, \chi \zeta \rangle_V,$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $x \in$ supp u and supp $\chi \subset V$. Hence if we take into account [7, Proposition 3.5, p. 109], [21, Theorems 4.3.2.1, p. 317 and 4.8.2, p. 332], and the fact that $\chi \zeta \in \mathring{W}_{p'}^{(s)}(V)$, it follows that

$$\left| \langle \left(A(x, D) - \lambda I_N\right) u, \zeta \rangle_{\mathbb{R}^n} \right| \leq |||\left(A(x, D) - \lambda I_N\right) u|||_{(-s),p,V} |||\chi \zeta|||_{(s),p',V}$$
$$\leq C_0 |||\left(A(x, D) - \lambda I_N\right) u|||_{(-s),p,V} |||\zeta|||_{(s),p',\mathbb{R}^n},$$

where the constant C_0 does not depend upon u and λ. This completes the proof of the Proposition. $\square$

Proof of Proposition 3.2. For $t > 0$ let $Q(t)$ denote the open cube in $\mathbb{R}^n$ with centre at the origin and with sides parallel to the coordinate axes and of length $2t$. Also let $\zeta, \chi \in C_0^\infty(\mathbb{R}^n)$ such that $\zeta(x) = 1$ for $x \in Q(1/2)$ and $\zeta(x) = 0$ for $x \in \mathbb{R}^n \setminus Q(3/4)$, while $\chi(x) = 1$ for $x \in Q(13/16)$ and $\chi(x) = 0$ for $x \in \mathbb{R}^n \setminus Q(14/16)$. Then for $d > 0$ and $\gamma \in \mathbb{Z}^n$, we let $Q_{\gamma, d} = \left\{ x \in \mathbb{R}^n \,\middle|\, |x - d\gamma| \in Q(d) \right\}$, $\zeta_{\gamma, d}(x) = \zeta((x - d\gamma)/d)$, and $\chi_{\gamma, d}(x) = \chi((x - d\gamma)/d)$.

Next let d_0 denote a constant satisfying $0 < d_0 < r_0/4n^{1/2}$ (see Proposition 3.5). Then we shall henceforth fix $d \leq d_0$ and let $\{\gamma_j\}_1^\infty$ denote an enumeration of the members of $\mathbb{Z}^n$ and put $Q_j = Q_{\gamma_j, d}, \zeta_j(x) = \zeta_{\gamma_j, d}(x)$ and $\chi_j(x) = \chi_{\gamma_j, d}(x)$. Hence if for $j \geq 1$ we let $\eta_j(x) = \zeta_j(x)/\sum_{j=1}^\infty \zeta_j(x)$, then $\{Q_j\}_1^\infty$ is an open covering of $\mathbb{R}^n$ and $\{\eta_j(x)\}_1^\infty$ a partition of unity subordinate to this covering. Note also that for any $x \in \mathbb{R}^n$, x can lie in at most 2^n of the Q_j.

Bearing in mind Proposition 3.5 let $u \in W_p^{(t)}(\mathbb{R}^n)$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_2$. Then we have

$$|||u|||_{(t),p,\mathbb{R}^n} \leq N^{1/p'} \left(\sum_{j=1}^N |||\sum_{\ell \geq 1} \eta_\ell u_j|||_{t_j,p,\mathbb{R}^n}^{(j)\,p} \right)^{1/p},$$

which leads to the inequality

$$|||u|||_{(t),p,\mathbb{R}^n} \leq 2^{n+1} N \left(\sum_{\ell \geq 1} |||\eta_\ell u|||_{(t),p,Q_\ell}^p \right)^{1/p},$$

and hence it follows from Proposition 3.5 and its proof that

$$(3.2) \qquad |||u|||_{(t),p,\mathbb{R}^n} \leq C \left(\sum_{\ell \geq 1} |||(A(x,D) - \lambda I_N)\eta_\ell u|||^p_{(-s),p,Q_\ell} \right)^{1/p},$$

where here and for the remainder of this proof C denotes a generic constant which may vary from inequality to inequality, but in all cases it does not depend upon u, λ and ℓ (see below). Note that here we have used the fact that under our present definitions of V and χ, we can take the constant C of Proposition 3.5 to be also independent of x^0.

Let us firstly fix our attention upon a particular $\ell \geq 1$. Then observing that

$$||| (A(x,D) - \lambda I_N) \eta_\ell u - \eta_\ell (A(x,D) - \lambda I_N) u|||_{(-s),p,Q_\ell}$$
$$\leq ||| (A(x,D) - \lambda I_N) \eta_\ell u - \eta_\ell (A(x,D) - \lambda I_N) u|||_{(-s),p,\mathbb{R}^n},$$

and referring to the proof of Proposition 3.1 for notation, we can argue as in that proof to show that

$$\left| \left\langle \left((A(x,D) - \lambda I_N)\eta_\ell u - \eta_\ell (A(x,D) - \lambda I_N)u \right), \zeta \right\rangle_{\mathbb{R}^n} \right|$$
$$\leq C \sum_{j=1}^{N} \sum_{k=1}^{N} \left(\| \chi_\ell u_k \|_{t_k-1,p,Q_\ell} \| \chi_\ell \zeta_j \|_{s_j,p',Q_\ell} + \| \chi_\ell u_k \|_{t_k,p,Q_\ell} \| \chi_\ell \zeta_j \|_{s_j-1,p',Q_\ell} \right).$$

Hence it follows from [4, Proposition 2.2] and the fact that χ_ℓ has support in Q_ℓ that

$$(3.3) \qquad \begin{aligned} &\left| \left\langle \left((A(x,D) - \lambda I_N)\eta_\ell u - \eta_\ell (A(x,D) - \lambda I_N) u \right), \zeta \right\rangle_{\mathbb{R}^n} \right| \\ &\leq C|\lambda|^{-1/m_1} |||u|||_{(t),p,Q_\ell} |||\zeta|||_{(s),p',Q_\ell}. \end{aligned}$$

Thus we conclude from (3.3) that

$$||| (A(x,D) - \lambda I_N) \eta_\ell u|||_{(-s),p,\mathbb{R}^n} \leq$$
$$|||\eta_\ell (A(x,D) - \lambda I_N) u|||_{(-s),p,\mathbb{R}^n} + C|\lambda|^{-1/m_1} |||u|||_{(t),p,Q_\ell},$$

and hence that

$$(3.4) \qquad \begin{aligned} &\sum_{\ell \geq 1} || (A(x,D) - \lambda I_N) \eta_\ell u|||^p_{(-s),p,\mathbb{R}^n} \\ &\leq 2^p \sum_{\ell \geq 1} |||\eta_\ell (A(x,D) - \lambda I_N) u|||^p_{(-s),p,\mathbb{R}^n} + 2^p C^p |\lambda|^{-p/m_1} \sum_{\ell \geq 1} |||u|||^p_{(t),p,Q_\ell}. \end{aligned}$$

We are now going to use (3.2) and (3.4) to complete the proof of the proposition. To this end, some further preparation is required. Accordingly, let us recall that we have so far equipped the space $W_{p'}^{s_j}(\mathbb{R}^n), 1 \leq j \leq N$, with the parameter dependent norm $||| \cdot |||_{s_j,p',\mathbb{R}^n}$. For our purposes it will be convenient now to introduce an equivalent norm, namely the norm $||| \cdot |||'_{s_j,p',\mathbb{R}^n}$ defined by

$$|||v|||'_{s_j,p',\mathbb{R}^n} = \left(\| v \|^{p'}_{s_j,p',R^n} + \left[(1 + |\lambda|^{s_j/m_j})^{p'} - 1 \right] \| v \|^{p'}_{0,p',\mathbb{R}^n} \right)^{1/p'}$$

for $v \in W_{p'}^{s_j}(\mathbb{R}^n)$. Note that by the equivalence of these two norms we mean that there are constants C_1 and C_2, not depending upon v and λ, such that $|||v|||_{s_j,p',\mathbb{R}^n} \leq C_1 |||v|||'_{s_j,p',\mathbb{R}^n}$ and $|||v|||'_{s_j,p',\mathbb{R}^n} \leq C_2 |||v|||_{s_j,p',\mathbb{R}^n}$.

Supposing now that λ is fixed and $W_{p'}^{s_j}(\mathbb{R}^n)$ is equipped with the norm $||| \cdot |||'_{s_j,p',\mathbb{R}^n}$, let ν_j denote the number of distinct multi-indices α satisfying $0 \leq |\alpha| \leq s_j$. Then to each $v \in W_{p'}^{s_j}(\mathbb{R}^n)$ we can associate the vector $\mathcal{P}_{j,p'} v \in L_{p'}(\Omega)^{\nu_j}$, where $\mathcal{P}_{j,p'} v$ denotes the ν_j − vector with components $\left\{ \Lambda_{j,\alpha} D^\alpha \right\}_{|\alpha| \leq s_j}$, where $\Lambda_{j,\alpha} = 1$ if $|\alpha| \geq 1$ and equals $1 + |\lambda|^{s_j/m_j}$ otherwise, and where the components of $\mathcal{P}_{j,p'} v$ are arranged so that if $\Lambda_{j,\alpha} D^\alpha v$ is the ℓ-th component of this vector and $\Lambda_{j,\beta} D^\beta v$ the $(\ell+1)$-th component, then $|\alpha| \leq |\beta|$. Hence

if we let $\mathcal{H}_{j,p'}$ denote the closed subspace of $L_{p'}(\Omega)^{\nu_j}$ spanned by the vectors $\mathcal{P}_{j,p'}v$ for $v \in W_{p'}^{s_j}(\mathbb{R}^n)$, then $\mathcal{H}_{j,p'}$ is isometrically isomorphic to $W_{p'}^{s_j}(\mathbb{R}^n)$.

Turning next to the space $W_{p'}^{(s)}(\mathbb{R}^n)$, we have so far equipped this space with the parameter dependent norm $|||\cdot|||_{(s),p',\mathbb{R}^n}$. Let us also equip $W_{p'}^{(s)}(\mathbb{R}^n)$ with the norm $|||\cdot|||'_{(s),p',\mathbb{R}^n}$, which is equivalent to $|||\cdot|||_{(s),p',\mathbb{R}^n}$, defined by

$$|||v|||'^{p'}_{(s),p'\,\mathbb{R}^n} = \sum_{j=1}^{N} \left(|||v_j|||'_{s_j,p',\mathbb{R}^n} \right)^{p'}$$

for $v = (v_1,\ldots,v_N)^T \in W_{p'}^{(s)}(\mathbb{R}^n)$. Note that the norms $|||\cdot|||_{(s),p',\mathbb{R}^n}$ and $|||\cdot|||'_{(s),p'\mathbb{R}^n}$ induce equivalent norms on the adjoint space of $W_{p'}^{(s)}(\mathbb{R}^n)$, $H_p^{(-s)}(\mathbb{R}^n)$, and hence if we denote by $|||\cdot|||'_{(-s),p,\mathbb{R}^n}$ the norm induced on $H_p^{(-s)}(\mathbb{R}^n)$ by the norm $|||\cdot|||'_{(s),p',\mathbb{R}^n}$, then $|||\cdot|||'_{(-s),p,\mathbb{R}^n}$ and $|||\cdot|||_{(-s),p,\mathbb{R}^n}$ are equivalent norms on this space. Note also that when $W_{p'}^{(s)}(\mathbb{R}^n)$ is equipped with the norm $|||\cdot|||'_{(s),p',\mathbb{R}^n}$, then $W_{p'}^{(s)}(\mathbb{R}^n)$ is isometrically isomorphic to $\mathcal{H}_{p'}$, the closed subspace of $L_{p'}(\mathbb{R}^n)^{\sum_{j=1}^{N}\nu_j}$ defined by $\mathcal{H}_{p'} = \prod_{j=1}^{N}\mathcal{H}_{j,p'}$. Hence it follows from [1, Theorem 3.8, p. 49] that for each $u \in H_p^{(-s)}(\mathbb{R}^n)$, there is a $f \in L_p(\mathbb{R}^n)^{\sum_{j=1}^{N}\nu_j}$ such that $|||u|||'_{(-s),p,\mathbb{R}^n} = \|f\|$, where $\|\cdot\|$ denotes the norm in $L_p(\mathbb{R}^n)^{\sum_{j=1}^{N}\nu_j}$.

Fixing our attention upon (3.4) and referring to the proof of Proposition 3.1 for notation, we have for $\ell \geq 1$,

$$|||\eta_\ell\,(A(x,D) - \lambda\,I_N)\,u|||_{(-s),p,\mathbb{R}^n} \leq C|||\eta_\ell\,(A(x,D) - \lambda\,I_N)\,u|||'_{(-s),p,\mathbb{R}^n}$$
$$= C\,\sup\left|\langle\,(A(x,D) - \lambda\,I_N)\,u, \eta_\ell\zeta\,\rangle_{\mathbb{R}^n}\right|$$
$$= C\,\sup\Big|\sum_{j=1}^{N}\sum_{k=1}^{\nu_j}(f_{j,k}, \Lambda_{j,\alpha}D^\alpha\eta_\ell\zeta_j)_{\mathbb{R}^n}\Big|$$

for some $f \in L_p(\mathbb{R}^n)^{\sum_{j=1}^{N}\nu_j}$, where the supremum is taken over the set $\big\{\,\zeta \in C_0^\infty(\mathbb{R}^n)^N \,\big|\, |||\zeta|||'_{0,p,\mathbb{R}^n} = 1\,\big\}$, and where we have written $f = (f_1,\ldots,f_N)^T$, $f_j = (f_{j,1},\ldots,f_{j,\nu_j})^T$, and $\Lambda_{j,\alpha}D^\alpha\eta_\ell\zeta_j$ denotes the k-th component of $\mathcal{P}_{j,p'}\eta_\ell\zeta_j$. Hence it follows that

$$(3.5) \quad \sum_{\ell\geq1}|||\eta_\ell\,(A(x,D) - \lambda\,I_N)\,u|||^p_{(-s),p,\mathbb{R}^n} \leq C^p\sum_{\ell\geq1}\sum_{j=1}^{N}\sum_{k=1}^{\nu_j}\|\,f_{j,k}\,\|^p_{0,p,Q_\ell}$$
$$\leq C'|||\,(A(x,D) - \lambda\,I_N)\,|||^p_{(-s),p,\mathbb{R}^n},$$

where the constant C' has the same properties as the constant C. Hence it follows from equations (3.2), (3.4), and (3.5) that we may choose $\lambda'(p)$ so that the assertion of the proposition holds. $\qquad\square$

To terminate the work of this section, we now present a result which will be used in the sequel. Here, for $\ell \in \mathbb{N}$, we use the notation $(t + \ell)$ to denote the multi-index whose entries are $t_j + \ell, j = 1,\ldots,N$. The multi-indices $(-s + \ell), (-r_j + \ell - 1/p)$ are defined analogously.

Proposition 3.6. *Suppose that the spectral problem* (1.1) *is parameter-elliptic in $\mathcal{L}$ and that $\ell_0 \in \mathbb{N}$. Suppose also that for each pair $j, k, 1 \leq j, k \leq N$, $a_\alpha^{jk} \in C^{s_j+\ell_0}(\overline{\mathbb{R}^n})$ for $|\alpha| \leq s_j + t_k$. Lastly suppose that $u \in W_p^{(t)}(\mathbb{R}^n), \lambda \in \mathcal{L}\setminus\{\,0\,\}$, and that f is defined by* (1.1). *Consequently, if $f \in H_p^{(-s+\ell_0)}(\mathbb{R}^n)$, then $u \in W_p^{(t+\ell_0)}(\mathbb{R}^n)$.*

Proof. Referring to the proof of Proposition 3.2 for terminology, it follows from (3.1) and from arguments similar to those used in the proof of Proposition 3.1 that for $\ell \geq 1$,

$$(3.6) \qquad \| \eta_\ell u \|_{(t),p,Q_\ell} \leq C \big(\| \big(A(x,D) - \lambda I_N \big) \eta_\ell u \|_{(-s),p,Q_\ell} \\ + \| \eta_\ell u \|_{(t-1),p,Q_\ell} \big),$$

where here and for the remainder of this proof, C denotes a generic constant which may vary from inequality to inequality, but in all cases it does not depend upon ℓ, u, and λ. By employing arguments similar to those used in the proof of Proposition 3.1, it is not difficult to verify that $(A(x,D) - \lambda I_N) \eta_\ell u \in H_p^{(-s+1)}(Q_\ell)$. Hence for $\ell \geq 1$ we can apply the differential quotient method to (3.6) as in [14, Proof of Theorem 3.1, p. 123] (see also [2]) to deduce that $\eta_\ell u \in W_p^{(t+1)}(Q_\ell)$ and

$$(3.7) \qquad \| \eta_\ell u \|_{(t+1),p,Q_\ell} \leq C \left(\| \big(A(x,D) - \lambda I_N \big) \eta_\ell u \|_{(-s+1),p,Q_\ell} + \| \eta_\ell u \|_{(t),p,Q_\ell} \right).$$

We conclude from the foregoing results and from arguments similar to those used in the proof of Proposition 3.2 that $u \in W_p^{(t+1)}(B(r))$ for every $r > 0$ where $B(r) = \big\{ x \in \mathbb{R}^n \,\big|\, |x| < r \big\}$. Let us now show that $u \in W_p^{(t+1)}(\mathbb{R}^n)$ and that

$$(3.8) \qquad \| u \|_{(t+1),p,\mathbb{R}^n} \leq C \big(\| \big(A(x,D) - \lambda I_N \big) u \|_{(-s+1),p,\mathbb{R}^n} \\ + \| u \|_{(t),p,\mathbb{R}^n} \big).$$

Accordingly, it is clear that

$$\| u \|_{(t+1),p,\mathbb{R}^n} \leq N^{1/p'} \left(\sum_{j=1}^N \| \sum_{\ell \geq 1} \eta_\ell u_j \|_{t_j+1,p,\mathbb{R}^n}^p \right)^{1/p},$$

which leads to the inequality

$$(3.9) \qquad \| u \|_{(t+1),p,\mathbb{R}^n} \leq 2^{n+1} N \left(\sum_{\ell \geq 1} \| \eta_\ell u_j \|_{(t+1),p,Q_\ell}^p \right)^{1/p}.$$

Furthermore, by appealing to (3.7) and by arguing in a manner similar to that in the proof of Proposition 3.2, we can also show that

$$(3.10) \qquad \sum_{\ell \geq 1} \| \eta_\ell u \|_{(t+1),p,Q_\ell}^p \leq C \sum_{\ell \geq 1} \left(\| \eta_\ell \big(A(x,D) - \lambda I_N \big) u \|_{(-s+1),p,Q_\ell}^p \\ + \| u \|_{(t),p,Q_\ell}^p \right).$$

When $s_N \geq 1$, then arguments similar to those used in the proof of Proposition 3.2 show that the expression on the right side of (3.10) is majorized by

$$C' \left(\| \big(A(x,D) - \lambda I_N \big) u \|_{(-s+1),p,\mathbb{R}^n}^p + \| u \|_{(t),p,\mathbb{R}^n}^p \right),$$

where the constant C' has the same properties as C, and hence it follows from (3.7), (3.9), and (3.10) that $u \in W_p^{(t+1)}(\mathbb{R}^n)$ and that the inequality (3.8) holds.

Suppose next that for some $r, 1 < r \leq d$, $s_j = 0$ for $j > k_{r-1}$ and $s_j > 0$ for $j \leq k_{r-1}$. Then

$$H_p^{(-s+1)}(\mathbb{R}^n) = H_p^{(-s+1)_1}(\mathbb{R}^n) \times W_p^1(\mathbb{R}^n)^{N-k_{r-1}},$$

where $H_p^{(-s+1)_1}(\mathbb{R}^n) = \prod_{j=1}^{k_{r-1}} H_p^{-s_j+1}(\mathbb{R}^n)$, and where now $H_p^{(-s+1)}(\mathbb{R}^n), H_p^{(-s+1)_1}(R^n)$ and its adjoint space $W_{p'}^{(s-1)_1}(\mathbb{R}^n) = \prod_{j=1}^{k_{r-1}} W_{p'}^{s_j-1}(\mathbb{R}^n)$, and $W_p^1(\mathbb{R}^n)^{N-k_{r-1}}$ are equipped with their ordinary norms, $\| \cdot \|_{(-s+1),p,\mathbb{R}^n}, \| \cdot \|_{(-s+1)_1,p,\mathbb{R}^n}, \| \cdot \|_{(s-1)_1,p',\mathbb{R}^n}$, and $\| \cdot \|_{(1)_1,p,\mathbb{R}^n}$, respectively, which are defined in a manner analogous to the way $W_p^{(\tau)}(G)$ and $H_p^{(-s)}(G)$ were defined at the beginning of this section. For our purposes it will

be convenient to impose equivalent norms on $W_{p'}^{(s-1)_1}(\mathbb{R}^n)$ and $W_p^1(\mathbb{R}^n)^{N-k_{r-1}}$ namely $\|\cdot\|'_{(s-1)_1,p',\mathbb{R}^n}$ and $\|\cdot\|'_{(1)_1,p,\mathbb{R}^n}$, respectively, where

$$\|u\|'_{(s-1)_1,p',\mathbb{R}^n} = \left(\sum_{j=1}^{k_{r-1}} \|u_j\|_{s_j-1,p',\mathbb{R}^n}^p\right)^{1/p}$$

for $u \in W_{p'}^{(s-1)_1}(\mathbb{R}^n)$ and

$$\|u\|'_{(1)_1,p,\mathbb{R}^n} = \left(\sum_{j=k_{r-1}+1}^{N} \|u_j\|_{1,p,\mathbb{R}^n}^p\right)^{1/p}$$

for $u \in W_p^1(\mathbb{R}^n)^{N-k_{r-1}}$. Note that the norms $\|\cdot\|_{(s-1)_1,p',\mathbb{R}^n}$ and $\|\cdot\|'_{(s-1)_1,p',\mathbb{R}^n}$ induce equivalent norms on the adjoint space of $W_{p'}^{(s-1)_1}(\mathbb{R}^n)$, $H_p^{(-s+1)_1}(\mathbb{R}^n)$, and hence if we denote by $\|\cdot\|'_{(-s+1)_1,p,\mathbb{R}^n}$ the norm induced on $H_p^{(-s+1)_1}(\mathbb{R}^n)$ by the norm $\|\cdot\|'_{(s-1)_1,p',\mathbb{R}^n}$, then $\|\cdot\|'_{(-s+1)_1,p,\mathbb{R}^n}$ and $\|\cdot\|_{(-s+1)_1,p,\mathbb{R}^n}$ are equivalent norms on this space.

Let us now fix our attention upon $W_{p'}^{(s-1)_1}(\mathbb{R}^n)$ and for $1 \le j \le k_{r-1}$ let $\tilde{\nu}_j$ denote the number of distinct multi-indices α for which $|\alpha| \le s_j - 1$. Then to each $v = (v_1 \dots, v_{k_{r-1}})^T \in W_{p'}^{(s-1)_1}(\mathbb{R}^n)$ we can associate the vector

$$\mathcal{P}_{p'}^{(1)} v = \left(\mathcal{P}_{1,p'}^{(1)} v_1, \dots, \mathcal{P}_{k_{r-1},p'}^{(1)} v_{k_{r-1}}\right) \in L_{p'}(\Omega)^{\sum_{j=1}^{k_{r-1}} \tilde{\nu}_j},$$

where for $1 \le j \le k_{r-1}$, $\mathcal{P}_{j,p'}^{(1)} v$ denotes the $\tilde{\nu}_j$-th vector with components $\{D^\alpha v_j\}_{|\alpha| \le s_j - 1}$, and where the components of $\mathcal{P}_{j,p'}^{(1)}$ are arranged in the same way as the entries of $\mathcal{P}_{j,p'}$ were arranged in the proof of Proposition 3.2. Hence if we let $\mathcal{H}_{p'}^{(1)}$ denote the closed subspace of $L_{p'}^{\sum_{j=1}^{k_{r-1}} \tilde{\nu}_j}(\mathbb{R}^n)$ spanned by the vectors $\mathcal{P}_{p'}^{(1)} v$ for $v \in W_{p'}^{(s-1)_1}(\mathbb{R}^n)$ and if we equip $W_{p'}^{(s-1)_1}(\mathbb{R}^n)$ with the norm $\|\cdot\|'_{(s-1)_1,p',\mathbb{R}^n}$, then $\mathcal{H}_{p'}^{(1)}$ is isometrically isomorphic to $W_{p'}^{(s-1)_1}(\mathbb{R}^n)$. Hence it follows, as in the proof of Proposition 3.2, that for each $u \in \mathcal{H}_p^{(-s+1)_1}(\mathbb{R}^n)$ there is an $f^{(1)} = \left(f_1^{(1)}, \dots, f_{k_{r-1}}^{(1)}\right) \in L_p(\mathbb{R}^n)^{\sum_{j=1}^{k_{r-1}} \tilde{\nu}_j}$ where $f_j^{(1)} = \left(f_{j,1}^{(1)}, \dots, f_{j,\tilde{\nu}_j}^{(1)}\right)$ such that

(3.11) $$\|u\|'_{(-s+1)_1,p,\mathbb{R}^n} = \|f^{(1)}\|_{\mathbb{R}^n},$$

where $\|\cdot\|_{\mathbb{R}^n}$ denotes the norm in $L_p(\mathbb{R}^n)^{\sum_{j=1}^{k_{r-1}} \tilde{\nu}_j}$. Consequently if we fix our attention again upon (3.10) and let $P_p^{(1)}$ denote the operator projecting $H_p^{(-s+1)}(\mathbb{R}^n)$ onto $H_p^{(-s+1)_1}(\mathbb{R}^n)$ along $W_p^1(\mathbb{R}^n)^{N-k_{r-1}}$, then there is an $f^{(1)} \in L_p(\mathbb{R}^n)^{\sum_{j=1}^{k_{r-1}} \tilde{\nu}_j}$ satisfying (3.11) with u there replaced by $P_p^{(1)}\big(A(x,D) - \lambda I_N\big)u$ such that for $\ell \ge 1$ we have

$$\begin{aligned}
\|P_p^{(1)} \eta_\ell (A(x,D) - \lambda I_N) u\|_{(-s+1)_1,p,Q_\ell} &\le \|P_p^{(1)} \eta_\ell (A(x,D) - \lambda I_N) u\|_{(-s+1)_1,p,\mathbb{R}^n} \\
&\le C\|P_p^{(1)} \eta_\ell (A(x,D) - \lambda I_N) u\|'_{(-s+1)_1,p,\mathbb{R}^n} \\
&= C \sup \left|\langle P_p^{(1)} \eta_\ell (A(x,D) - \lambda I_N) u, \zeta\rangle_{\mathbb{R}^n}\right| \\
&= C \sup \left|\sum_{j=1}^{k_{r-1}} \sum_{k=1}^{\tilde{\nu}_j} \left(f_{j,k}^{(1)}, D^\alpha \eta_\ell \zeta_j\right)_{\mathbb{R}^n}\right| \le C_1 \|f^{(1)}\|_{Q_\ell},
\end{aligned}$$

where in each case the supremum is over the set $\left\{ \zeta \in C_0^\infty(\mathbb{R}^n)^{k_{r-1}} \,\middle|\, \|\zeta\|_{(s-1)_1,p',\mathbb{R}^n} = 1 \right\}$, the constant C_1 has the same properties as the constant C, $f^{(1)} \in L_p(\mathbb{R}^n)^{\sum_{j=1}^{k_{r-1}} \tilde{\nu}_j}$ denotes the norm preserving extension of $P_p^{(1)}(A(x,D) - \lambda I_N)u$ to a linear functional over $L_{p'}(\mathbb{R}^n)^{\sum_{j=1}^{k_{r-1}} \tilde{\nu}_j}$, and the terminology used here is analogous to that used in the proof of Proposition 3.2. If we now let $f^{(2)} = \left(f_{k_{r-1}+1}^{(2)}, \ldots, f_N^{(2)} \right) \in L_p(\mathbb{R}^n)^{(N-k_{r-1})(n+1)}$, where $f_j^{(2)} = \left(f_{j,1}^{(2)}, \ldots, f_{j,n+1}^{(2)} \right)$ and $f_{j,k}^{(2)} = D_{k-1}u_j$, with $D_0 u_j = u_j$, then we conclude from the foregoing results that

$$
\sum_{\ell \geq 1} \| \eta_\ell (A(x,D) - \lambda I_N)u \|_{(-s+1),p,Q_\ell}^p
$$

$$
\leq C \sum_{\ell \geq 1} \left(\sum_{j=1}^{k_{r-1}} \sum_{k=1}^{\tilde{\nu}_j} \| f_{j,k}^{(1)} \|_{0,p,Q_\ell}^p + \sum_{j=k_{r-1}+1}^{N} \sum_{k=1}^{n+1} \| f_{j,k}^{(2)} \|_{0,p,Q_\ell}^p \right)
$$

$$
\leq C' \| (A(x,D) - \lambda I_N)u \|_{(-s+1),p,\mathbb{R}^n}^p,
$$

where the constant C' has the same properties as the constant C. In light of these last inequalities, (3.9), and (3.10), we conclude that $u \in W_p^{(t+1)}(\mathbb{R}^n)$ and that the inequality (3.1) holds with t and $-s$ replaced by $t+1$ and $-s+1$, respectively.

Suppose next that $s_j = 0$ for $j = 1, \ldots, N$. Then $H_p^{(-s+1)}(\mathbb{R}^n) = W_p^1(\mathbb{R}^n)^N$, and for this case the proposition can be proved by arguing with $W_p^1(\mathbb{R}^n)^N$ as we argued with $W_p^1(\mathbb{R}^n)^{N-k_{r-1}}$ in the previous case.

If $\ell_0 = 1$, then the proof of the proposition is complete. Otherwise we complete the proof by proceeding by induction.   □

4. Fredholm theory

In this section we are going to use the results of Section 3 to derive information pertaining to the Fredholm theory for the Banach space operators induced by the spectral problem (1.1). Furthermore, when in the sequel we refer to $W_p^{(\tau)}(G), \tau = t$ or s and $H_p^{(-s)}(G)$ as Banach spaces (see Section 3 for terminology), then it is to be understood that we are equipping these spaces with their ordinary norms. If X and Y are Banach spaces, then we shall also use the notation $L(X,Y)$ to denote the space of bounded linear operators mapping X into Y and equipped with its usual norm.

Next let A_p denote the operator on $H_p^{(-s)}(\mathbb{R}^n)$ that acts like $A(x,D)$ and has domain $D(A_p) = W_p^{(t)}(\mathbb{R}^n)$.

We note for later use that that if we suppose that the hypotheses of Proposition 3.2 hold, then

$$
\| u \|_{(t),p,\mathbb{R}^n} \leq C \left(\| A_p u \|_{(-s),p,\mathbb{R}^n} + \| u \|_{(-s),p,\mathbb{R}^n} \right) \quad \text{for} \quad u \in D(A_p),
$$

where the constant C does not depend upon u. Thus we conclude that the operator $A_p : D(A_p) \to H_p^{(-s)}(\mathbb{R}^n)$ is closed. Note also from Proposition 3.1 that $A_p : D(A_p) \to H_p^{(-s)}(\mathbb{R}^n)$ is bounded.

Referring to [12, pp. 242–243] for terminology, we now have the following result.

Theorem 4.1. *Suppose that the spectral problem* (1.1) *is weakly smooth and parameter-elliptic in* $\mathcal{L}$. *Then there exists the number* $\lambda^\dagger \in \mathbb{R}_+$, *where* $\lambda^\dagger$ *does not depend upon* p, *such that* $A_p - \lambda I_N \in L\left(D(A_p), H_p^{(-s)}(\mathbb{R}^n) \right)$ *and is Fredholm for* $\lambda \in \mathcal{L}^\dagger = \left\{ \lambda \in \mathcal{L} \middle| |\lambda| \geq \lambda^\dagger \right\}$. *Furthermore, there exists the number* $\lambda^0(p) \geq \lambda^\dagger$ *such that* λ *belongs to the resolvent set of* A_p *for* $\lambda \in \mathcal{L}_p^\dagger = \left\{ \lambda \in \mathcal{L}^\dagger \middle| |\lambda| \geq \lambda^0(p) \right\}$, *and hence* $\text{index}(A_p - \lambda I_N) = 0$

for $\lambda \in \mathcal{L}^\dagger$. Finally, that part of the spectrum of A_p which is contained in $\mathcal{L}^\dagger$ consists solely of most a finite number of distinct eigenvalues, with each being of finite algebraic multiplicity.

Proof. We know from the proof of Proposition 3.4 that for $x, \xi \in \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$,

$$|\det \left(\mathring{A}(x, \xi) - \lambda I_N \right)| \geq C_0 \prod_{j=1}^{N} \langle \xi, \lambda \rangle_j^{m_j},$$

where the constant C_0 does not depend upon x, ξ, and λ. Furthermore, it is not difficult to verify that for these values of x, ξ, and λ,

$$|\det \left(\mathring{A}(x, \xi) - \lambda I_N \right) - \det \left(A(x, \xi) - \lambda I_N \right)| \leq C|\lambda|^{-1/m_1} \prod_{j=1}^{N} \langle \xi, \lambda \rangle_j^{m_j},$$

where the constant C does not depend upon x, ξ, and λ. Hence it follows that we can choose the number $\lambda^\dagger \in \mathbb{R}_+$ such that

$$|\det \left(A(x, \xi) - \lambda I_N \right)| \geq C_0/2 \prod_{j=1}^{N} \langle \xi, \lambda \rangle_j^{m_j}$$

for $x, \xi \in \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^\dagger$. Thus all but the final assertions of the proposition now follow from this last result, [19], Proposition 3.3, from what was said in the text preceding the statement of this theorem, and from the fact that $\mathcal{L}^\dagger$ is contained in a component of the Fredholm domain of A_p (see [12, pp. 242–243]).

Turning now to the final assertions of the proposition, let $\lambda_1, \lambda_2 \in \mathcal{L}^\dagger$ with $|\lambda_1| \geq \lambda^0(p)$ and $|\lambda_2| < \lambda^0(p)$. Then there is a polygonal arc $\gamma = \left\{ \gamma(t), 0 \leq t \leq 1 \right\}$ joining λ_1 to λ_2 and lying entirely in $\mathcal{L}^\dagger$ such that $\gamma(0) = \lambda_1$ and $\gamma(1) = \lambda_2$. But this implies that either $N(t) = \dim \ker \left(A_p - \gamma(t) I_N \right) = 0$ for $0 \leq t < 1$ or there is a $\tau_1, 0 < \tau_1 < 1$ such that $N(t) = 0$ for $0 \leq t < \tau_1, N(\tau_1) > 0$. However for either case we know from [12, Theorem 5.3.1, p. 241] that for some $\epsilon > 0$, $\dim \ker(A_p - \lambda I_N) = 0$ for $0 < |\lambda - \lambda_2| < \epsilon$ if the first case occurs and for $0 < |\lambda - \gamma(\tau_1)| < \epsilon$ if the second case occurs. Furthermore, if the second case occurs, then either $N(t) = 0$ for $0 \leq t < 1$ except for $t = \tau_1$, or there is a $\tau_2, 0 < \tau_1 < \tau_2 < 1$, such that $N(t) = 0$ for $0 \leq t \leq \tau_2$ except for $t = \tau_j, j = 1, 2$. But as before, we know that in either case there is an $\epsilon > 0$ such that $\dim \ker(A_p - \lambda I_N)) = 0$ for $0 < |\lambda - \lambda_2| < 0$ if the first case holds and for $0 < |\lambda - \gamma(\tau_2)| < \epsilon$ if the second case holds. Carrying on in this manner we finally arrive at the situation where either there is a finite sequence $\left\{ \tau_j \right\}_1^k, 0 < \tau_1 < \cdots < \tau_k < 1$ such that $N(t) = 0$ for $0 \leq t < 1$ except for $t = \tau_j, j = 1, \ldots, k$, or there is an infinite sequence $\left\{ \tau_j \right\}_1^\infty, 0 < \tau_1 < \tau_2 < \cdots < 1$, such that $N(t) = 0$ for $0 \leq t < \tau$ except for $t = \tau_j, j \geq 1$, where $\tau = \lim_{j \to \infty} \tau_j$. Then we see from [12] that the second case is not possible, while for the first case there is an $\epsilon > 0$ such that $\dim \ker(A_p - \lambda I_N) = 0$ for $0 < |\lambda - \lambda_2| < \epsilon$. Thus we have shown that if $\mu \in \mathcal{L}^\dagger$ and $\dim \ker(A_p - \mu I_N) > 0$, then there is an $\epsilon > 0$ such that $\dim \ker(A_p - \lambda I_N) = 0$ for $0 < |\lambda - \mu| < \epsilon$. The final assertions of the theorem follows from this last result, and [12], which concludes the proof. $\square$

We are now going to investigate how the eigenvalues, if any, of A_p which lie in $\mathcal{L}^\dagger$ vary with p.

Theorem 4.2. *Suppose that the spectral problem* (1.1) *is weakly smooth and parameter-elliptic in $\mathcal{L}$. Suppose in addition that $a_\alpha^{jk} \in C^{s_j+n}(\overline{\mathbb{R}^n})$ for $|\alpha| \leq s_j + t_k, 1 \leq j, k \leq N$. Lastly suppose that $\lambda_1 \in \mathcal{L}^\dagger$ is an eigenvalue of A_p and $u^{(1)}$ a corresponding eigenvector. Then λ_1 is an eigenvalue and $u^{(1)} \in W_q^{(t)}(\mathbb{R}^n)$ a corresponding eigenvector of A_q for every q satisfying $p < q < \infty$. Consequently $\ker \left(A_p - \lambda_1 I_N \right) \subset \ker \left(A_q - \lambda_1 I_N \right)$.*

Proof. As a consequence of Proposition 3.6 we see that $u^{(1)} \in W_p^{(t+n)}(\mathbb{R}^n)$. Hence it follows from the Sobolev embedding theorem (see [1, Theorem 5.4, p. 97]) that $u^{(1)} \in W_q^{(t)}(\mathbb{R}^n)$ for every q satisfying $p < q < \infty$, and all the assertions of the theorem are immediate consequences of this fact.   □

Under some further restrictions Theorem 4.2 can be improved.

Theorem 4.3. *Suppose that the hypotheses of Theorem 4.2 hold with $s_j = 0$ and t_j even for $j = 1, \ldots, N$. Suppose in addition that for $1 \le j, k \le N$, $a_\alpha^{jk} \in C^{|\alpha|+n}(\overline{\mathbb{R}^n}) \cap C^{|\alpha|+1,0}(\overline{\mathbb{R}^n})$ for $|\alpha| \le t_k$ with $a_\alpha^{jk}(x) = 0$ for $x \in \mathbb{R}^n$ if $j > k, t_j < t_k$, and $|\alpha| \ge t_j$ and also if $j < k, t_j > t_k$, and $|\alpha| = t_k$. Then the eigenvalues of A_p lying in $\mathcal{L}^\dagger$, as well as their geometric and algebraic multiplicities, are the same for all these values of p.*

In order to prove the theorem, a preliminary result is required.

Proposition 4.4. *Suppose that the hypotheses of Theorem 4.3 hold and let $1 < q < p$. Also let $\lambda_1 \in \mathcal{L}^\dagger$ and put $n_p = \dim \ker(A_p - \lambda_1 I_N)$, $d_p = \dim \operatorname{coker}(A_p - \lambda_1 I_N)$. Then*
   (1) $\ker(A_p - \lambda_1 I_N) \subset \ker(A_q - \lambda_1 I_N)$ if $0 \le d_p \le n_p < \infty$;
   (2) $\ker(A_p - \lambda_1 I_N) \subset \ker(A_q - \lambda_1 I_N)$ if $0 \le n_p < d_p < \infty$.

Proof. To begin with let us introduce the multiplication operators $T_p(\tau)$ and $S_p(\tau)$, $\tau \ge 0$, acting on $D(A_p)$ and $L_p(\mathbb{R}^n)^N$ $(= H_p^{(-s)}(\mathbb{R}^n))$, respectively, where $T_p(\tau)u(x) = e^{-\tau \langle x \rangle}u(x)$ for $u \in D(A_p)$, $S_p(\tau)v(x) = e^{-\tau \langle x \rangle}v(x)$ for $v \in L_p(\mathbb{R}^n)^N$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

Let $u \in D(A_p)$, denote by u_k its k-th component, and let α denote a multi-index such that $|\alpha| \le t_k$. Then it follows from the Leibnitz formula that

$$D^\alpha e^{\pm\tau \langle x \rangle} u_k(x) = e^{\pm\tau \langle x \rangle} D^\alpha u_k(x) + \sum_{\substack{\beta \le \alpha \\ |\beta| > 0}} \binom{\alpha}{\beta} \left(D^\beta e^{\pm\tau \langle x \rangle)} \right) D^{\alpha-\beta} u_k(x),$$

while direct calculations show that

$$D^\beta e^{\pm\tau \langle x \rangle} = e^{\pm\tau \langle x \rangle} \sum {}' \prod {}' (\pm\tau D^{\gamma_j} \langle x \rangle)$$

and that

$$|D^{\gamma_j} \langle x \rangle| \le C(\gamma_j) \langle x \rangle^{1-|\gamma_j|},$$

where $\prod'$ indicates that the product is taken over a set of distinct multi-indices γ_j for which $|\gamma_j| > 0$ and $\sum_j |\gamma_j| = |\beta|$, $\sum'$ indicates that the sum is taken over all such sets, and $C(\gamma_j)$ denotes a constant depending upon γ_j. In light of these facts it is not difficult to deduce that for $\tau \in [0, \infty)$, $T_p(\tau) \in L(D(A_p), D(A_p))$ and $S(\tau) \in L(L_p(\mathbb{R}^n)^N, L_p(\mathbb{R}^n)^N)$ are C_0 semigroups such that for each $\tau, T(\tau)$ and $S(\tau)$ are injective,while $S(\tau)^{-1} A_p T(\tau) \in L(D(A_p), L_p(\mathbb{R}^n)^N)$ and that $S(\tau)^{-1} A_p T(\tau)$ converges in norm to A_p as $\tau \to 0$.

Let us next fix our attention upon the assertions of the proposition, and to begin with let us consider assertion (1) with $n_p > 0$ and $d_p = 0$. Then it follows from [12, Theorem 5.22, p. 236] that for some $\tau_0 > 0, V = S(\tau_0)^{-1}(A_p - \lambda_1 I_N)T(\tau_0)$ is semi-Fredholm with $\dim \ker V = n_p$, and $\dim \operatorname{coker} V = 0$. Hence by arguing as in [19] and applying Holder's inequality, we readily deduce that $\ker(A_p - \lambda_1 I_N) \subset \ker(A_q - \lambda_1 I_N)$. Obviously the same result is true if we assume instead that $n_p = 0$.

Finally assertions (1), with $d_p > 0$, as well as assertion (2), can be proved by modifying the above arguments as in [19].   □

Proof of Theorem 4.3. Referring to Proposition 4.4 for notation, let us firstly fix our attention upon the case $1 < q < p$ and prove all but the final assertions of the theorem for this case. Accordingly, to begin with, let us observe from Theorem 4.2 and Proposition 4.4 that $\ker(A_q - \lambda_1 I_N) = \ker(A_p - \lambda_1 I_N)$.

We now turn our attention to cokernels. Then it is not difficult to show that under our assumptions the spectral problem formally adjoint to the spectral problem (1.1),

$$(4.1) \qquad A'(x,D)u(x) - \overline{\lambda_1}u(x) = f(x) \quad \text{for} \quad x \in \mathbb{R}^n,$$

is well defined. Here $A'(x,D) = \left(A'_{jk}(x,D) \right)_{j,k=1}^N$ is the formal adjoint of $A(x,D)$, the $A'_{jk}(x,D)$ are linear differential operators defined on $\mathbb{R}^n$ of order not exceeding $s'_j + t'_k$ where $s'_j = 0, t'_k = t_k$ for $1 \le j, k \le N$. Note also that the top order operators $\mathring{A}(x,D)$ and $\mathring{A}'(x,D)$ are block diagonal. Then it is not difficult to verify that the analogues of Propositions 3.1–3.3 for the spectral problem (4.1) also hold and we can take the constant $\lambda^\sharp$ of Proposition 3.1 to remain the same. We henceforth let $A'_{p'}$ denote the operator on $L_{p'}(\mathbb{R}^n)^N$ that acts like $A'(x,D)$ and has domain $D(A'_{p'}) = W_{p'}^{(t)}(\mathbb{R}^n)$; and we can readily verify that the analogues of Theorems 4.1, 4.2 and Proposition 4.4 hold for $A'_{p'}$. Hence we can now argue as above with $A'_{q'} - \overline{\lambda_1}\, I_N$ and $A'_{p'} - \overline{\lambda_1}\, I_N$ in place of $A_p - \lambda_1\, I_N$ and $A_q - \lambda_1\, I_N$, respectively, to deduce that $\operatorname{coker}(A_q - \lambda_1\, I_N) = \operatorname{coker}(A_p - \lambda_1\, I_N)$. This proves all but the final assertions of the theorem for the case $q < p$, and the analogous result for the case $q > p$ can be proved in a similar fashion (i.e., by arguing with the adjoint operators).

Turning to the final assertions of the theorem, let $\lambda_1 \in \mathcal{L}^\dagger$ be an eigenvalue of A_p and $u^{(1)}$ a corresponding eigenvector. Then it is clear from what has already been proved that we need only prove that the algebraic multiplicity of λ_1 is the same for all p. Accordingly, let us suppose firstly that $q > p$ and let $\left\{ u^{(1,j)} \right\}_{j=0}^{m-1}$ be a chain of length m consisting of the eigenvector $u^{(1,0)} = u^{(1)}$ and the associated vectors $\left\{ u^{(1j)} \right\}_{j=1}^{m-1}$ corresponding to the eigenvalue λ_1 of A_p (see [16, pp. 60–61]). Thus it follows from Proposition 3.6 and the Sobolev embedding theorem (see [1, Theorem 5.4, p. 97]) that $\left\{ u^{(1,j)} \right\}_{j=0}^{m-1}$ is a chain of length m consisting of the eigenvector $u^{(1)} = u^{(1,0)}$ and the associated vectors $\left\{ u^{(1,j)} \right\}_{j=1}^{m-1}$ corresponding to the eigenvalue λ_1 of A_q. We conclude immediately that the algebraic multiplicity of λ_1 as an eigenvalue of A_p does not exceed the algebraic multiplicity of λ_1 as an eigenvalue of A_q. On the other hand we can appeal to the analogue of Proposition 3.6 for the spectral problem (4.1) to show that the algebraic multiplicity of $\overline{\lambda_1}$ as an eigenvalue of $A'_{q'}$ does not exceed the algebraic multiplicity of $\overline{\lambda_1}$ as an eigenvalue of $A'_{p'}$. Hence we conclude that the algebraic multiplicity of λ_1 as an eigenvalue of A_p and of A_q are the same (see [12, p. 184]). Since similar arguments give the same result for $q < p$, the proof of the theorem is complete. $\qquad \square$

5. An example

In this section we fix our attention upon the spectral problem (1.1) with $A(x,D)$ a 2×2 matrix operator whose entries $A_{jk}(x,D)$ are linear differential operators defined on $\mathbb{R}^n$ of order not exceeding $s_j + t_k$, where $s_j = 0$ for $j = 1, 2$, and $t_1 = 4, t_2 = 2$. To be more precise we now take

$$(5.1) \qquad A(x,D) = A_0(x,D) - cI_2 + \left(\tilde{A}_{jk}(x,D) \right)_{j,k=1}^2,$$

where $A_0(x,D) = \operatorname{diag}\left(\Delta^2, -\Delta \right)$, Δ denotes the Laplacian in $\mathbb{R}^n$, $c \in \overline{\mathbb{R}_+}$ denotes a constant, $\tilde{A}_{jk}(x,D) = \sum_{|\alpha| \le \sigma_{jk}} a_\alpha^{jk}(x)D^\alpha$ with $\sigma_{jk} = 1$ if $j \ne k$, $\sigma_{11} = 3$, $\sigma_{22} = 1$, and the $a_\alpha^{jk} \in C_0^\infty(\mathbb{R}^n)$. Then with $A(x,D)$ given by (5.1), let us now investigate the spectral theory connected with the problem (1.1).

Accordingly, let us fix an ϵ satisfying $0 < \epsilon < \pi/2$ and let $\mathcal{L}$ denote the sector in the complex plane with vertex at the origin determined by the inequalities $\epsilon \le \arg \lambda \le 2\pi - \epsilon$.

Then we can readily verify that the hypotheses of Theorem 4.3 concerning the spectral problem (1.1) are satisfied, and hence if we let A_p denote the operator on $L_p(\mathbb{R}^n)^2$ that acts like $A(x, D)$ and has domain $D(A_p) = W_p^{(t)}(\mathbb{R}^n)$, the all the assertions of Theorems 4.1 and 4.3 hold.

In order to derive more information concerning the spectral properties of A_p, we are now going to fix our attention upon the case $p = 2$, and let $A_2^{(1)}$ (resp. $A_2^{(2)}$) denote the operator on $L_2(\mathbb{R}^n)$ that acts like Δ^2 (resp. $-\Delta$) and with domain $D(A_2^{(1)}) = W_2^4(\mathbb{R}^n)$ (resp. $D(A_2^{(2)}) = W_2^2(\mathbb{R}^n)$. Then direct calculations show that $A_2^{(1)}$ is a symmetric operator on $L_2(\mathbb{R}^n)$ whose numerical range is contained in the interval $[0, \infty)$. Furthermore, since it is shown in [10] that the analogue of Theorem 3.3 holds when the spectral problem (1.1) is replaced by a scalar spectral problem, we conclude from these results that $A_2^{(1)}$ is a selfadjoint operator on $L_2(\mathbb{R}^n)$ whose spectrum is contained in the interval $[0, \infty)$. Similarly we can show that $A_2^{(2)}$ is a selfadjoint operator on $L_2(\mathbb{R}^n)$ whose spectrum is contained in $[0, \infty)$, and furthermore, we know from [6, p. 416] (see also [20, p. 158] for the case $n = 2$) that its spectrum is precisely $[0, \infty)$.

Thus we have shown that if we let $A_{0,2} = \text{diag}(A_2^{(1)}, A_2^{(2)})$, then $A_{0,2} - c\,I_2$ is a selfadjoint operator on $L_2(\mathbb{R}^n)^2$ with domain $D(A_{0,2}) = W_2^{(t)}(\mathbb{R}^n)$ and whose spectrum is precisely $[-c, \infty)$. Furthermore, we can appeal to Proposition 3.2 to show that $A_2 - (A_{0,2} - c\,I_2)$ is relatively compact with respect to $A_{0,2} - c\,I_2$, and hence it follows from [12, Theorem 5.35, p. 244] that A_2 is a closed operator on $L_2(\mathbb{R}^n)^2$ with essential spectrum $[-c, \infty)$ and with semi-Fredholm domain $\mathbb{C} \setminus [-c, \infty)$. Hence referring again to Theorem 4.1, it is clear that we must have $\lambda^\dagger > c$, while it also follows from the arguments used in the proof of the final assertion of that theorem that the Fredholm domain of A_2 is precisely $\mathcal{C} \setminus [-c, \infty)$, that $\mathcal{C} \setminus [-c, \infty)$ consists of only one component, and that $\text{index}(A_2 - \lambda\,I_2) = 0$ for λ lying in this component.

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