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Nabil Machrafi, Aziz Elbour, Mohammed Moussa

Link to this article: http://mfat.imath.kiev.ua/article/?id=893

How to cite this article:

Nabil Machrafi, Aziz Elbour, and Mohammed Moussa, *Some results on order bounded almost weak Dunford-Pettis operators*, Methods Funct. Anal. Topology **22** (2016), no. 3, 256-265.

SOME RESULTS ON ORDER BOUNDED ALMOST WEAK DUNFORD-PETTIS OPERATORS

NABIL MACHRAFI, AZIZ ELBOUR, AND MOHAMMED MOUSSA

ABSTRACT. We give some new characterizations of almost weak Dunford-Pettis operators and we investigate their relationship with weak Dunford-Pettis operators.

1. INTRODUCTION AND NOTATIONS

Throughout this paper, X, Y will denote real Banach spaces, and E, F will denote real Banach lattices. B_X is the closed unit ball of X. We mean by operator between Banach spaces, a bounded linear application.

A real vector space E is said to be a *partially ordered vector space* whenever it is equipped with a partial order relation $\geq$ (i.e., a reflexive, antisymmetric, and transitive binary relation on E) that is compatible with the algebraic structure of E in the sense that it satisfies the following two axioms:

(1) If $x \geq y$, then $x + z \geq y + z$ holds for all $z \in E$.
(2) If $x \geq y$, then $\lambda x \geq \lambda y$ holds for all $\lambda \geq 0$.

An alternative notation for $x \geq y$ is $y \leq x$. The *positive cone* of E, denoted by E^+, is the set of all positive vectors of E, i.e., $E^+ := \{x \in E : x \geq 0\}$. If furthermore, every set $\{x, y\} \subset E$ has a supremum $\sup \{x, y\} = x \vee y$ (or equivalently it has an infimum $\inf \{x, y\} = x \wedge y$) then E is called a *vector lattice* (or *Riesz space*). The elements

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = x \vee (-x)$$

are called the *positive part*, *negative part*, and *modulus* of the element x, respectively. Note that $x = x^+ - x^-$ and $|x| = x^+ + x^-$. If $x, y \in E$ and $x \leq y$, then the *order interval* $[x, y]$ is defined by

$$[x, y] = \{z \in E : x \leq z \leq y\}.$$

A subset $A \subset E$ is said to be *order bounded* if it is contained in some order interval. It is said to be *solid* if conditions $|x| \leq |a|$ and $a \in A$ imply $x \in A$. The smallest solid set containing a set A is called the *solid hull* of A and denoted by $sol(A)$. We have

$$sol(A) = \{x \in E : |x| \leq |a| \text{ for some } a \in A\}.$$

The elements $x, y \in E$ are called *disjoint* if $|x| \wedge |y| = 0$. A generalized sequence (x_α) in a vector lattice E (i.e. a function $\alpha \to x_\alpha$ from some an upward directed set I to E) is called disjoint, if $|x_\alpha| \wedge |x_\beta| = 0$, $\alpha \neq \beta$. We will use the notation $x_\alpha \perp x_\beta$ to mean that the generalized sequence (x_α) is disjoint. A collection $(e_i) \subset E^+ \setminus \{0\}$ is called a *complete disjoint system* if

$$e_i \wedge e_j = 0, \quad i \neq j \quad \text{and} \quad e_i \wedge |x| = 0 \quad \text{for all } i \text{ implies} \quad x = 0.$$

A positive non-zero element x of an *Archimedean vector lattice* E (i.e., such vector lattices that $\inf_n \left\{ \frac{1}{n} x \right\} = 0$ for every $x \in E^+$) is called *discrete*, if

$$u, v \in [0, x] \quad \text{and} \quad u \wedge v = 0 \quad \text{imply} \quad u = 0 \quad \text{or} \quad v = 0.$$

An Archimedean vector lattice E is called *discrete* (or *atomic*), if E has a complete disjoint system consisting of discrete elements, or equivalently, every non-trivial interval $[0, x]$ contains a discrete element. A norm $\|.\|$ on a vector lattice $(E, \leq)$ is said to be a *lattice norm* whenever

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\|.$$

A vector lattice equipped with a lattice norm is known as a *normed vector lattice*. If a normed vector lattice is also norm complete, then it is referred to as a *Banach lattice*. Note that If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|.\|$ of a Banach lattice E is called *order continuous* if for each generalized sequence $(x_\alpha) \subset E$, $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \to 0$, where the notation $x_\alpha \downarrow 0$ means that the generalized sequence (x_α) is decreasing and $\inf \{x_\alpha\} = 0$. A Banach lattice E is said to be a *Kantorovich–Banach space* (briefly *KB-space*) whenever every increasing norm bounded sequence of E^+ is norm convergent. Note that every KB-space has order continuous norm. The lattice operations in a Banach lattice E are said to be *sequentially weakly continuous*, if for every weakly null sequence (x_n) in E we have $|x_n| \to 0$ for $\sigma(E, E')$. For a sequence $(x_n) \subset E$ of a Banach lattice, the following fact will be used throughout this paper (see [2, Theorem 4.34]):

$$x_n \overset{\sigma(E,E')}{\to} 0, \quad x_n \perp x_m \quad \text{implies} \quad |x_n| \overset{\sigma(E,E')}{\to} 0.$$

Recall that a subset A of a Banach space X is called a *Dunford-Pettis (DP) set*, if each weakly null sequence (f_n) in X' converges uniformly to zero on A. In his paper [9], T.Leavelle considered the dual version of DP sets, so-called *(L) sets*, that is, each subset B of the topological dual X', on which every weakly null sequence (x_n) in X converges uniformly to zero. Recently, the authors of [5] and [3] considered respectively, the disjoint versions of DP sets and (L)-sets, which are respectively called *almost Dunford-Pettis* (almost DP) sets and *almost (L)-sets*. From [5] (resp. [3]), a norm bounded subset $A \subset E$ (resp. $B \subset E'$) is said to be almost DP (resp. almost (L)-) set, if every disjoint weakly null sequence $(f_n) \subset E'$ (resp. $(x_n) \subset E$) converges uniformly to zero on A (resp. B). Clearly, every DP set in a Banach lattice (resp. (L)-set in the topological dual of a Banach lattice) is an almost DP (resp. almost (L)-) set . But the converse is false in general for the two latter classes of sets.

A Banach lattice E is said to have the (*positive*) *Schur property* if every (positive) weakly null sequence $(x_n) \subset E$ is norm null. Furthermore, A Banach space X is said to have the *Dunford-Pettis (DP) property* if each relatively weakly compact set in X is a DP set, alternatively, $f_n(x_n) \to 0$ for every weakly null sequences $(x_n) \subset X$, $(f_n) \subset X'$.

An operator $T : E \to F$ between two vector lattices is called an *order bounded operator*, if it maps order bounded subsets of E into an order bounded ones of F. It is *positive* if $T(E^+) \subset F^+$. The positive operators between two vector lattices generate the vector space of all *regular operators*, i.e., operators that are written as a difference of two positive operators. Note that a regular operator between two vector lattices need not be order bounded (see example of H. P. Lotz [2, Example 1.16]). An operator $T : X \to Y$ between two Banach spaces is said to be *Dunford-Pettis*, if T carries each weakly null sequence (x_n) in X to a norm null one in Y, equivalently, T carries relatively weakly compact subsets of X to a relatively compact ones in Y. The following weak versions of Dunford-Pettis operators are considered in the Banach lattice setting. An operator $T : E \to F$ is

- *weak Dunford-Pettis* (wDP), if $f_n(Tx_n) \to 0$ whenever (x_n) converges weakly to 0 in E and (f_n) converges weakly to 0 in F', equivalently, T maps relatively weakly compact sets in E to a Dunford–Pettis sets in F (see Theorem 5.99 of [2]).
- *almost Dunford-Pettis* (almost DP), if $\|Tx_n\| \to 0$ whenever $(x_n) \subset E$ is a disjoint weakly null sequence, equivalently, $\|Tx_n\| \to 0$ for every weakly null sequence $(x_n) \subset E$ consisting of positive terms [4, Theorem 2.2].

The class of wDP operators (between two Banach spaces) was introduced by C. D. Aliprantis and O. Burkinshaw in [1]. It extends the notions of DP operator and the DP property of a Banach space, in the sense that every DP operator $T : X \to Y$ is wDP, and a Banach space X has the DP property iff the identity operator on X is wDP. Next, J. A. Sanchez [13] introduced the class of almost DP operators (from a Banach lattice into a Banach space) which also extends, in the Banach lattice setting, the notions of DP operator and the positive Schur property, since every DP operator $T : E \to Y$ is almost DP, and a Banach lattice E has the positive Schur property iff the identity operator on E is almost DP. Also, F. Räbiger [12] distinguished a class of Banach lattices with a weak version of the DP property. A Banach lattice E is said to have the *weak Dunford-Pettis (wDP) property* if every weakly compact operator on E is almost DP, equivalently (see [15, Proposition 1]), for all sequences $(x_n) \subset E'_+$, $(f_n) \subset E'$,

$$(*) \qquad x_n \xrightarrow{\sigma(E,E')} 0, \quad x_n \perp x_m \quad \text{and} \quad f_n \xrightarrow{\sigma(E,E')} 0 \quad \text{imply} \quad f_n(x_n) \to 0.$$

Inspired by the preceding facts, K. Bouras and M. Moussa introduced naturally in their recent paper [6] the class of *almost weak Dunford-Pettis* (awDP) operators, as a class of operators that extends both the notions of wDP operator, almost DP operator, and the wDP property of a Banach lattice. An operator $T : E \to F$ is said to be awDP, if for all sequences $(x_n) \subset E$, $(f_n) \subset F'$,

$$x_n \xrightarrow{\sigma(E,E')} 0, \quad x_n \perp x_m \quad \text{and} \quad f_n \xrightarrow{\sigma(E,E')} 0, \quad f_n \perp f_m \quad \text{imply} \quad f_n(Tx_n) \to 0.$$

It follows from [6, Theorem 2.1(3)] combined with (*) that a Banach lattice E has the wDP property iff the identity operator on E is awDP. Note that the authors gave in [6, Theorem 2.1] some sequence characterisations of positive almost weak Dunford-Pettis operators, which allowed them to establish some new sequence characterisations of the wDP property of a Banach lattice (see [6, Corollary 2.1]). The class of awDP operators contains strictly that of wDP operators as well as that of almost DP operators, that is, every wDP (resp. almost DP) operator is awDP, but there exists an awDP operator which is not wDP nor almost DP. The example of such operator is the identity operator on a Banach lattice Φ with the wDP property but without the DP property nor the positive Schur property. W. Wnuk gave in [15, p. 231] an example of such Banach lattice:

Example. Let ω be a positive non-increasing continuous function on $(0,1)$ so that

$$\lim_{t \to 0} \omega(t) = \infty \quad \text{and} \quad \int_0^1 \omega(t)\, dt = 1.$$

The Lorentz function space $E = \wedge(\omega, 1)$ is the space of all measurable functions f on $[0,1]$ for which

$$\|f\|_{\omega,1} = \int_0^1 f^*(t)\, \omega(t)\, dt < \infty,$$

where f^* denotes the decreasing rearrangement of $|f|$ (cf. [10, p. l17, p. 120]). E is a Banach lattice under the norm $\|.\|_{\omega,1}$ and the standard almost everywhere pointwise order (i.e., $f \le g$ if $f(x) \le g(x)$ a.e.). Note that, since E has the Fatou property, then it is a maximal rearrangement invariant space [10, Definition 2.a.1], and therefore E has

a predual, that is, $E = \Phi'$, where Φ is the closed linear span of the simple functions in E'_i. Here, E'_i stands for the linear subspace of E' consisting of all integrals on E, i.e., functionals $\varphi_g \in E'$ defined by

$$\varphi_g(f) = \int_0^1 f(t) g(t) \, dt,$$

where g is any measurable function on $[0, 1]$ so that $gf \in L_1(0, 1)$ for every $f \in E$ (for details about the preceding facts, see [10, p. 29, p. 118, p. 121]). Now, it follows from [15, p. 231] that Φ has the wDP property but not the DP property nor the positive Schur property.

Moreover, for the class of wDP operators and that of almost DP ones, each one of the two classes is not included in the other. For instance, as the Lorentz space $\wedge(\omega, 1)$ has the positive Schur property without the DP property (see [14, Remark 3]), the identity operator on $\wedge(\omega, 1)$ is almost DP but not wDP. Conversely, since c_0 (the space of real sequences (x_n) with $\lim x_n = 0$) has the DP property without the the positive Schur property, the identity operator on c_0 is wDP but not almost DP.

The present paper is devoted to the class of awDP operators. W. Wnuk noted in [15, Example 4 p. 230] that a positive operator $T : E \to F$ is almost DP if and only if it is a DP operator, provided that F is discrete with order continuous norm. Motivated by this fact, we look at its weak alternative, that is, when an awDP operator is wDP? As a response, we prove the following theorem.

Theorem A. *Let E and F be two Banach lattices. Then, an order bounded operator $T : E \to F$ is almost weak Dunford-Pettis if and only if it is weak Dunford-Pettis, whenever one of the following holds:*

    (i) E has sequentially weakly continuous lattice operations.
    (ii) F' has sequentially weakly continuous lattice operations.
    (iii) T is positive and F is discrete with order continuous norm.

For that purpose, we present some new characterisations of awDP operators through almost DP (resp. almost (L)-) sets and some lattice approximations (Sect. 2). Next, we give the proof of Theorem A and we derive some consequences (Sect. 3).

We refer the reader to [2, 11] for more details on Banach lattice theory and positive operators.

2. Characterisation of almost weak Dunford-Pettis operators

We start this paper by the following lemma which is just a particular case of Theorem 2.4 of [8].

Lemma 2.1. *Let E be a Banach lattice, and let $(f_n) \subset E'$ be a sequence with $|f_n| \xrightarrow{w^*} 0$. If $A \subset E$ is a norm bounded and solid set such that $f_n(x_n) \to 0$ for every disjoint sequence $(x_n) \subset A^+ := A \cap E^+$, then $\sup_{x \in A} |f_n|(x) \to 0$.*

We will use throughout this paper the following lemma.

Lemma 2.2. *Let $T : E \to F$ be an order bounded operator between two Banach lattices, and let A and B be respectively a norm bounded solid subsets of E and F'. Then, the following holds:*

    (1) If the sequence $(f_n) \subset F'$ satisfy $|f_n| \xrightarrow{w^*} 0$ and $f_n(Tx_n) \to 0$ for every disjoint sequence $(x_n) \subset A^+$, then (f_n) converges uniformly to zero on $T(A)$.
    (2) If the sequence $(x_n) \subset E$ satisfy $|x_n| \xrightarrow{w} 0$ and $f_n(Tx_n) \to 0$ for every disjoint sequence $(f_n) \subset B^+$, then (x_n) converges uniformly to zero on $T'(B)$.

Proof. (1) We claim that $|T'(f_n)| \xrightarrow{w^*} 0$ holds in E'. Let $x \in E^+$ and pick some $y \in F^+$ such that $T[-x, x] \subseteq [-y, y]$. Thus

$$|T'(f_n)|(x) = \sup\{|T'(f_n)(u)| : |u| \le x\} = \sup\{|f_n(T(u))| : |u| \le x\} \le |f_n|(y).$$

Since $|f_n| \xrightarrow{w^*} 0$, we have $|f_n|(y) \to 0$ and hence $|T'(f_n)|(x) \to 0$. Therefore $|T'(f_n)| \xrightarrow{w^*} 0$. On the other hand, by hypothesis $T'(f_n)(x_n) = f_n(Tx_n) \to 0$ for every disjoint sequence $(x_n) \subset A^+$. Then, applying Lemma 2.1, we get $\sup_{x \in A} |T'(f_n)|(x) \to 0$. Now, from the inequality

$$\sup_{y \in T(A)} |f_n(y)| = \sup_{x \in A} |f_n(Tx)| \le \sup_{x \in A} |T'(f_n)|(x),$$

we conclude that $\sup_{y \in T(A)} |f_n(y)| \to 0$ and we are done.

(2) We claim that $|Tx_n| \xrightarrow{w} 0$ holds in F. Let $f \in (F')^+$. By Theorem 1.73 of [2], $T' : F' \to E'$ is order bounded. So there exists some $g \in (E')^+$ such that $T'[-f, f] \subseteq [-g, g]$. For each n pick $|f_n| \le f$ with $f(|Tx_n|) = f_n(Tx_n) = T'(f_n)(x_n)$ (see Theorem 1.23 [2]). Thus, for each n, we have

$$f(|T(x_n)|) = f_n(Tx_n) \le |T'(f_n)|(|x_n|) \le g(|x_n|).$$

Since $|x_n| \xrightarrow{w} 0$, we have $g(|x_n|) \to 0$ and hence $f(|Tx_n|) \to 0$. Therefore $|Tx_n| \xrightarrow{w} 0$ holds in F. On the other hand, if $j : F \to F''$ is the natural embedding, then $|j(Tx_n)| = j(|Tx_n|) \xrightarrow{w^*} 0$ holds in F''. Also, by hypothesis $j(Tx_n)(f_n) = f_n(Tx_n) \to 0$ for every disjoint sequence $(f_n) \subset B^+$. Then, applying Lemma 2.1 for the sequence $(j(Tx_n)) \subset F''$ and the solid subset $B \subset F'$, we get $\sup_{f \in B} |j(Tx_n)|(f) \to 0$. Now, from

$$\sup_{g \in T'(B)} |g(x_n)| = \sup_{f \in B} |f(Tx_n)| \le \sup_{f \in B} f(|T(x_n)|) \quad = \sup_{f \in B} (j|Tx_n|)(f)$$

$$= \sup_{f \in B} |j(Tx_n)|(f),$$

we conclude that $\sup_{g \in T'(B)} |g(x_n)| \to 0$. This completes the proof. $\square$

The next result gives a new characterisations of order bounded awDP operators between Banach lattices, through the almost DP (resp. almost (L)-) sets.

Theorem 2.3. *Let $T : E \to F$ be an order bounded operator between two Banach lattices. Then the following assertions are equivalent:*

(1) *T is an awDP operator.*
(2) *T carries the solid hull of each relatively weakly compact subset of E to an almost DP set in F.*
(3) *T carries each relatively weakly compact subset of E to an almost DP set in F.*
(4) *T' carries the solid hull of each relatively weakly compact subset of F' to an almost (L)-set in E'.*
(5) *T' carries each relatively weakly compact subset of F' to an almost (L)-set in E'.*

Proof. (1) $\Rightarrow$ (2) Let A be a relatively weakly compact subset of E and let $(f_n) \subset F'$ be a disjoint weakly null sequence. By Theorem 4.34 of [2] if $(x_n) \subset (\text{sol}(A))^+$ is a disjoint sequence, then $x_n \xrightarrow{w} 0$ and hence by our hypothesis $f_n(Tx_n) \to 0$. Now, since $|f_n| \xrightarrow{w} 0$ by Lemma 2.2 we conclude that $\sup_{x \in T(\text{sol}(A))} |f_n(x)| \to 0$. Therefore $T(\text{sol}(A))$ is an almost DP set.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let $(x_n) \subset E$, $(f_n) \subset F'$ be two disjoint weakly null sequences. It follows that the set $\{Tx_n : n \in \mathbb{N}\}$ is an almost DP. Therefore, $\sup_k |f_n(Tx_k)| \to 0$ as $n \to \infty$. Now, from the inequality $\sup_k |f_n(Tx_k)| \ge |f_n(Tx_n)|$ we see that $f_n(Tx_n) \to 0$, and hence T is an awDP operator.

(1) $\Rightarrow$ (4) Let B be a relatively weakly compact subset of F' and let $(x_n) \subset E$ be a disjoint weakly null sequence. Similarly, we have $|x_n| \xrightarrow{w} 0$ and for each disjoint sequence $(f_n) \subset (\mathrm{sol}(B))^+$, $f_n \xrightarrow{w} 0$ and Thus by hypothesis $f_n(Tx_n) \to 0$. We infer by Lemma 2.2 that $\sup_{f \in T'(\mathrm{sol}(B))} |f(x_n)| \to 0$, i.e., $T'(\mathrm{sol}(B))$ is an almost (L)-set.

(4) $\Rightarrow$ (5) Obvious.

(5) $\Rightarrow$ (1) Let $(x_n) \subset E$, $(f_n) \subset F'$ be two disjoint weakly null sequences. Since the set $\{T'(f_n) : n \in \mathbb{N}\}$ is an almost (L)-set, it follows by the same justification in (3) $\Rightarrow$ (1) that $f_n(Tx_n) \to 0$, and hence T is an awDP operator. $\qquad\square$

The set characterisations in the above theorem enable us to derive the following result.

Corollary 2.4. *In the class of all order bounded operators from E into E, the order bounded awDP operators from E into E form a closed two-sided ideal.*

Proof. It is easy to see from the characterisation (3) of Theorem 2.3 that if for two operators $T, S : E \to E$, S is an awDP operator thus the product ST is so. Now, if T is an awDP operator, let B be a relatively weakly compact subset of E'. Hence, $S'(B)$ is also a relatively weakly compact subset of E'. As T is an awDP operator then by Theorem 2.3(5), $T'S'(B)$ is an almost (L)-set in E'. This shows that $(ST)' = T'S'$ carries each relatively weakly compact subset of E' to an almost (L)-set in E'. Thus, by Theorem 2.3(5) again, ST is an awDP operator and we are done. $\qquad\square$

In our following result, we show that order bounded awDP operators satisfy some lattice approximations.

Theorem 2.5. *Let $T : E \to F$ be an order bounded awDP operator between two Banach lattices. Then, the following assertions hold:*

(1) *For each relatively weakly compact subsets $A \subset E$, $B \subset F'$, and for every $\varepsilon > 0$, there exists some $u \in E^+$ satisfying*

$$|f|\left(T\left(|x| - u\right)^+\right) \leq \varepsilon$$

for all $x \in A$ and all $f \in B$.

(2) *For each relatively weakly compact subsets $A \subset E$, $B \subset F'$, and for every $\varepsilon > 0$, there exists some $g \in (F')^+$ satisfying*

$$(|f| - g)^+ (T|x|) \leq \varepsilon$$

for all $x \in A$ and all $f \in B$.

Proof. Note that the proof is similar for the two assertions, so we present only that of the first one. To do this, we proceed in two steps:

Step 1: For every disjoint sequence (x_n) in the solid hull of A, the sequence (Tx_n) converges uniformly to zero on the solid hull of B. Indeed, the sequence (x_n) is in this case, a disjoint weakly null sequence (Theorem 4.34 of [2]). Thus, by Theorem 2.3(4) we have

$$\sup_{f \in \mathrm{sol}(B)} |f(Tx_n)| = \sup_{g \in T'(\mathrm{sol}(B))} |g(x_n)| \to 0$$

as desired.

Step 2: Assume by way of contradiction that there exists a relatively weakly compact subsets $A \subset E$, $B \subset F'$ and some $\varepsilon > 0$ such that for each $u \in E^+$ we have

$$|f|\left(T\left(|x| - u\right)^+\right) > \varepsilon$$

for at least two elements $x \in A$ and $f \in B$. In particular, an easy inductive argument shows that there exists a sequences $(x_n) \subset A$, $(f_n) \subset B$ such that

$$(**) \qquad |f_n| \left(T \left(\left| x_{n+1} \right| - 4^n \sum_{i=1}^{n} |x_i| \right)^+ \right) > \varepsilon$$

holds for each n. Put $y = \sum_{n=1}^{\infty} 2^{-n} |x_n|$, $y_n = \left(|x_{n+1}| - 4^n \sum_{i=1}^{n} |x_i| \right)^+$ and $z_n = \left(|x_{n+1}| - 4^n \sum_{i=1}^{n} |x_i| - 2^{-n} y \right)^+$. From Lemma 4.35 of [2] the sequence (z_n) is disjoint. Also, since $0 \le z_n \le |x_n + 1|$ holds, we see that $(z_n) \subset sol(A)$, and so by Step 1 $\sup_{f \in sol(B)} |f(Tz_n)| \to 0$. In particular, $|f_n|(Tz_n) \to 0$. On the other hand, we have $0 \le y_n - z_n \le 2^{-n} y$ from which we get $\|y_n - z_n\| \le 2^{-n} \|y\|$. In particular, we infer that $|f_n|(T(y_n - z_n)) \to 0$. Therefore, we see that

$$|f_n|(Ty_n) = |f_n|(T(y_n - z_n)) + |f_n|(Tz_n) \to 0,$$

which contradicts $(**)$. This completes the proof. $\qquad \square$

Corollary 2.6. *If $T : E \to F$ is a positive operator between two Banach lattices, then the following assertions are equivalent:*

(1) *T is an awDP operator.*
(2) *For each relatively weakly compact subsets $A \subset E$, $B \subset F'$, and for every $\varepsilon > 0$, there exists some $u \in E^+$ satisfying*

$$|f| \left(T \left(|x| - u \right)^+ \right) \le \varepsilon$$

for all $x \in A$ and all $f \in B$.
(3) *For each relatively weakly compact subsets $A \subset E$, $B \subset F'$, and for every $\varepsilon > 0$, there exists some $g \in (F')^+$ satisfying*

$$\left(|f| - g \right)^+ (T |x|) \le \varepsilon$$

for all $x \in A$ and all $f \in B$.

Proof. Note that the proof is similar for the two equivalences $1 \Leftrightarrow 2$ and $1 \Leftrightarrow 3$. The implication $1 \Rightarrow 2$ is exactly Theorem 2.5(1). For the reciprocal one, let $(x_n) \subset E$, $(f_n) \subset F'$ be two disjoint weakly null sequences, and let $\varepsilon > 0$. Put $A = \{x_n : n \in \mathbb{N}\}$, $B = \{f_n : n \in \mathbb{N}\}$. By hypothesis there exists some $u \in E^+$ so that $|f| \left(T \left(|x| - u \right)^+ \right) \le \varepsilon$ holds for all $x \in A$ and all $f \in B$. In particular, $|f_n| \left(T \left(|x_n| - u \right)^+ \right) \le \varepsilon$ for all n. Now, as $|f_n| \xrightarrow{w} 0$ choose some natural number m such that $|f_n|(Tu) \le \varepsilon$ holds for every $n \ge m$. Thus, for every $n \ge m$ we get

$$\begin{aligned} |f_n(Tx_n)| &\le |f_n|(T|x_n|) \\ &\le |f_n| \left(T \left(|x_n| - u \right)^+ \right) + |f_n|(Tu) \le 2\varepsilon. \end{aligned}$$

This shows that $f_n(Tx_n) \to 0$, and then T is an awDP operator. $\qquad \square$

Corollary 2.7. *For a Banach lattice E, the following assertions are equivalent:*

(1) *E has the wDP property.*
(2) *For each relatively weakly compact subsets $A \subset E$, $B \subset F'$, and for every $\varepsilon > 0$, there exists some $u \in E^+$ satisfying*

$$|f| \left(\left(|x| - u \right)^+ \right) \le \varepsilon$$

for all $x \in A$ and all $f \in B$.

(3) *For each relatively weakly compact subsets $A \subset E$, $B \subset F'$, and for every $\varepsilon > 0$, there exists some $g \in (F')^+$ satisfying*

$$(|f| - g)^+ (|x|) \leq \varepsilon$$

for all $x \in A$ and all $f \in B$.

3. Proof of Theorem A

(*i*) **E has sequentially weakly continuous lattice operations.** Let $(x_n) \subset E$, $(f_n) \subset F'$ be two weakly null sequences. We shall see that $f_n (Tx_n) \to 0$. To this end, put $A = sol \{x_n : n \in \mathbb{N}\}$ and $B = sol \{f_n : n \in \mathbb{N}\}$. We proceed in two steps:

Step 1: We claim that $g_n (Tx_n) \to 0$ for every disjoint sequence $(g_n) \subset B^+$. Let $(g_n) \subset B^+$ be such a sequence. Thus, we have $g_n \xrightarrow{w} 0$. As T is an awDP operator, it follows by Theorem 2.3(2) that (g_n) converges uniformly to zero on $T(A)$, that is, $\sup_{x \in A} |g_n (Tx)| \to 0$. From the inequality $|g_n (Tx_n)| \leq \sup_{x \in A} |g_n (Tx)|$, we conclude that $g_n (Tx_n) \to 0$.

Step 2: Since the lattice operations in E are sequentially weakly continuous, we have $|x_n| \xrightarrow{w} 0$. Thus, taking into account Step 1, we see by Lemma 2.2(2), that (x_n) converges uniformly to zero on $T'(B)$, i.e., $\sup_{f \in B} |f(Tx_n)| \to 0$. From $|f_n(Tx_n)| \leq \sup_{f \in B} |f(Tx_n)|$, we conclude that $f_n(Tx_n) \to 0$. Therefore T is a wDP operator.

(*ii*) **F' has sequentially weakly continuous lattice operations.** Note that the two situations (*i*) and (*ii*) are symmetric. Let $(x_n) \subset E$, $(f_n) \subset F'$ be two weakly null sequences. We shall see that $f_n (Tx_n) \to 0$. To this end, put $A = sol \{x_n : n \in \mathbb{N}\}$ and $B = sol \{f_n : n \in \mathbb{N}\}$. We proceed as in (*i*):

Step 1: We claim that $f_n (Ty_n) \to 0$ for every disjoint sequence $(y_n) \subset A^+$. Let $(y_n) \subset A^+$ be such a sequence. Thus, we have $y_n \xrightarrow{w} 0$. As T is an awDP operator, it follows by Theorem 2.3(4) that (y_n) converges uniformly to zero on $T'(B)$, that is, $\sup_{g \in B} |(T'g)(y_n)| \to 0$. From the inequality $|f_n(Ty_n)| = |T'(f_n)(y_n)| \leq \sup_{g \in B} |(T'g)(y_n)|$, we conclude that $f_n(Ty_n) \to 0$.

Step 2: Since the lattice operations in F' are sequentially weakly continuous, we have $|f_n| \xrightarrow{w} 0$. Thus, taking into account Step 1, we see by Lemma 2.2(1), that (f_n) converges uniformly to zero on $T(A)$, i.e., $\sup_{x \in A} |f_n(Tx)| \to 0$. From $|f_n(Tx_n)| \leq \sup_{x \in A} |f_n(Tx)|$, we conclude that $f_n(Tx_n) \to 0$. Therefore T is a wDP operator.

(*iii*) **T is positive and F is discrete with order continuous norm.** Let us recall that an operator $T : E \to Y$ is said to be order weakly compact, if T carries each order bounded subset of E to a relatively weakly compact one in Y, equivalently, $T([0, x])$ is relatively weakly compact in Y, for every $x \in E^+$. We need to prove the following claim.

Claim. *The product ST is a wDP operator, for every positive order weakly compact operator $S : F \to G$ into an arbitrary Banach lattice.*

Proof. Let $(x_n) \subset E$, $(f_n) \subset G'$ be a weakly null sequences. We shall see that $f_n (ST(x_n)) \to 0$. To this end, let $\varepsilon > 0$. As T is an awDP operator, then ST is so (Corollary 2.4). Therefore, by Theorem 2.5, pick some $u \in E^+$ such that

$$|f_n| \left(ST (|x_n| - u)^+ \right) < \varepsilon$$

holds for all n. Now, from the inequalities

$$
\begin{aligned}
|Tx_n| - Tu &\leq T|x_n| - T(|x_n| \wedge u) \\
&\leq |T|x_n| - T(|x_n| \wedge u)| = T\left((|x_n| - u)^+\right)
\end{aligned}
$$

we see that $(|Tx_n| - Tu)^+ \leq T\left((|x_n| - u)^+\right)$ holds for all n. Thus, for every n we have

$$
\begin{aligned}
|f_n\left(ST\left(x_n\right)\right)| &\leq |f_n|\left(S\left|T\left(x_n\right)\right|\right) \\
&\leq |f_n|\left(S\left((|T\left(x_n\right)| - Tu)^+\right)\right) + |f_n|\left(S\left(|T\left(x_n\right)| \wedge Tu\right)\right) \\
&\leq |f_n|\left(ST\left(|x_n| - u\right)^+\right) + |f_n|\left(S\left(|T\left(x_n\right)| \wedge Tu\right)\right) \\
&\leq \varepsilon + |f_n|\left(S\left(|T\left(x_n\right)| \wedge Tu\right)\right).
\end{aligned}
$$

Or, as F is discrete with order continuous norm, then it follows from [11, Proposition 2.5.23] that the lattice operations in F are sequentially weakly continuous, and then the sequence $(|T\left(x_n\right)| \wedge Tu)$ is order bounded weakly null in F^+. It follows from [11, Corollary 3.4.9] that $\|S\left((|T\left(x_n\right)| \wedge Tu)\right)\| \to 0$. This shows that $\limsup |f_n\left(ST\left(x_n\right)\right)| \leq \varepsilon$. As $\varepsilon > 0$ is arbitrary, we infer that $f_n\left(ST\left(x_n\right)\right) \to 0$ as desired. $\square$

Turning to the proof of the theorem, since the norm of F is order continuous then by [2, Theorem 4.9] each order interval of F is weakly compact. Thus, the identity operator $I : F \to F$ is order weakly compact. Now, it follows from the preceding claim that $T = IT$ is a wDP operator as desired.

Corollary 3.1. *Let E be a Banach lattice such that the lattice operations in E (resp. in E') are sequentially weakly continuous. Then, E has the wDP property if and only if it has the DP property.*

In case the range space is a discrete KB-space, the DP operators and their three weak classes considered in this paper coincide on positive operators. The details follow.

Corollary 3.2. *Let E and F be two Banach lattices such that F is a discrete KB-space. Then, for a positive operator $T : E \to F$ the following statements are equivalent:*

(1) *T is an awDP operator.*
(2) *T is a wDP operator.*
(3) *T is an almost DP operator.*
(4) *T is a DP operator.*

Proof. It suffices to show that a positive wDP operator $T : E \to F$ is DP. To this end, let $(x_n) \subset E$ be a weakly null sequence. Since F is a discrete KB-space then it is a dual (see [11, Exercise 5.4.E2]), that is, $F = G'$ for some Banach lattice G. To see that $\|Tx_n\| \to 0$ it suffices by [8, Corollary 2.7] to show that $|Tx_n| \xrightarrow{w^*} 0$ and $(Tx_n)(y_n) \to 0$ for every disjoint bounded sequence $(y_n) \subset G^+$. Note that, since the lattice operations in F are sequentially weakly continuous then we have $|Tx_n| \xrightarrow{w} 0$. Now, if $(y_n) \subset G^+$ is a disjoint bounded sequence, then by Theorem 2.4.14 of [11] $y_n \xrightarrow{w} 0$ as the norm of $G' = F$ is order continuous. By the lattice embedding $G \hookrightarrow G''$ we see that $y_n \xrightarrow{w} 0$ in $G'' = F'$. Since T is a wDP operator, then $(Tx_n)(y_n) = y_n(Tx_n) \to 0$ as desired. This completes the proof. $\square$

In particular, we obtain a result noted by W. Wnuk in [15, Proposition 6].

Corollary 3.3. *For a discrete KB-space E the following statements are equivalent:*

(1) *E has the wDP property.*
(2) *E has the DP property.*
(3) *E has the positive Schur property.*
(4) *E has the Schur property.*

References

1. C. D. Aliprantis and O. Burkinshaw, *Dunford-Pettis operators on Banach lattices*, Trans. Amer. Math. Soc. **274** (1982), no. 1, 227–238.
2. Charalambos D. Aliprantis and Owen Burkinshaw, *Positive operators*, Springer, Dordrecht, 2006, Reprint of the 1985 original.
3. Belmesnaoui Aqzzouz and Khalid Bouras, *(L) sets and almost (L) sets in Banach lattices*, Quaest. Math. **36** (2013), no. 1, 107–118.
4. Belmesnaoui Aqzzouz and Aziz Elbour, *Some characterizations of almost Dunford-Pettis operators and applications*, Positivity **15** (2011), no. 3, 369–380.
5. Khalid Bouras, *Almost Dunford-Pettis sets in Banach lattices*, Rend. Circ. Mat. Palermo (2) **62** (2013), no. 2, 227–236.
6. Khalid Bouras and Mohammed Moussa, *On the class of positive almost weak Dunford-Pettis operators*, Positivity **17** (2013), no. 3, 589–600.
7. Z. L. Chen and A. W. Wickstead, *Relative weak compactness of solid hulls in Banach lattices*, Indag. Math. (N.S.) **9** (1998), no. 2, 187–196.
8. P. G. Dodds and D. H. Fremlin, *Compact operators in Banach lattices*, Israel J. Math. **34** (1979), no. 4, 287–320.
9. T. Leavelle, *The reciprocal dunford-pettis property*, Ann. Mat. Pura Appl., to appear.
10. Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 97, Springer-Verlag, Berlin-New York, 1979, Function spaces.
11. Peter Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991.
12. Frank Räbiger, *Beiträge zur Strukturtheorie der Grothendieck-Räume*, Sitzungsberichte der Heidelberger Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse [Reports of the Heidelberg Academy of Science. Section for Mathematics and Natural Sciences], vol. 85, Springer-Verlag, Berlin, 1985.
13. J. A. Sanchez, *Operators on banach lattices (spanish)*, Ph.D. thesis, Complutense University, Madrid, 1985.
14. Witold Wnuk, *Banach lattices with properties of the Schur type – a survey*, Confer. Sem. Mat. Univ. Bari (1993), no. 249, 25.
15. Witold Wnuk, *Banach lattices with the weak Dunford-Pettis property*, Atti Sem. Mat. Fis. Univ. Modena **42** (1994), no. 1, 227–236.

Université Ibn Tofaïl, Faculté des Sciences, Département de Mathématiques, B.P. 133, Kénitra 14000, Maroc.
E-mail address: nmachrafi@gmail.com

Université Moulay Ismaïl, Faculté des Sciences et Techniques, Département de Mathématiques, B.P. 509, Errachidia 52000, Maroc.
E-mail address: azizelbour@hotmail.com

Université Ibn Tofaïl, Faculté des Sciences, Département de Mathématiques, B.P. 133, Kénitra 14000, Maroc.
E-mail address: mohammed.moussa09@gmail.com