MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

Generalized solutions of Riccati equalities and inequalities

D. Z. Arov, M. A. Kaashoek, D. R. Pik

Methods Funct. Anal. Topology, Volume 22, Number 2, 2016, pp. 95-116

Methods of Functional Analysis and Topology
Vol. 22 (2016), no. 2, pp. 95–116

GENERALIZED SOLUTIONS OF RICCATI EQUALITIES AND INEQUALITIES

D. Z. AROV, M. A. KAASHOEK, AND D. R. PIK

Dedicated to Yurii Makarovich Berezanskii on the occasion of his 90th birthday

Abstract. The Riccati inequality and equality are studied for infinite dimensional linear discrete time stationary systems with respect to the scattering supply rate. The results obtained are an addition to and based on our earlier work on the Kalman–Yakubovich–Popov inequality in [6]. The main theorems are closely related to the results of Yu. M. Arlinskiĭ in [3]. The main difference is that we do not assume the original system to be a passive scattering system, and we allow the solutions of the Riccati inequality and equality to satisfy weaker conditions.

1. Introduction and main theorems

This paper is an addition to [6]. Throughout $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a shorthand notation for the linear discrete time-invariant system

$$\Sigma \begin{cases} x_{n+1} &= Ax_n + Bu_n \\ y_n &= Cx_n + Du_n \end{cases} \qquad (n = 0, 1, 2, \ldots). \tag{1.1}$$

Here $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$ and $D : \mathcal{U} \to \mathcal{Y}$ are bounded linear operators acting between separable Hilbert spaces. The operator A is called the *state operator*, B and C are referred to as *input operator* and *output operator*, respectively, and D is called the *feed through operator*. The spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are called *state space*, *input space*, and *output space*, respectively. By definition the *transfer function* of the system Σ is the operator-valued function

$$\theta_\Sigma(\lambda) = D + \lambda C(I - \lambda A)^{-1}B.$$

Note that θ_Σ is an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function which is defined and analytic on the open set consisting of all $\lambda \in \mathbb{C}$ such that $I - \lambda A$ is boundedly invertible. In particular, θ_Σ is analytic in an open neighborhood of zero.

With the system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ we associate the linear manifolds $\operatorname{Im}(A|B)$ and $\operatorname{Ker}(C|A)$ which are defined as follows

$$\operatorname{Im}(A|B) = \operatorname{span}\{\operatorname{Im} A^n B \mid n \geq 0\}, \quad \operatorname{Ker}(C|A) = \bigcap_{n\geq 0} \operatorname{Ker} CA^n. \tag{1.2}$$

Recall that Σ is *minimal* if $\operatorname{Im}(A|B)$ is dense in $\mathcal{X}$ (i.e., Σ is *controllable*) and $\operatorname{Ker}(C|A) = \{0\}$ (i.e., Σ is *observable*); cf., Theorem 2.1 in [6]. Finally, we denote by $M(\Sigma)$ the *system matrix associated with* Σ, that is, $M(\Sigma)$ is the 2×2 operator matrix defined by

$$M(\Sigma) := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}. \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary 47A48, 47A62; Secondary 47A56, 93C55.
Key words and phrases. Discrete time-invariant systems, scattering supply rate, passive systems, Riccati equality, Riccati inequality, Kalman–Yakubovich–Popov inequality.

In this paper we are interested in systems that are *passive* (or, in an other terminology, *dissipative*) with respect to the scattering supply rate function $w(u, y) = \|u\|^2 - \|y\|^2$. The latter means that for each initial condition x_0 and each input sequence $u_0, u_1, u_2, \dots$ we have

$$\|x_{n+1}\|^2 - \|x_n\|^2 \leq \|u_n\|^2 - \|y_n\|^2, \quad n = 0, 1, 2, \dots,$$

where x_{n+1} and y_n are determined from u_n and x_n via the system equations in (1.1) In that case the associate system matrix is a contractive operator from $\mathcal{X} \oplus \mathcal{U}$ into $\mathcal{X} \oplus \mathcal{Y}$. The converse is also true. In other words, the system Σ is passive if and only if the operator M_Σ is a contraction. Moreover, in that case its transfer function θ_Σ is a Schur class function.

Our main theorems given below concern the Riccati equality and Riccati inequality for discrete time systems with a scattering supply rate. Analogous results may be obtained for other supply rates, e.g., impedance and transmission supply rates, and for continuous time systems. For these different supply rate functions see, e.g., the papers [7], [8] and the references therein.

Definition 1.1. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. A (possibly unbounded) selfadjoint operator H in $\mathcal{X}$ is said to be a* generalized solution of the Riccati equation associated to Σ *if the following four conditions are satisfied:*

- (C1) *the operator H is positive as a selfadjoint operator, i.e, $\langle Hx, x\rangle > 0$ for each $0 \neq x \in \mathcal{D}(H)$;*
- (C2) *$A\mathcal{D}(H^{1/2}) \subset \mathcal{D}(H^{1/2})$ and $B\mathcal{U} \subset \mathcal{D}(H^{1/2})$;*
- (C3) *the operator $\delta_\Sigma(H) = I_\mathcal{U} - D^*D - (H^{1/2}B)^*H^{1/2}B$ is bounded and nonnegative, and*

$$\Big(D^*C + (H^{1/2}B)^*H^{1/2}A\Big)\mathcal{D}(H^{1/2}) \subset \delta_\Sigma(H)^{1/2}\mathcal{U}; \tag{1.4}$$

- (C4) *for each $x \in \mathcal{D}(H^{1/2})$ we have*

$$\begin{aligned} &\|H^{1/2}x\|^2 - \|H^{1/2}Ax\|^2 - \|Cx\|^2 \\ &\qquad = \|\Big(\delta_\Sigma(H)^{1/2}\Big)^{[-1]}\Big(D^*C + (H^{1/2}B)^*H^{1/2}A\Big)x\|^2. \end{aligned} \tag{1.5}$$

Here and in the sequel $\mathcal{D}(H)$ stands for the *domain* of the operator H. Since H is a positive selfadjoint operator, we know from the theory of operators (possibly unbounded) on Hilbert spaces (see, e.g., Chapter XII in [16]) that $H^{1/2}$ is well-defined and a positive selfadjoint operator too. Moreover,

$$\begin{aligned} \mathcal{D}(H) &= \{x \in \mathcal{D}(H^{1/2}) \mid H^{1/2}x \in \mathcal{D}(H^{1/2})\}, \\ Hx &= H^{1/2}\big(H^{1/2}x\big) \quad \big(x \in \mathcal{D}(H)\big). \end{aligned}$$

The latter two properties define $H^{1/2}$ uniquely.

Note that (C1) and (C2) imply that the operator $H^{1/2}B$ is a bounded operator from $\mathcal{U}$ into $\mathcal{X}$, and the hence the operator $\delta_\Sigma(H)$ defined in (C3) is automatically bounded.

The symbol $[-1]$ appearing in the right hand side of the inequality (1.7) means that the term involved is the Moore-Penrose pseudo-inverse of the nonnegative bounded operator $\delta_\Sigma(H)^{1/2}$. See the final paragraph of the present section for the definition of this notion. Note that $\delta_\Sigma(H)$ can be a zero operator (see Theorem 5.4).

In what follows we refer to (1.5) as the *Riccati equality* associated to Σ. By RE_Σ we shall denote the set of all generalized solutions H of the Riccati equation associated to Σ. If $H \in \mathrm{RE}_\Sigma$, then

$$\mathrm{Im}\,(A|B) \subset \mathcal{D}(H^{1/2}) \quad \text{and} \quad \mathrm{Im}\,(A^*|C^*) \subset \mathcal{D}(H^{-1/2}). \tag{1.6}$$

The first inclusion follows from condition (C2). The second inclusion in (1.6) requires a proof which will be given in the next section; see Lemma 2.3.

By RE_Σ° we denote the subset of RE_Σ consisting of all $H \in \mathrm{RE}_\Sigma$ such that the following two additional conditions are satisfied:

(a) both $H^{1/2}\mathrm{Im}\,(A|B)$ and $(H^{-1/2})\mathrm{Im}\,(A^*|C^*)$ are dense in $\mathcal{X}$;
(b) the linear manifold $\mathrm{Im}\,(A|B)$ is a core for the operator $H^{1/2}$.

By definition (see, e.g., Section III.5.2 in [19]) condition (b) means that the linear manifold $\{(u, H^{1/2}u) \mid u \in \mathrm{Im}\,(A|B)\}$ is dense in the graph of $H^{1/2}$ with respect to the graph norm. Note that the sets $H^{1/2}\mathrm{Im}\,(A|B)$ and $(H^{-1/2})\mathrm{Im}\,(A^*|C^*)$ are well defined because of (1.6). For a better understanding of condition (a) we refer to Lemma 2.3 in Section 2 below. We shall prove the following theorems.

Theorem 1.2. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system. If the set RE_Σ is nonempty, then the transfer function θ_Σ coincides with a Schur class function in a neighborhood of zero.*

Theorem 1.3. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system, and assume that its transfer function coincides with a Schur class function in a neighborhood of zero. Then the set RE_Σ° is nonempty and this set contains a minimal element with respect to the usual partial ordering of (possibly unbounded) nonnegative selfadjoint operators.*

Let us recall (see [19, page 330] or [6, Section 5]) the definition of the ordering referred to in the previous theorem. Let H_1, H_2 be non-negative selfadjoint operators acting in a Hilbert space $\mathcal{X}$. Then, by definition, $H_1 \prec H_2$ means that

$$\mathcal{D}(H_2^{1/2}) \subset \mathcal{D}(H_1^{1/2}) \quad \text{and} \quad \|H_1^{1/2}x\| \leq \|H_2^{1/2}x\| \quad \Big(x \in \mathcal{D}(H_2^{1/2})\Big).$$

If H_1 and H_2 are bounded, then $H_1 \prec H_2$ is equivalent to $H_1 \leq H_2$.

To prove the above two theorems it will be convenient first to consider the Riccati inequality associated to Σ. This inequality appears when the equality sign in (1.5) is replaced by a "greater than equal to" sign. In other words condition (C4) in Definition 1.1 is replaced by

(CI4) *for each $x \in \mathcal{D}(H^{1/2})$ we have*

$$\begin{aligned} &\|H^{1/2}x\|^2 - \|H^{1/2}Ax\|^2 - \|Cx\|^2 \\ &\qquad \geq \|\Big(\delta_\Sigma(H)^{1/2}\Big)^{[-1]}\Big(D^*C + (H^{1/2}B)^*H^{1/2}A\Big)x\|^2, \quad x \in \mathcal{D}(H^{1/2}). \end{aligned} \tag{1.7}$$

We shall say that a selfadjoint operator H acting in $\mathcal{X}$ is a *generalized solution of the Riccati inequality associated to* Σ when conditions (C1), (C2), (C3), and (CI4) are satisfied. By RI_Σ we shall denote the set of all generalized solutions H of the Riccati inequality associated to Σ. Furthermore, RI_Σ° will denote the subset of RI_Σ consisting of all $H \in \mathrm{RI}_\Sigma$ such the two additional conditions (a) and (b) above are satisfied. Clearly, the following inclusions hold:

$$\mathrm{RE}_\Sigma \subset \mathrm{RI}_\Sigma, \quad \mathrm{RE}_\Sigma^\circ \subset \mathrm{RI}_\Sigma^\circ. \tag{1.8}$$

These inclusions will allow us to derive Theorems 1.2 and 1.3 as corollaries of the following two results.

Theorem 1.4. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system. Then the set RI_Σ is nonemepty if and only if the transfer function of θ_Σ coincides with a Schur class function in a neighborhood of zero.*

Theorem 1.5. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system, and assume that its transfer function coincides with a Schur class function in a neighborhood of zero. Then the set RI_Σ° is nonempty and this set contains a minimal element $H_\circ$ and a maximal* *element $H_\bullet$ with respect to the usual ordering of nonnegative operators. Furthermore, the minimal element $H_\circ$ in RI°_Σ belongs to the set RE°_Σ.*

In Section 3 we shall show that the Riccati inequality is closely related to the Kalman–Yakubovich–Popov inequality. This allows us to prove (see the first paragraph after Theorem 3.1 in Section 3) that Theorem 1.4 is equivalent to Theorem 1.2 in [6], and that Theorem 1.5, except for its final statement, is equivalent to Theorem 5.1 in [6]. The final statement of Theorem 1.5 will be proved in Section 4.

As we mentioned, Theorems 1.2 and 1.3 appear as corollaries of Theorems 1.4 and 1.5. Indeed, if RE_Σ is nonempty, then the same holds true for RI_Σ because of the first inclusion in (1.8). But then Theorem 1.4 tells us that θ_Σ coincides with a Schur class function in a neighborhood of zero, which proves Theorem 1.2. Thus Theorem 1.2 is covered by the "if part" of Theorem 1.4. In a similar way, using the second inclusion in (1.8) and the final statement of Theorem 1.5, one sees that Theorem 1.3 is covered by Theorem 1.5.

The paper consists of seven sections including the present introduction and an appendix. In Section 2 the set RI_Σ is related to the set of H-passive systems. Furthermore, given $H \in \mathrm{RI}_\Sigma$ we give a necessary and sufficient condition on H in order that $H \in \mathrm{RE}_\Sigma$. In Section 3 we make explicit the relation between the Riccati inequality and the Kalman-Yakubovic-Popov inequality which allows us to show that Theorem 1.4 is equivalent to Theorem 1.2. in [6] and Theorem 1.5 (except for the final statement) is equivalent to Theorem 5.1 in [6]. The final statement in Theorem 1.5 is proved in Section 4. In Section 5, using the last part of Theorem 7.1 in [6], we present a necessary and sufficient condition for RI°_Σ to consist of a single element only, and we specify this result for the case when θ is an inner or a co-inner function. Examples illustrating the general theory are given in Section 6. In the Appendix we review a number of results regarding 2×2 nonnegative operator matrices and related Schur complements that are used in the present paper.

Moore–Penrose pseudo-inverse. Let A be a bounded selfadjoint operator on a Hilbert space $\mathcal{X}$. Put $\mathcal{X}_1 = \overline{A\mathcal{X}}$ and $\mathcal{X}_2 = \mathcal{X} \ominus \mathcal{X}_1$. Since A is selfadjoint, $\mathcal{X}_2$ is the null space of A. It follows that relative to the Hilbert space orthogonal direct sum $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ the operator A has the following 2×2 operator matrix representation:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}. \tag{1.9}$$

The fact that $\mathcal{X}_2$ is the null space of A, implies that the operator A_1 maps $\mathcal{X}_1$ in one-to-one way into itself and $A_1\mathcal{X}_1$ is equal to the range of A which is dense in $\mathcal{X}_1$. By $A^{[-1]}$ we denote the closed linear operator given by

$$A^{[-1]} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{Im} A_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}.$$

We call $A^{[-1]}$ the Moore–Penrose pseudo-inverse of A. Its domain $\mathcal{D}(A^{[-1]})$ is the linear space $\operatorname{Im} A_1 \oplus \mathcal{X}_2$. Note that $A^{[-1]}$ is a selfadjoint operator, possibly unbounded. Furthermore, $A^{[-1]}$ is a zero operator if and only if A is a zero operator.

Now assume that A is a bounded selfadjoint operator on $\mathcal{X}$ which is nonnegative. Then $A^{[-1]}$ is nonnegative too, and the square roots $A^{1/2}$ and $(A^{[-1]})^{1/2}$ are well-defined. Note that the spaces $\overline{A\mathcal{X}}$ and $\overline{A^{1/2}\mathcal{X}}$ coincide. Using the latter and the operator matrix representation (1.9), it is not difficult to show that

$$(A^{1/2})^{[-1]} = (A^{[-1]})^{1/2}. \tag{1.10}$$

In particular, these two operators have the same domain.

2. The set RI_Σ and related H-passive systems

Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a linear discrete time-invariant system, and let $H \in \mathrm{RI}_\Sigma$. Since H is a positive operator, the same is true for $H^{1/2}$, and both H and $H^{1/2}$ are one-to-one. It follows (cf., the first paragraph of Subsection 4.1 in [6]) that the following operators are well defined:

$$A_H : \operatorname{Im} H^{1/2} \to \mathcal{X}, \quad A_H H^{1/2} x = H^{1/2} A x \quad \big(x \in \mathcal{D}(H^{1/2})\big), \tag{2.1}$$

$$C_H : \operatorname{Im} H^{1/2} \to \mathcal{Y}, \quad C_H H^{1/2} x = C x \quad \big(x \in \mathcal{D}(H^{1/2})\big), \tag{2.2}$$

$$B_H : \mathcal{U} \to \mathcal{X}, \quad B_H u = H^{1/2} B u \quad (u \in \mathcal{U}). \tag{2.3}$$

From condition (CI4) we see that

$$\|z\|^2 - \|A_H z\|^2 - \|C_H z\|^2 \geq 0 \quad (z \in \operatorname{Im} H^{1/2}).$$

Thus A_H and C_H are bounded in norm by one on $\operatorname{Im} H^{1/2}$. Since $\operatorname{Im} H^{1/2}$ is dense in $\mathcal{X}$, we can extend A_H and C_H by continuity to contractions on $\mathcal{X}$ which also will be denoted by A_H and C_H. From the second part of condition (C2) it follows that B_H is well defined bounded operator, and the first part of condition (C3) implies that B_H is a contractive operator mapping $\mathcal{U}$ into $\mathcal{X}$. Put

$$\Sigma_H = (A_H, B_H, C_H, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}). \tag{2.4}$$

We shall call Σ_H *the system associated with* Σ *and* H. Recall that the system matrix $M(\Sigma_H)$ associated with Σ_H is given by

$$M(\Sigma_H) = \begin{bmatrix} A_H & B_H \\ C_H & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}.$$

Definition 2.1. *In the sequel the system Σ will be called* H-passive *when Σ_H is passive. In other words, Σ is H-passive if and only if $M(\Sigma_{\mathcal{H}})$ is contractive.*

Theorem 2.2. *Let $H \in \mathrm{RI}_\Sigma$. Then the system Σ is H-passive. Furthermore, $H \in \mathrm{RE}_\Sigma$ if and only if*

$$\inf \left\{ \| \begin{bmatrix} x \\ u \end{bmatrix} \|^2 - \|M(\Sigma_H) \begin{bmatrix} x \\ u \end{bmatrix} \|^2 \mid u \in \mathcal{U} \right\} = 0 \quad (x \in \mathcal{X}). \tag{2.5}$$

Proof. We split the proof into two parts. First we show that the system Σ is H-passive.

Part 1. Using the definitions in (2.1), (2.2), and (2.3) we see that condition (C3) can be rephrased as

(C3') the operator $\delta_\Sigma(H) = I_{\mathcal{U}} - D^* D - B_H^* B_H$ is bounded and nonnegative, and

$$(D^* C_H + B_H^* A_H) z \in \delta_\Sigma(H)^{1/2} \mathcal{U} \quad (z \in \operatorname{Im} H^{1/2}). \tag{2.6}$$

Similarly, (CI4) can be rephrased as

$$\begin{aligned} &\|z\|^2 - \|A_H z\|^2 - \|C_H z\|^2 \\ &\qquad \geq \| \left(\delta_\Sigma(H)^{1/2}\right)^{[-1]} (D^* C_H + B_H^* A_H) z\|^2 \quad (z \in \operatorname{Im} H^{1/2}). \end{aligned} \tag{2.7}$$

Next, put

$$\alpha = I_{\mathcal{X}} - A_H^* A_H - C_H^* C_H, \quad \beta = -A_H^* B_H - C_H^* D, \quad \delta = \delta_\Sigma(H).$$

Then

$$R := I_{\mathcal{X} \oplus \mathcal{U}} - M(\Sigma_H)^* M(\Sigma_H) = \begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix}. \tag{2.8}$$

In order to prove that the system Σ is H-passive we have to show that the 2×2 operator matrix in the right hand side of (2.8) is nonnegative. To do this we apply Proposition A.1.

Note that

$$\begin{aligned}\langle \alpha z, z\rangle &= \langle z, z\rangle - \langle A_H^* A_H z, z\rangle - \langle C_H^* C_H z, z\rangle\\ &= \|z\|^2 - \|A_H z\|^2 - \|C_H z\|^2 \geq 0 \quad (z \in \operatorname{Im} H^{1/2}).\end{aligned}$$

Since $\operatorname{Im} H^{1/2}$ is dense in $\mathcal{X}$ and the operators A_H and C_H are bounded, the preceding inequality shows, by continuity, that

$$\langle \alpha x, x\rangle = \|x\|^2 - \|A_H x\|^2 - \|C_H x\|^2 \geq 0 \quad (x \in \mathcal{X}).$$

Hence $\alpha \geq 0$. We already know that $\delta = \delta_\Sigma(H)$ is nonnegative too. Next, note that (2.6) and (2.7) yield

$$\beta^* z \in \delta^{1/2}\mathcal{U} \quad (z \in \operatorname{Im} H^{1/2}), \tag{2.9}$$

$$\langle \alpha z, z\rangle \geq \|(\delta^{1/2})^{[-1]}\beta^* z\|^2 \quad (z \in \operatorname{Im} H^{1/2}). \tag{2.10}$$

Recall that δ is bounded and nonnegative. Thus $\delta_0 := \delta|_{\overline{\operatorname{Im}\delta}}$ is a one-to-one operator on $\overline{\operatorname{Im}\delta}$ and the range $\operatorname{Im}\delta_0$ is dense in $\overline{\operatorname{Im}\delta}$. Since

$$(\delta^{1/2})^{[-1]} = (\delta^{[-1]})^{1/2} = \begin{bmatrix}\delta_0^{-1/2} & 0\\ 0 & 0\end{bmatrix} : \begin{bmatrix}\overline{\operatorname{Im}\delta}\\ \operatorname{Ker}\delta\end{bmatrix} \to \begin{bmatrix}\overline{\operatorname{Im}\delta}\\ \operatorname{Ker}\delta\end{bmatrix},$$

we conclude that the range of $(\delta^{1/2})^{[-1]}$ is a subset of $\overline{\operatorname{Im}\delta}$. Now define

$$\begin{aligned}&\Gamma_0 : \alpha^{1/2}(\operatorname{Im} H^{1/2}) \to \overline{\operatorname{Im}\delta},\\ &\Gamma_0(\alpha^{1/2} z) = (\delta^{1/2})^{[-1]}\beta^* z, \quad z \in \operatorname{Im} H^{1/2}.\end{aligned} \tag{2.11}$$

According to the identity (2.10) the operator Γ_0 is well defined and Γ_0 is a contraction. Observe that

$$\overline{\alpha^{1/2}(\operatorname{Im} H^{1/2})} = \overline{\alpha^{1/2}\left(\overline{\operatorname{Im} H^{1/2}}\right)} = \overline{\alpha^{1/2}\mathcal{X}} = \overline{\alpha\mathcal{X}} = \overline{\operatorname{Im}\alpha}.$$

But then, by continuity, the contraction Γ_0 extends to a contraction $\widetilde{\Gamma_0}$ mapping $\overline{\operatorname{Im}\alpha}$ into $\overline{\operatorname{Im}\delta}$ and such that

$$\delta^{1/2}\widetilde{\Gamma_0}\alpha^{1/2} z = \delta^{1/2}\Gamma_0\alpha^{1/2} z = \delta^{1/2}(\delta^{1/2})^{[-1]}\beta^* z = \beta^* z \quad (z \in \operatorname{Im} H^{1/2}).$$

Here we used that $\delta^{1/2}(\delta^{1/2})^{[-1]}$ is the orthogonal projection onto $\overline{\operatorname{Im}\delta^{1/2}}$ and the fact that $\operatorname{Im}\beta^* \subset \overline{\operatorname{Im}\delta^{1/2}}$ which follows from (2.9). Since $\operatorname{Im} H^{1/2}$ is dense in $\mathcal{X}$ and the operators $\delta^{1/2}\widetilde{\Gamma_0}\alpha^{1/2}$ and β^* are bounded operators, we conclude, by continuity, that $\beta^* = \delta^{1/2}\widetilde{\Gamma_0}\alpha^{1/2}$. Finally, define $\Gamma : \mathcal{X} \to \mathcal{U}$ by

$$\Gamma|_{\overline{\operatorname{Im}\alpha}} = \widetilde{\Gamma_0} \quad \text{and} \quad \Gamma|_{\operatorname{Ker}\alpha} = 0. \tag{2.12}$$

Then $\Gamma : \mathcal{X} \to \mathcal{U}$ is a contraction satisfying conditions (a) and (b) in Proposition A.1, and hence we can apply Proposition A.1 with $T = R$ to show that the operator R in (2.8) is nonnegative. Hence $M(\Sigma_H)$ is a contraction, and the first part of the proposition is proved.

Part 2. In this part given $H \in \mathrm{RI}_\Sigma$ we show that $H \in \mathrm{RE}_\Sigma$ if and only if (2.5) holds. Since $H \in \mathrm{RI}_\Sigma$ we can freely use the operators introduced in the previous part. In particular, R is the operator defined by (2.8) and Γ is the contraction defined by (2.12).

First we assume that $H \in \mathrm{RE}_\Sigma$. This implies (see condition (C4)) that we have equality in (2.7) and in (2.10), and hence the operator Γ_0 defined in (2.11) is an isometry. But then, following the reasoning in the previous part of the proof, we see that $\widetilde{\Gamma_0}$, the continuous extension of Γ_0 to $\overline{\operatorname{Im}\alpha}$, is an isometry too, and thus the operator Γ defined by (2.12) is a partial isometry with initial space $\operatorname{Im}\alpha$. But then the Schur complement $Z = \alpha^{1/2}(I - \Gamma^*\Gamma)\alpha^{1/2}$ is the zero operator, and we can apply Proposition A.2 to show that (2.5) holds.

The converse implication follows in a similar way reversing the arguments. Indeed, assume (2.5) holds. Then Proposition A.2 tells us that the Schur complement of R supported by $\mathcal{X}$ is equal to zero. Here R is given by (2.8). Thus $\alpha^{1/2}(I - \Gamma^*\Gamma)\alpha^{1/2} = 0$, where Γ is the minimal contraction determined by R, which in our case is the contraction defined by (2.12). Thus Γ is a partial isometry with initial space $\overline{\operatorname{Im}\alpha}$. It follows that Γ_0 defined by (2.11) also is an isometry. But then we have equality in (2.10) and hence also in (2.7), Thus condition (C4) is satisfied which implies that $H \in \mathrm{RE}_\Sigma$. □

We conclude this section with the following lemma. For the definition of the notion of pseudo-similarity we refer to [6, Section 3].

Lemma 2.3. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system, and let $H \in \mathrm{RI}_\Sigma$. Then the systems Σ and Σ_H are pseudo-similar and $H^{1/2}$ is a pseudo-similarity from Σ to Σ_H. Furthermore, the inclusions in* (1.6) *are satisfied, and Σ_H is minimal if and only if both $H^{1/2}\operatorname{Im}(A|B)$ and $(H^{1/2})^{-1}\operatorname{Im}(A^*|C^*)$ are dense in $\mathcal{X}$.*

Proof. Put $S = H^{1/2}$. To prove that S is a pseudo-similarity from Σ to Σ_H we have to check (see formulas (3.1)–(3.4) in [6]) the following properties:

$$\overline{\mathcal{D}(S)} = \mathcal{X}, \qquad \overline{\operatorname{Im} S} = \mathcal{X}; \tag{2.13}$$

$$A\mathcal{D}(S) \subset \mathcal{D}(S), \qquad SAx = A_H Sx, \quad x \in \mathcal{D}(S); \tag{2.14}$$

$$B\mathcal{U} \subset \mathcal{D}(S), \qquad SB = B_H; \tag{2.15}$$

$$Cx = C_H Sx, \quad x \in \mathcal{D}(S). \tag{2.16}$$

Condition (C1) in Definition 1.1 implies that $H^{1/2}(\mathcal{X} \to \mathcal{X})$ is a closed, injective, densely defined operator, and its range is dense in $\mathcal{X}$. Since $S = H^{1/2}$, it follows that (2.13) holds. Formulas (2.14) and (2.15) follow from condition (C2) in Definition 1.1 using the definitions of A_H and B_H in (2.1) and (2.3), respectively. Formula (2.16) follows from the definition of C_H in (2.2). Thus $S = H^{1/2}$ is a pseudo-similarity from Σ to Σ_H.

The identities in the right hand side of (2.14) and (2.15) tell us that $\operatorname{Im}(A|B)$ is a subset of $\mathcal{D}(H^{1/2})$. Thus the the first inclusion in (1.6) holds true. Furthermore, we have

$$\begin{aligned}\operatorname{Im}(A_H|B_H) &= \operatorname{span}\{\operatorname{Im} A_H^n B_H \mid n = 0, 1, 2, \ldots\}\\ &= \operatorname{span}\{\operatorname{Im} H^{1/2}A^n B \mid n = 0, 1, 2, \ldots\} = H^{1/2}\operatorname{Im}(A|B).\end{aligned}$$

This implies that Σ_H is controllable if and only if $H^{1/2}\operatorname{Im}(A|B)$ is dense in $\mathcal{X}$.

Next we apply the final part of Proposition 3.1 in [6]. It follows that $S^{-1} = H^{-1/2}$ is a pseudo–similarity from Σ_H to Σ. But then $(S^{-1})^* = H^{-1/2}$ is a pseudo–similarity from Σ^* to$(\Sigma_H)^*$, where

$$\Sigma^* = (A^*, C^*, B^*, D^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}), \tag{2.17}$$

$$(\Sigma_H)^* = (A_H^*, C_H^*, B_H^*, D^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}). \tag{2.18}$$

In particular, using (2.14) and (2.15), we have

$$\begin{aligned}A^*\mathcal{D}(H^{-1/2}) \subset \mathcal{D}(H^{-1/2}), &\qquad H^{-1/2}A^*x = A_H^* H^{-1/2}x, \quad x \in \mathcal{D}(H^{-1/2});\\ C^*\mathcal{Y} \subset \mathcal{D}(H^{-1/2}), &\qquad H^{-1/2}C^* = C_H^*.\end{aligned}$$

Thus $\operatorname{Im}(A^*|C^*) \subset \mathcal{D}(H^{-1/2})$, and hence the second inclusion in (1.6) holds true. Furthermore, using the same calculation for A_H^*, C_H^* as for A_H, B_H in the previous paragraph, we obtain $\operatorname{Im}(A_H^*|C_H^*) = H^{-1/2}\operatorname{Im}(A^*|C^*)$, which shows that Σ_H is observable if and only if the space $H^{-1/2}\operatorname{Im}(A^*|C^*)$ is dense in $\mathcal{X}$. This completes the proof. □

The system Σ^* defined by (2.17) is called the *adjoint* of the system Σ. Using the main results of the next section we shall derive some further properties of the adjoint system at the end of Section 4.

3. The Kalman–Yakubovich–Popov inequality

The Riccati inequality is closely related to the Kalman–Yakubovich–Popov inequality (for short, KYP inequality). Recall (see Section 1 of [6]) that a (possibly unbounded) selfadjoint operator H acting in $\mathcal{X}$ is called a *generalized solution of the KYP inequality associated to* Σ if conditions (C1) and (C2) are satisfied, and

$$K_\Sigma(H)\begin{bmatrix} x \\ u \end{bmatrix} \geq 0, \quad x \in \mathcal{D}(H^{1/2}), \quad u \in \mathcal{U}, \tag{3.1}$$

where

$$K_\Sigma(H)\begin{bmatrix} x \\ u \end{bmatrix} = \| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_\mathcal{U} \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \|^2 - \| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \|^2. \tag{3.2}$$

Note that condition (C2) tells us that $Ax + Bu \in \mathcal{D}(H^{1/2})$ whenever x and u are as in (3.1). Thus $K_\Sigma(H)$ is well defined. See [10] for continuous time analogues of the results in [6].

In what follows KYP_Σ denotes the set of all generalized solution of the KYP inequality associated to Σ. We shall prove the following theorem.

Theorem 3.1. *A selfadjoint operator H acting on $\mathcal{X}$ belongs to RI_Σ if and only if H belongs to KYP_Σ, that is, $\mathrm{RI}_\Sigma = \mathrm{KYP}_\Sigma$.*

When Theorem 3.1 is proved, then Theorem 1.4 is proved too. In fact, if Theorem 3.1 is proved, then Theorem 1.4 is equivalent to Theorem 1.2. in [6]. Analogously, Theorem 1.5 (except for the final sentence) is equivalent to Theorem 5.1 in [6]. The statement in the final sentence of Theorem 1.5 will be proved in the next section.

We shall denote by $\mathrm{KYP}_\Sigma^\circ$ the set of all H in KYP_Σ that satisfy the additional conditions (a) and (b) appearing in the paragraph preceding Theorem 1.2. From Lemma 2.3 we know that condition (a) just means that Σ_H is minimal. It follows that $\mathrm{KYP}_\Sigma^\circ$ coincides with the set which in [6] is denoted by $\mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$; see [6, formulas (5.1) and (5.2)]. Using Theorem 3.1 and the definitions of the sets RI_Σ° and $\mathrm{KYP}_\Sigma^\circ$ we obtain the following corollary.

Corollary 3.2. *A selfadjoint operator H acting on $\mathcal{X}$ belongs to RI_Σ° if and only if H belongs to $\mathrm{KYP}_\Sigma^\circ$, that is, $\mathrm{RI}_\Sigma^\circ = \mathrm{KYP}_\Sigma^\circ$.*

In order to prove Theorem 3.1 we need some preliminaries. Assume that $H \in \mathrm{KYP}_\Sigma$. By specifying (3.1) for the vectors $(x, 0)$ and $(0, u)$ we see that

$$\|H^{1/2}x\|^2 - \|H^{1/2}Ax\|^2 - \|Cx\|^2 \geq 0 \quad \big(x \in \mathcal{D}(H^{1/2})\big),$$

$$\|u\|^2 - \|Du\|^2 - \|H^{1/2}Bu\|^2 \geq 0 \quad (u \in \mathcal{U}).$$

As we proved in Subsection 4.1 of [6], this allows one to define operators A_H, B_H and C_H in the same way as in the paragraphs preceding Proposition 2.2. Also in this setting the resulting system Σ_H, defined as in (2.4), is called the *the system associated with* Σ *and* H. The following lemma, which is the analogue of the first part of Proposition 2.2 with $H \in \mathrm{KYP}_\Sigma$ in place of $H \in \mathrm{RI}_\Sigma$, is covered by Proposition 4.2 in [6].

Lemma 3.3. *Let $H \in \mathrm{KYP}_\Sigma$. Then the system Σ_H associated with Σ and H is passive.*

Proof of Theorem 3.1. We split the proof into to parts. In the first part we show that $H \in \mathrm{RI}_\Sigma$ implies that $H \in \mathrm{KYP}_\Sigma$. The second part proves the reverse implication.

Part 1. Let $H \in \mathrm{RI}_\Sigma$, and let $\Sigma_H = (A_H, B_H, C_H, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be the system associated with Σ and $H \in \mathrm{RI}_\Sigma$. In particular, H satisfies conditions (C1) and (C2). Thus it remains to prove (3.1). In order to that, fix $x \in \mathcal{D}(H^{1/2})$ and $u \in \mathcal{U}$. Then

$$\begin{aligned} K_\Sigma(H)\begin{bmatrix} x \\ u \end{bmatrix} &= \Big\| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_\mathcal{U} \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \Big\|^2 - \Big\| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \Big\|^2 \\ &= \Big\| \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \Big\|^2 - \Big\| \begin{bmatrix} H^{1/2}A & H^{1/2}B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \Big\|^2 \\ &= \Big\| \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \Big\|^2 - \Big\| \begin{bmatrix} A_H & B_H \\ C_H & D \end{bmatrix}\begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \Big\|^2. \end{aligned}$$

But, by Theorem 2.2, the system matrix $M(\Sigma_H)$ is a contraction. It follows that

$$\Big\| \begin{bmatrix} A_H & B_H \\ C_H & D \end{bmatrix}\begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \Big\| \leq \Big\| \begin{bmatrix} H^{1/2}x \\ u \end{bmatrix} \Big\|, \quad x \in \mathcal{D}(H^{1/2}), \quad u \in \mathcal{U}.$$

Thus (3.1) holds true.

Part 2. Let $H \in \mathrm{KYP}_\Sigma$, and let $\Sigma_H = (A_H, B_H, C_H, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be the system associated with Σ and $H \in \mathrm{KYP}_\Sigma$. Since $H \in \mathrm{KYP}_\Sigma$, we know that conditions (C1) and (C2) are satisfied. It remains to check (C3) and (CI4). According to Lemma 3.3, the system matrix $M(\Sigma_H)$ is a contraction. This implies that the operator T defined by

$$T = \begin{bmatrix} I_\mathcal{X} - A_H^*A_H - C_H^*C_H & -A_H^*B_H - C_H^*D \\ -B_H^*A_H - D^*C_H & I_\mathcal{U} - B_H^*B_H - D^*D \end{bmatrix} \tag{3.3}$$

is a bounded nonnegative operator on the Hilbert space direct sum $\mathcal{X} \oplus \mathcal{U}$. This allows us to apply Proposition A.1 with

$$\alpha = I_\mathcal{X} - A_H^*A_H - C_H^*C_H, \quad \beta = -A_H^*B_H - C_H^*D, \tag{3.4}$$

$$\delta = I_\mathcal{U} - B_H^*B_H - D^*D. \tag{3.5}$$

Since T defined by (3.3) is nonnegative, Proposition A.1 tells us that α and δ are nonnegative, and there exists a contraction Γ mapping $\mathcal{X}$ into $\mathcal{U}$ such that

$$\operatorname{Ker}\Gamma \supset \operatorname{Ker}\alpha, \quad \operatorname{Im}\Gamma \subset \overline{\operatorname{Im}\delta}, \quad \beta^* = \delta^{1/2}\Gamma\alpha^{1/2}. \tag{3.6}$$

Since $H^{1/2}B = B_H$ is a well-defined bounded operator (see (2.3)), we have

$$\delta_\Sigma(H) = I_\mathcal{U} - D^*D - (H^{1/2}B)^*H^{1/2}B = I_\mathcal{U} - D^*D - B_H^*B_H = \delta,$$

and hence $\delta_\Sigma(H) = \delta$ is bounded and nonnegative because T given by (3.3) is bounded and nonnegative. Furthermore, the inclusion (1.4) follows from the identity in the third part of (3.6). To see this, note the equality $\beta^* = \delta^{1/2}\Gamma\alpha^{1/2}$ implies that $\operatorname{Im}\beta^* \subset \operatorname{Im}\delta^{1/2}$. Specifying this inclusion for β and δ given by (3.4) and (3.5), respectively, and using $\delta_\Sigma(H) = \delta$ we obtain

$$\begin{aligned} &\Big(D^*C + (H^{1/2}B)^*H^{1/2}A\Big)\mathcal{D}(H^{1/2}) \\ &\quad = \Big(A_H^*B_H + C_H^*D\Big)\operatorname{Im}H^{1/2} \subset \operatorname{Im}(A_H^*B_H + C_H^*D) \subset \operatorname{Im}\delta_\Sigma(H)^{1/2}. \end{aligned}$$

This proves the inclusion (1.4). Thus (C3) is satisfied.

It remains to prove the inequality (1.7). To do this we first observe that with our choice of H, the inequality (1.7) is equivalent to

$$\begin{aligned} &\|z\|^2 - \|A_Hz\|^2 - \|C_Hz\|^2 \\ &\qquad \geq \|(\delta_\Sigma(H)^{1/2})^{[-1]}(D^*C_H + B_H^*A_H)z\|^2, \quad z \in \operatorname{Im}H^{1/2}. \end{aligned}$$

Thus, using the two identities in (3.4) and $\delta_\Sigma(H) = \delta$, in order to prove (1.7) we have to show that

$$\|\alpha^{1/2}z\| \geq \|(\delta^{1/2})^{[-1]}\beta^*z\|, \quad z \in \operatorname{Im}H^{1/2}. \tag{3.7}$$

But $\beta^* = \delta^{1/2}\Gamma\alpha^{1/2}$ yields $(\delta^{1/2})^{[-1]}\beta^* = \Gamma\alpha^{1/2}$. Since Γ is a contraction, we see that the inequality in (3.7) holds for any $z \in \mathcal{X}$. Thus condition (CI4) is also satisfied. □

Theorem 3.4. *The set* $\mathrm{RI}_\Sigma \neq \emptyset$ *if and only if* Σ *is pseudo-similar to a passive system.*

Proof. We know (Theorem 4.1 in [6]) that this is true for KYP_Σ in place of RI_Σ. By Theorem 3.1, we have $\mathrm{RI}_\Sigma = \mathrm{KYP}_\Sigma$. Hence the result is also true for RI_Σ in place of KYP_Σ. □

4. Proof of the final statement in Theorem 1.5

The following proposition covers the final statement in Theorem 1.5.

Proposition 4.1. *Let* $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *be a minimal system, and assume that its transfer function coincides with a Schur class function in a neighborhood of zero. If* $H_\circ$ *is a minimal element in* RI°_Σ *with respect to the usual ordering of nonnegative operators, then* $H_\circ \in \mathrm{RE}^\circ_\Sigma$.

For the proof of the above proposition we need Lemma 4.2 below which is an addition to [4, Theorem 5.1]. Recall (cf., Section 2 of [6]) that a discrete time linear system Σ is called a *realization* of a Schur class function θ whenever the transfer function of Σ coincides with θ in a neighborhood of zero. For the definition of an optimal passive system we refer to Section 3 in [4].

Lemma 4.2. *Let* $\Sigma = (A, B, C, D, \mathcal{X}, \mathcal{U}, \mathcal{Y})$ *be a minimal and optimal passive discrete time linear system. Then*

$$\inf\left\{ \| \begin{bmatrix} x \\ u \end{bmatrix} \|^2 - \|M(\Sigma) \begin{bmatrix} x \\ u \end{bmatrix} \|^2 \mid u \in \mathcal{U} \right\} = 0 \quad (x \in \mathcal{X}). \tag{4.1}$$

The above lenma has been established in item (1) of [3, Corollary 7.3] using results of M. G. Kreĭn on shorted operators; cf., the final paragraph of the appendix (Section A). In the present paper we give a proof based on the functional model of minimal passive optimal systems derived in [6].

Proof. Let $\Sigma = (A, B, C, D, \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal and optimal, and let θ_Σ be its transfer function. Since Σ is passive, θ_Σ belongs to the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$, that is, θ_Σ is analytic on the open unit disc $\mathbb{D}$ and $\|\theta_\Sigma(z)\| \leq 1$ for all $z \in \mathbb{D}$. This allows us to replace Σ by its restricted shift model. Indeed, let $\theta = \theta_\Sigma$, and let $\Sigma_\circ = (A_\circ, B_\circ, C_\circ, D, \mathcal{X}_\circ, \mathcal{U}, \mathcal{Y})$ be the minimal and optimal realization of θ given by Theorem 5.1 in [4]. Then Σ and $\Sigma_\circ$ are unitary equivalent by Theorem 3.2 in [4], and hence it suffices to prove Lemma 4.2 for $\Sigma_\circ$ in place of Σ.

Let us recall the construction of $\Sigma_\circ$ given in the paragraph preceding Theorem 5.1 in [4]. For this purpose we need the de Branges-Rovnyak space $\mathcal{H}(\theta) := \{f \in H^2(\mathcal{Y}) \mid \|f\|_{\mathcal{H}(\theta)} < \infty\}$, where $H^2(\mathcal{Y})$ is the standard Hardy spaces of $\mathcal{Y}$-valued functions on the open unit disc $\mathbb{D}$ with square summable Taylor coefficients and

$$\|f\|^2_{\mathcal{H}(\theta)} = \sup\{\|f + \theta\eta\|^2_{H^2(\mathcal{Y})} - \|\eta\|^2_{H^2(\mathcal{U})} \mid \eta \in H^2(\mathcal{U})\}. \tag{4.2}$$

Let us list a few properties (see, e.g., [1, Chapter 2] and [12, Section 2]) of the space $\mathcal{H}(\theta)$:

(a) the space $\mathcal{H}(\theta)$ a Hilbert space with the Hilbert space norm $\|\cdot\|_{\mathcal{H}(\theta)}$ being given by (4.2) and $\mathcal{H}(\theta)$ is contractively embedded in $H^2(\mathcal{Y})$;

(b) the space $\mathcal{H}(\theta)$ is invariant under the backward-shift operator on $H^2(\mathcal{Y})$, that is, if $f \in \mathcal{H}(\theta)$, then the function $\tilde{f}$, $\tilde{f}(z) = z^{-1}(f(z) - f(0))$, also belongs to $\mathcal{H}(\theta)$;

(c) for each $u \in \mathcal{U}$ the function $\tilde{\theta}(\cdot)u$, where $\tilde{\theta}(z) = z^{-1}(\theta(z) - \theta(0))$, belongs to $\mathcal{H}(\theta)$.

Furthermore, we need the Hankel operator G_θ mapping $K^2(\mathcal{U})$ into $H^2(\mathcal{U})$. Here

$$K^2(\mathcal{U}) = L^2(U) \ominus H^2(\mathcal{U}),$$

and for any separable Hilbert space $\mathcal{F}$ we denote by $L^2(\mathcal{F})$ the Hilbert space of measurable $\mathcal{F}$-valued functions f on the unit circle $\mathbb{T}$ such that $\|f(\cdot)\|^2$ is Lebesgue integrable on $\mathbb{T}$, and with the norm on $L^2(\mathcal{F})$ being defined by

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \|f(e^{it})\|^2\, dt.$$

The action of G_θ is given by

$$G_\theta f = P_{H^2(\mathcal{Y})}\theta f, \quad f \in K^2(\mathcal{U}),$$

where $P_{H^2(\mathcal{Y})}$ is the orthogonal projection of $L^2(Y)$ onto $H^2(\mathcal{Y})$. Using the norm (4.2) and items (b) and (c) above it follows (see, e.g., [4, Lemma 5.2]) that the range of the Hankel operator G_θ is contained in the model space $\mathcal{H}(\theta)$.

We are now ready to define the system $\Sigma_\circ$. By definition, the state space $\mathcal{X}_\circ$ is the closure of $\operatorname{Im} G_\theta$ in $\mathcal{H}(\theta)$ and

$$\begin{array}{ll}
A_\circ : \mathcal{X}_\circ \to \mathcal{X}_\circ, & (A_\circ x)(z) = z^{-1}(x(z) - x(0)) \quad (x \in \mathcal{X}_\circ);\\
B_\circ : \mathcal{U} \to \mathcal{X}_\circ, & (B_\circ u)(z) = z^{-1}(\theta(z) - \theta(0))u \quad (u \in \mathcal{U});\\
C_\circ : \mathcal{X}_\circ \to \mathcal{Y}, & C_\circ x = x(0) \quad (x \in \mathcal{X}_\circ);\\
D : \mathcal{U} \to \mathcal{Y}, & Du = \theta(0)u \quad (u \in \mathcal{U}).
\end{array}$$

These operators are all well defined, and $\Sigma_\circ = (A_\circ, B_\circ, C_\circ, D, \mathcal{X}_\circ, \mathcal{U}, \mathcal{Y})$ is the minimal and optimal realization of θ given by Theorem 5.1 in [4].

Now let us prove Lemma 4.2 with $\Sigma_\circ$ in place of Σ. Let $\eta \in H^2(\mathcal{U})$. We decompose η as $\eta(z) = u + z\tilde\eta(z)$, where $u = \eta(0)$ and $\tilde\eta(z) = z^{-1}(\eta(z) - \eta(0))$. Note that the constant function u and the function $z\tilde\eta(z)$ are perpendicular in $H^2(\mathcal{U})$, and thus

$$\|\eta\|^2_{H^2(\mathcal{U})} = \|u\|^2 + \|\tilde\eta\|^2_{H^2(\mathcal{U})}. \tag{4.3}$$

Next observe that

$$\begin{aligned}(x + \theta\eta)(z) = x(0) + \theta(0)\eta(0) &+ \big(x(z) - x(0)\big)\\
&+ \big(\theta(z) - \theta(0)\big)\eta(0) + \theta(z)\big(\eta(z) - \eta(0)\big).\end{aligned}$$

Furthermore, using $\eta(z) = u + z\tilde\eta(z)$ and the definitions of the operators $A_\circ, B_\circ, C_\circ, D$ given above we see that

$$\begin{aligned}
&x(0) + \theta(0)\eta(0) = C_\circ x + Du, \quad x(z) - x(0) = z(A_\circ x)(z),\\
&\big(\theta(z) - \theta(0)\big)\eta(0) = z(B_\circ u)(z),\\
&\theta(z)\big(\eta(z) - \eta(0)\big) = z(\theta\tilde\eta)(z), \quad z \in \mathbb{D}.
\end{aligned}$$

It follows that

$$\|x + \theta\eta\|^2_{H^2(\mathcal{Y})} = \|C_\circ x + Du\|^2 + \|A_\circ x + B_\circ u + \theta\tilde\eta\|^2_{H^2(\mathcal{Y})}. \tag{4.4}$$

Using the identities (4.3) and (4.4) we see that

$$\begin{aligned}&\|x + \theta\eta\|^2_{H^2(\mathcal{Y})} - \|\eta\|^2_{H^2(\mathcal{U})}\\
&\qquad = \big(\|C_\circ x + Du\|^2 - \|u\|^2\big) + \Big(\|A_\circ x + B_\circ u + \theta\tilde\eta\|^2_{H^2(\mathcal{Y})} - \|\tilde\eta\|^2_{H^2(\mathcal{U})}\Big).\end{aligned}$$

But then, using the definition of the norm $\|\cdot\|_{\mathcal{H}(\theta)}$ in (4.2), we obtain

$$\begin{aligned}\|x\|^2_{\mathcal{H}(\theta)} &= \sup\Big\{ \big(\|C_\circ x + Du\|^2 - \|u\|^2\big) \\ &\qquad + \Big(\|A_\circ x + B_\circ u + \theta\tilde{\eta}\|^2_{H^2(\mathcal{Y})} - \|\tilde{\eta}\|^2_{H^2(\mathcal{U})}\Big) \,|\, u \in \mathcal{U},\ \tilde{\eta} \in H^2(\mathcal{U})\Big\} \\ &= \sup\Big\{\|C_\circ x + Du\|^2 + \|A_\circ x + B_\circ u\|^2_{\mathcal{H}(\theta)} - \|u\|^2 \mid u \in \mathcal{U}\Big\}.\end{aligned}$$

We conclude that

$$\inf\Big\{\|x\|^2_{\mathcal{H}(\theta)} + \|u\|^2 - \|M(\Sigma_\circ)\begin{bmatrix} x \\ u\end{bmatrix}\|^2_{\mathcal{H}(\theta)\oplus\mathcal{U}} \mid u \in \mathcal{U}\Big\} = 0.$$

This proves the lemma for $\Sigma_\circ$, and hence we are done. □

Proof of Proposition 4.1. Let $H_\circ$ be a minimal element in RI°_Σ with respect to the usual ordering of nonnegative operators. It suffices to show that $H_\circ \in \mathrm{RE}_\Sigma$. Recall that $\mathrm{RI}^\circ_\Sigma = \mathrm{KYP}^\circ_\Sigma$ by Corollary 3.2, and that $\mathrm{KYP}^\circ_\Sigma$ coincides with the set $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ used in Section 5 of [6]; see the paragraph before Corollary 3.2. These facts allow us to use the final part of item (ii) in [4, Propositon 5.8]. It follows that $\Sigma_{H_\circ}$ is a minimal and optimal passive system. But then we know from Lemma 4.2 that equation (4.1) holds with $\Sigma_{H_\circ}$ in place of Σ, and we can apply Proposition 2.2 to conclude that $H_\circ \in \mathrm{RE}_\Sigma$. □

The equalities $RI_\Sigma = KYP_\Sigma$ and $RI^\circ_\Sigma = KYP^\circ_\Sigma$, proved in Section 3, Theorem 3.1 and Corollary 3.2, allow us to extend results proved in Section 4 of [6] to the setting considered in the present paper. Among other things this provides the following addition to Theorem 1.5 for the adjoint system.

Theorem 4.3. *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a discrete time-invariant system, and let $\Sigma^* = (A^*, C^*, B^*, D^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$ be its adjoint system. Then the transfer function of Σ^* is given by $\theta_{\Sigma^*} = \theta^\sim_\Sigma$, where $\theta^\sim_\Sigma(\lambda) = \theta_\Sigma(\bar{\lambda})^*$, and Σ is minimal if and only if Σ^* is minimal. Furthermore, assuming Σ is minimal and RI_Σ is non-empty, we have*

$$\mathrm{RI}^\circ_{\Sigma^*} = \{H^{-1} \mid H \in \mathrm{RI}^\circ_\Sigma\}. \tag{4.5}$$

Finally, if $H_\circ$ and $H_\bullet$ are the minimal and maximal elements in RI°_Σ, then $H_\bullet^{-1}$ and $H_\circ^{-1}$ are the minimal and maximal elements in $\mathrm{RI}^\circ_{\Sigma^*}$

The analogue of (4.5) for the Riccati equality in place of the Riccati inequality, i.e., with RI replaced by RE, does not hold. See (6.8) in the final paragraph of Example 6.1.

5. A criterion for uniqueness and inner functions

Let Σ be a minimal realization of an inner function θ. In this section we show that in that case RI°_Σ consists of a single element, $H_\circ$ say, and we prove that $\delta_\Sigma(H_\circ) = 0$. Since RI°_Σ is equal to the set $\mathcal{GK}^{\min}_{\Sigma,\mathrm{core}}$ appearing in [6], we shall show that the first statement can be obtained as a corollary of the final part of Theorem 7.1 in [6]. The second statement is proved in the second part of this section.

Let us recall the final part of [6, Theorem 7.1]. This requires some preliminaries, which we take from [8, pages 164, 165] with some minor changes. Let θ be an arbitrary function in $\mathcal{S}(\mathcal{U}, \mathcal{Y})$, not necessarily inner. It is known [21, Section V.4] that there exist a Hilbert space $\mathcal{F}_r \subset \mathcal{U}$ and a function $\varphi_r \in \mathcal{S}(\mathcal{U}, \mathcal{F}_r)$ with the following three properties:

(a) $\varphi_r(z)^*\varphi_r(z) \le I_\mathcal{U} - \theta(z)^*\theta(z)$ for each $z \in \mathbb{D}$;

(b) for any Schur class function $\varphi \in \mathcal{S}(\mathcal{U}, \mathcal{G})$, where $\mathcal{G}$ is a Hilbert space, such that $\varphi(z)^*\varphi(z) \le I_\mathcal{U} - \theta(z)^*\theta(z)$ for each $z \in \mathbb{D}$, we have

$$\varphi(z)^*\varphi(z) \le \varphi_r(z)^*\varphi_r(z) \quad \text{for each} \quad z \in \mathbb{D};$$

(c) $\overline{\mathrm{Im}\,\varphi_r(0)} = \mathcal{F}_r$.

Here, the inequalities are understood in the sense of bounded selfadjoint operators on Hilbert spaces. The function φ_r can be normalized by the condition $\varphi_r(0)|_{\mathcal{F}_r}$ is positive. With this additional normalization, the function φ_r is uniquely defined (see [21]). From [21] we also know that properties (a), (b), (c) imply that the function $\varphi_r(z)$ is outer.

In a similar way one defines a maximal factor φ_l from the left. Indeed, there exist a Hilbert space $\mathcal{F}_l \subset \mathcal{Y}$ and a function $\varphi_l \in \mathcal{S}(\mathcal{F}_l, \mathcal{Y})$ with the following three properties:

(a') $\varphi_l(z)\varphi_l(z)^* \leq I_{\mathcal{Y}} - \theta(z)\theta(z)^*$ for each $z \in \mathbb{D}$;

(b') for any Schur class function $\psi \in \mathcal{S}(\mathcal{G}', \mathcal{Y})$, where $\mathcal{G}'$ is a Hilbert space, such that $\psi(z)\psi(z)^* \leq I_{\mathcal{Y}} - \theta(z)\theta(z)^*$ for each $z \in \mathbb{D}$, we have

$$\psi(z)\psi(z)^* \leq I_{\mathcal{Y}} - \theta(z)\theta(z)^* \quad \text{for each} \quad z \in \mathbb{D};$$

(c') $\overline{\operatorname{Im} \varphi_l(0)^*} = \mathcal{F}_l$.

In this case the function $\varphi_l(\bar{z})^*$ is an outer function, and normalization is obtained by requiring $\varphi_l(0)^*|_{\mathcal{F}_l}$ to be a positive operator.

The functions φ_r and φ_l are called the *right and left defect functions* of θ; see [2, page 213] and the references given therein.

Given $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and the defect functions $\varphi_r \in \mathcal{S}(\mathcal{U}, \mathcal{F}_r)$ and $\varphi_l \in \mathcal{S}(\mathcal{F}_l, \mathcal{Y})$, we know from [13] that there exists a function h_0 in the space $L^\infty(\mathcal{F}_l, \mathcal{F}_r)$ of bounded measurable operator-valued functions defined on the unit circle with values in $\mathcal{L}(\mathcal{F}_l, \mathcal{F}_r)$ such that the block operator matrix

$$\Theta(\zeta) = \begin{bmatrix} \varphi_l(\zeta) & \theta(\zeta) \\ h_0(\zeta) & \varphi_r(\zeta) \end{bmatrix} : \begin{bmatrix} \mathcal{F}_l \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{F}_r \end{bmatrix} \tag{5.1}$$

is contractive almost everywhere for $\zeta \in \mathbb{T}$. Moreover, according to [13], the operator function h_0 defined above is unique. We call h_0 the *coupling function* defined by θ.

Let $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, and let h_0 be the coupling function defined above. Using [8, Theorem 1.1] and $\mathrm{RI}_\Sigma^\circ = \mathcal{GK}_{\Sigma,\mathrm{core}}^{\min}$, the final part of Theorem 7.1 in [6] yields the following theorem.

Theorem 5.1. *Let $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, and let Σ be a minimal realization of θ. Then RI_Σ° consists of a single element if and only if the following condition is satisfied:*

(C) *the coupling function h_0 defined by θ is the boundary value of a function from the Schur class $\mathcal{S}(\mathcal{F}_l, \mathcal{F}_r)$.*

Here $\mathcal{F}_l$ and $\mathcal{F}_r$ are the Hilbert space appearing in (5.1).

Condition (C) above is item (iii) in [8, Theorem 1.1]. Note that condition (C) does not depend on the particular choice of the minimal system Σ. As the proof of Theorem 7.1 in [6] shows, Theorem 5.1 above can be viewed as a corollary of the equivalence of items (i) and (iii) in [8, Theorem 1.1].

Remark 5.2. *If one of the spaces $\mathcal{F}_l$ and $\mathcal{F}_r$ consists of the zero vector only, then the coupling function h_0 defined by θ is zero. Hence condition* (C) *is trivially satisfied and, by Theorem 5.1, the set RI_Σ° consists of one element only for any minimal realization of θ.*

Corollary 5.3. *Let θ be a scalar Schur class function. Then the defect functions φ_r and φ_l coincide. Furthermore, $\varphi_r = \varphi_l = 0$ if and only if the function $\log(1 - |\theta(\cdot)|)$ is not Lebesgue integrable on the unit circle, and in that case the set RI_Σ° consists of one element only for any minimal realization of θ.*

Proof. The fact that φ_r and φ_l coincide follows directly from the fact that scalar functions commute. Now assume that $\varphi_r \neq 0$. Then $\log|\varphi_r(\cdot)| \in L^1(\mathbb{T})$; see, e.g., [14, Theorem 1.2]. Using $\log|\varphi(\cdot)|^2 = 2\log|\varphi(\cdot)|$ for any φ, we see that $\log|\varphi_r(\cdot)|^2 \in L^1(\mathbb{T})$. But then, since

$$|\varphi_r(z)|^2 \leq 1 - |\theta(z)|^2 \quad (z \in \mathbb{D}) \Longrightarrow |\varphi_r(\zeta)|^2 \leq 1 - |\theta(\zeta)|^2 \quad (\zeta \in \mathbb{T}\ a.e.),$$

it follows that $\log(1 - |\theta(\cdot)|^2)$ belongs to $L^1(\mathbb{T})$. Next use

$$\log(1 - |\theta(\cdot)|^2) = \log(1 - |\theta(\cdot)|) + \log(1 + |\theta(\cdot)|).$$

The preceding identity together with the fact that $\log(1 + |\theta(\cdot)|)$ belongs to $L^1(\mathbb{T})$ shows that $\log(1 - |\theta(\cdot)|) \in L^1(\mathbb{T})$.

Conversely, assume that $\log(1 - |\theta(\cdot)|) \in L^1(\mathbb{T})$. Then the factorization problem $|\varphi(z)|^2 \leq 1 - |\theta(z)|^2$ has a nonzero solution φ in H^∞ by Theorem 1.2 in [14] or Proposition V.7.1 (b) in [21], and hence, φ_r is not zero.

We conclude that $\varphi_r = 0$ if and only if $\log(1 - |\theta(\cdot)|) \notin L^1(\mathbb{T})$. The final part of the corollary now follows directly from Remark 5.2 above. □

Now assume that $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is inner. Then $I_\mathcal{U} - \theta(\zeta)^*\theta(\zeta) = 0$ almost everywhere for $\zeta \in \mathbb{T}$, and hence the space $\mathcal{F}_r$ consists of the zero element only. Thus, by the above remark, the set RI°_Σ consists of one element only. This proves the first part of the following theorem.

Theorem 5.4. *Let Σ be a minimal realization of the inner function $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$. Then RI°_Σ consists of a single element, $H_\circ$ say, and $\delta_\Sigma(H_\circ) = 0$.*

Proof. It remains to prove $\delta_\Sigma(H_\circ) = 0$. Since the function θ is inner, we know from the Sz-Nagy–Foias model theory [21] that θ has an observable realization

$$\Sigma_1 = (A_1, B_1, C_1, D; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$$

such that its system matrix $M(\Sigma_1)$ is unitary. Now put $\mathcal{X}_{10} = \overline{\mathrm{Im}\,(A_1|B_1)}$. Relative to the Hilbert space direct sum $\mathcal{X}_1 = \mathcal{X}_{10} \oplus \mathcal{X}_{10}^\perp$ the operators A_1, B_1, C_1 admit the following block matrix representations:

$$A_1 = \begin{bmatrix} A_{10} & \star \\ 0 & \star \end{bmatrix} : \begin{bmatrix} \mathcal{X}_{10} \\ \mathcal{X}_{10}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_{10} \\ \mathcal{X}_{10}^\perp \end{bmatrix}, \tag{5.2}$$

$$B_1 = \begin{bmatrix} B_{10} \\ 0 \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{X}_{10} \\ \mathcal{X}_{10}^\perp \end{bmatrix}, \quad C_1 = \begin{bmatrix} C_{10} & \star \end{bmatrix} : \begin{bmatrix} \mathcal{X}_{10} \\ \mathcal{X}_{10}^\perp \end{bmatrix} \to \mathcal{Y}. \tag{5.3}$$

Put $\Sigma_{10} = (A_{10}, B_{10}, C_{10}, D; \mathcal{X}_{10}, \mathcal{U}, \mathcal{Y})$. The above construction implies that Σ_{10} is controllable. Furthermore, since Σ_1 is observable, the same holds true for Σ_{10}. Thus Σ_{10} is a minimal system. Moreover , the transfer function of Σ_{10} is equal to the transfer function of Σ_1. Thus Σ_{10} is a minimal realization of θ.

Using the terminology of Section 2.1 in [4], the system Σ_{10} is the first minimal restriction of the system Σ. But then, by [4, Theorem 3.2], the system Σ_{10} is a minimal and optimal realization of θ.

We claim that $M(\Sigma_{10})$ is an isometry. To see this note that

$$\begin{aligned} I_{\mathcal{X}_1} - A_1^*A_1 - C_1^*C_1 &= \begin{bmatrix} I_{\mathcal{X}_{10}} - A_{10}^*A_{10} - C_{10}^*C_{10} & \star \\ \star & \star \end{bmatrix}, \\ A_1^*B_1 + C_1^*D &= \begin{bmatrix} A_{10}^*B_{10} + C_{10}^*D \\ \star \end{bmatrix}, \\ I_\mathcal{U} - B_1^*B_1 - D^*D &= I_\mathcal{U} - B_{10}^*B_{10} - D^*D. \end{aligned}$$

Since M_{Σ_1} is unitary, the operators in the left hand side of the three identities above are all zero. Thus

$$I_{\mathcal{X}_{10}} - A_{10}^*A_{10} - C_{10}^*C_{10} = 0, \quad A_{10}^*B_{10} + C_{10}^*D = 0, \quad I_\mathcal{U} - B_{10}^*B_{10} - D^*D = 0.$$

This shows that $M(\Sigma_{10})$ is an isometry.

Now use that the systems Σ_{10} and $\Sigma_{H_\circ}$ are unitarily equivalent (see Theorem 3.2 in [4]). It follows that $M(\Sigma_{H_\circ})$ is an isometry which implies that $\delta_\Sigma(H_\circ) = 0$. □

Remark 5.5. *Using Proposition 4 in* [9] *and taking into account Theorem 5.1, it can be shown that the two statements in Theorem 5.4 remain true if the condition θ is inner is replaced by the condition that the right defect function φ_r of θ is zero or, equivalently, that $\mathcal{F}_r = \{0\}$. In fact, with some minor changes the same proof can be used to derive this more general result. Indeed, from the Sz-Nagy-Foias model theory we know that θ is the transfer function of a simple conservative realization Σ_1. Here* conservative *means that the system matrix M_{Σ_1} is unitary. Furthermore, it is known* (*item* (a) *in* [9, Proposition 4]) *that the condition $\mathcal{F}_r = \{0\}$ implies that Σ_1 is observable. But then, as in the proof of Theorem 5.4, we construct the system Σ_{10}, show that $M_{\Sigma_{10}}$ is an isometry, and conclude that $\delta_\Sigma(H_\circ) = 0$.*

Corollary 5.6. *Let $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ be co-inner, and let Σ be a minimal realization of θ. Then RI_Σ° consists of a single element, $H_\bullet$ say, and $\delta_{\Sigma^*}(H_\bullet^{-1}) = 0$.*

Proof. Assume $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is co-inner. Then $I_\mathcal{Y} - \theta(\zeta)\theta(\zeta)^* = 0$ almost everywhere for $\zeta \in \mathbb{T}$, and hence the space $\mathcal{F}_l$ consists of the zero element only. The latter implies (see Remark 5.2) that RI_Σ° consists of a single element.

Next we use Theorem 4.3. Recall that $\theta^\sim(\lambda) = \theta(\bar{\lambda})^*$ for $\lambda \in \mathbb{D}$. The fact that θ is co-inner, implies that $\theta^\sim$ is inner. Indeed, we have

$$\begin{aligned} \theta \text{ is co-inner } &\iff \theta(\zeta)\theta(\zeta)^* = I \text{ almost everywhere on } \mathbb{T} \\ &\iff \theta(\bar{\zeta})\theta(\bar{\zeta})^* = I \text{ almost everywhere on } \mathbb{T} \\ &\iff \theta^\sim(\zeta)^*\theta^\sim(\zeta) = I \text{ almost everywhere on } \mathbb{T} \\ &\iff \theta^\sim \text{ is inner.} \end{aligned}$$

Since Σ is a minimal realization of θ, the system Σ^* is a minimal realization for $\theta^\sim$. Now let $H_\bullet$ be the (unique) element in RI_Σ°. From (4.5) it follows that $H_\bullet^{-1}$ belongs to $\mathrm{RI}_{\Sigma^*}^\circ$. But Σ^* is a minimal realization of an inner function. Hence, $\delta_{\Sigma^*}(H_\bullet^{-1}) = 0$ by Theorem 5.4. □

Note that the first statement in the above corollary can also be proved by using the duality argument used in the second paragraph of the above proof.

In general, the second part of Theorem 5.4 is not true for a co-inner function. See Example 6.3 in the next section.

Remark 5.7. *Finally, again with minor changes, one can prove that Corollary 5.6 remains true if the condition θ is co-inner is replaced by the condition that $\mathcal{F}_l = \{0\}$.*

6. Examples

In this section we present a few examples. Throughout θ is a Schur class function and $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a minimal realization of θ. In the first three examples the state space $\mathcal{X}$ will be finite dimensional. In that case a positive operator on $\mathcal{X}$ will be bounded and boundedly invertible, and the Riccati equality can be rewritten as

$$\alpha_\Sigma(H) - \beta_\Sigma(H)^*\delta_\Sigma(H)^{[-1]}\beta_\Sigma(H) = 0, \tag{6.1}$$

where

$$\alpha_\Sigma(H) = H - A^*HA - C^*C, \quad \beta_\Sigma(H) = D^*C + B^*HA,$$
$$\delta_\Sigma(H) = I - D^*D - B^*HB.$$

Furthermore, if $\mathcal{X}$ is finite dimensional, then $H \in \mathrm{RE}_\Sigma$ if and only if H is a positive operator on $\mathcal{X}$, the operator $\delta_\Sigma(H)$ is nonnegative, and H satisfies (6.1). Similarly, if $\mathcal{X}$ is finite dimensional, then $H \in \mathrm{RI}_\Sigma$ if and only if H is a positive operator on $\mathcal{X}$, the operator $\delta_\Sigma(H)$ is nonnegative, and

$$\alpha_\Sigma(H) - \beta_\Sigma(H)^*\delta_\Sigma(H)^{[-1]}\beta_\Sigma(H) \geq 0. \tag{6.2}$$

As before, the symbol $[-1]$ denotes the Moore–Penrose inverse.

Example 6.1. We present a simple scalar example showing that the maximal solution in RI_Σ may not belong to RE_Σ. To do this we use the scalar function θ given by [4, eq. (3.3)], i.e.,

$$\theta(\lambda) = (2\lambda + 4)(\lambda + 8)^{-1}.$$

From [4] we know that θ is a Schur class function (in fact, $|\theta(\lambda)| \leq 6/7 < 1$ for all $\lambda \in \mathbb{D}$) and a minimal realization of θ is given by

$$\Sigma = (-\frac{1}{8}, 1, \frac{3}{16}, \frac{1}{2}; \mathbb{C}, \mathbb{C}, \mathbb{C}). \tag{6.3}$$

For this choice of Σ the set RE_Σ is a singleton and RI_Σ is an interval

$$\mathrm{RE}_\Sigma = \left\{\frac{3}{64}\right\} \quad \text{and} \quad \mathrm{RI}_\Sigma = \left[\frac{3}{64}, \frac{3}{4}\right]. \tag{6.4}$$

In particular, the maximal solution $H_\bullet$ of the Riccati inequality does not belong RE_Σ.

To prove (6.4) let h be a positive real number viewed as a positive operator on $\mathbb{C}$. Then

$$\alpha_\Sigma(h) = \frac{9}{64}\left(7h - \frac{1}{4}\right), \quad \beta_\Sigma(h) = \frac{1}{8}\left(\frac{3}{4} - h\right), \quad \delta_\Sigma(h) = \frac{3}{4} - h.$$

Note that $\delta_\Sigma(h) \geq 0$ if and only if $h \leq 3/4$. The Moore-Penrose inverse of $\delta(h)$ is given by

$$\delta_\Sigma(h)^{[-1]} = \begin{cases} \left(\frac{3}{4} - h\right)^{-1} & (h \neq 3/4) \\ 0 & (h = 3/4) \end{cases} \quad : \ \mathbb{C} \to \mathbb{C}.$$

For $h = 3/4$ the right hand sides of both (6.1) and (6.2) are zero, and the left hand sides are strictly positive. Thus $3/4 \notin \mathrm{RE}_\Sigma$ and $3/4 \in \mathrm{RI}_\Sigma$. Next, let $0 < h < 3/4$. Then, respectively, (6.1) and (6.2) reduce to

$$\frac{9}{64}\left(7h - \frac{1}{4}\right) - \frac{1}{64}\left(\frac{3}{4} - h\right)^2\left(\frac{3}{4} - h\right)^{-1} = 0, \tag{6.5}$$

$$\frac{9}{64}\left(7h - \frac{1}{4}\right) - \frac{1}{64}\left(\frac{3}{4} - h\right)^2\left(\frac{3}{4} - h\right)^{-1} \geq 0. \tag{6.6}$$

Equation (6.5) has $h = 3/64$ as its unique solution in the interval $0 < h < 3/4$, which proves the first equality in (6.4). All solutions h of (6.6) are given by $h \geq 3/64$. Together with $0 < h \leq 3/4$ this yields the second equality in (6.4).

Let Σ^* be the adjoint of the system Σ given by (6.3), i.e.,

$$\Sigma^* = (-\frac{1}{8}, \frac{3}{16}, 1, \frac{1}{2}; \mathbb{C}, \mathbb{C}, \mathbb{C}).$$

For this choice we have the following analogue of (6.4)

$$\mathrm{RE}_{\Sigma^*} = \left\{\frac{4}{3}\right\} \quad \text{and} \quad \mathrm{RI}_\Sigma = \left[\frac{4}{3}, \frac{64}{3}\right]. \tag{6.7}$$

By Theorem 4.3 the second identity in (6.7) follows from the second identity in (6.4). The first identity in (6.4) cannot be obtained in this way but this identity is proved in a similar way as the first identity in (6.7) is proved. We omit the further details. Note that in this case

$$\mathrm{RE}_{\Sigma^*} \neq \{H^{-1} \mid H \in \mathrm{RE}_\Sigma\}. \tag{6.8}$$

Example 6.2. We consider the scalar function

$$\theta(\lambda) = \frac{\lambda ab}{1-\lambda^2 ab}, \quad \text{where} \quad 0 < a < b < 1, \quad a^2 + b^2 = 1.$$

The function θ is a Schur clas function, and θ is the transfer function of the system $(A, B, C, D; \mathbb{C}^2, \mathbb{C}, \mathbb{C})$, where

$$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ a \end{bmatrix}, \quad C = \begin{bmatrix} 0 & b \end{bmatrix}, \quad D = 0.$$

The system Σ is a passive minimal realization of θ, and hence $H_1 = I_{\mathbb{C}^2}$ is a solution of the Riccati equality associated to Σ. We shall see that there are three other solutions, namely:

$$H_2 = \frac{1}{a^2}\begin{bmatrix} (1-ab)\,\frac{b}{a} & (b-a)\sqrt{\frac{b}{a}} \\ (b-a)\sqrt{\frac{b}{a}} & 1-ab \end{bmatrix}, \quad H_3 = \frac{1}{a^2}\begin{bmatrix} (1-ab)\,\frac{b}{a} & -(b-a)\sqrt{\frac{b}{a}} \\ -(b-a)\sqrt{\frac{b}{a}} & 1-ab \end{bmatrix},$$

$$H_4 = \frac{1}{a^4}\begin{bmatrix} b^4 & 0 \\ 0 & a^2b^2 \end{bmatrix}.$$

We shall see that $H_1 \le H_j \le H_4$, $j = 2, 3$. It follows from Theorem 1.5 that $H_1 = I_{\mathbb{C}^2}$ is the minimal optimal solution, and thus H_1 is the minimal element in RI_Σ. It turns out that H_4 also belongs to RI_Σ and is the maximal element in RI_Σ.

To derive the results mentioned above, put

$$H = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

By assumption H is positive definite. In particular, $\overline{x_3} = x_2$. In this case we have

$$\alpha_\Sigma(H) = \begin{bmatrix} x_1 - b^2x_4 & x_2 - abx_3 \\ x_3 - abx_2 & x_4 - a^2x_1 - b^2 \end{bmatrix}, \quad \beta_\Sigma(H) = \begin{bmatrix} abx_4 & a^2x_3 \end{bmatrix},$$

$$\delta_\Sigma(H) = 1 - a^2x_4.$$

Recall that $\delta_\Sigma(H) = 1 - a^2x_4$ is required to be non-negative, and the associate Riccati equality is the identity

$$\begin{bmatrix} x_1 - b^2x_4 & x_2 - abx_3 \\ x_3 - abx_2 & x_4 - a^2x_1 - b^2 \end{bmatrix} - (1-a^2x_4)^{[-1]}\begin{bmatrix} a^2b^2x_4^2 & a^3bx_4x_3 \\ a^3bx_2x_4 & a^4x_2x_3 \end{bmatrix} = 0. \tag{6.9}$$

Since H is positive definite, $x_4 > 0$. Together with $1 - a^2x_4 \ge 0$ this implies that $0 < x_4 \le a^{-2}$. But $x_4 = a^{-2}$ is excluded, because in that case the Riccati equation (6.9) has no solution which can be proved by direct checking. Therefore we may assume that $0 < x_4 < a^{-2}$, and hence the Moore-Penrose inverse in (6.9) is a usual inverse. But then, with elementary computations or using the computer algebra program Mathematica, it is straightforward to show that the matrices H_j, $j = 1, 2, 3, 4$, are the only solutions of the Riccati equality (6.9).

Since H_2 and H_3 have the same diagonal entries, neither $H_2 \le H_3$ nor $H_3 \le H_2$. Indeed, if $H_2 \le H_3$, then $H_3 - H_2$ is a nonnegative operator of the form (A.1) with zero diagonal entries. But then, by Proposition A.1, the off diagonal entrties are zero too, and hence $H_2 = H_3$ which is not true. In a similar way one shows that $H_3 \le H_2$ is excluded.

The fact that H_4 is the maximal element in RI_Σ can be obtained from Theorem 4.3 by showing that H_4^{-1} is the minimal element of RI_{Σ^*}. Note that is this case

$$\Sigma^* = \left(\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix}, \begin{bmatrix} 0 & a \end{bmatrix}, 0; \mathbb{C}^2, \mathbb{C}, \mathbb{C} \right).$$

Example 6.3. Let $\theta(z) = \begin{bmatrix} z & 0 \end{bmatrix}$. Then $\theta(z)\theta(z)^* = 1$ for each $z \in \mathbb{T}$, and thus θ is co-inner. We show that the statement in the second part of Theorem 5.4 does not hold for this co-inner function θ. To do this put

$$A = 0 : \mathbb{C} \to \mathbb{C}, \quad B = \begin{bmatrix} 1 & 0 \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}, \quad C = 1 : \mathbb{C} \to \mathbb{C},$$
$$D = \begin{bmatrix} 0 & 0 \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}.$$

Then the system $\Sigma = (A, B, C, D; \mathbb{C}, \mathbb{C}^2, \mathbb{C})$ is a minimal realization of θ and its system matrix

$$M(\Sigma) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{is a co-isometry.}$$

Thus, by Corollary 5.6 above, $\mathrm{RI}_\Sigma^\circ = \{1\}$. But in this case

$$\delta_\Sigma(1) = I_{\mathbb{C}^2} - D^*D - B^*B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $\delta_\Sigma(1)$ is non-zero.

Example 6.4. We present an example of a minimal passive system Σ such that $H \in \mathrm{RI}_\Sigma$ while Σ_H is not minimal. In particular, RI_Σ° will be a proper subset of RI_Σ. The transfer function θ of the system involved will be of the form $\theta(\lambda) = \lambda K$, where K is a contraction. The latter allows us to use results from [5, Section 2.3].

Throughout ℓ^2_+ is the Hilbert space of all complex valued sequences that are square summable in absolute value. Furthermore, R and S are the linear operators acting in ℓ^2_+ defined by

$$\mathcal{D}(R) = \{x \in \ell^2_+ \mid (x_0, 2x_1, 3x_2, \ldots) \in \ell^2_+\}, \quad Rx = (0, x_0, 2x_1, 3x_2, \ldots); \tag{6.10}$$
$$\mathcal{D}(S) = \{\lambda v + x \mid \lambda \in \mathbb{C},\ v = (1, 1/2, 1/3, \ldots),\quad x \in \mathcal{D}(R)\}, \quad \text{and}$$
$$S(\lambda v + x) = \lambda e_0 + Rx, \quad \text{where} \quad e_0 = (1, 0, 0, 0, \ldots). \tag{6.11}$$

The operators R and S are both closed densely defined linear operators and both are one-to-one. Furthermore,

$$\mathcal{D}(R) \subset \mathcal{D}(S), \quad S|_{\mathcal{D}(R)} = R, \quad \mathcal{D}(R) \neq \mathcal{D}(S) \neq \ell^2_+, \quad \mathrm{Im}\, S = \overline{\mathrm{Im}\, S} = \ell^2_+.$$

Since S is densely defined, its adjoint S^* is well defined. In what follows $\mathcal{U}$ and $\mathcal{Y}$ denote the spaces $\mathcal{D}(R)$ and $\mathcal{D}(S^*)$ endowed with the corresponding graph norms. Thus

$$\|x\|_{\mathcal{U}} = \left(\|x\|^2 + \|Rx\|^2\right)^{1/2}, \quad x \in \mathcal{D}(R),$$
$$\|x\|_{\mathcal{Y}} = \left(\|x\|^2 + \|S^*x\|^2\right)^{1/2}, \quad x \in \mathcal{D}(S^*).$$

Next, put $\mathcal{X} = \ell^2_+$, and define the canonical embeddings

$$\tau_{\mathcal{U}} : \mathcal{U} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix}, \quad \tau_{\mathcal{U}} u = \begin{bmatrix} u \\ Ru \end{bmatrix}, \quad u \in \mathcal{D}(R),$$
$$\tau_{\mathcal{Y}} : \mathcal{Y} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix}, \quad \tau_{\mathcal{Y}} y = \begin{bmatrix} y \\ S^*y \end{bmatrix}, \quad y \in \mathcal{D}(S^*).$$

Note that both $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{Y}}$ are isometries. We also need the projections

$$\Pi_1 = \begin{bmatrix} I & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix} \to \mathcal{X} \quad \text{and} \quad \Pi_2 = \begin{bmatrix} 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix} \to \mathcal{X}.$$

Given we these operators we consider the system

$$\Sigma = (0, B, C, 0; \mathcal{X}, \mathcal{U}, \mathcal{Y}), \quad \text{where}$$

$$B = \Pi_1 \tau_{\mathcal{U}} : \mathcal{U} \to \mathcal{X} \quad \text{and} \quad C = (\Pi_2 \tau_{\mathcal{Y}})^* : \mathcal{X} \to \mathcal{Y}. \tag{6.12}$$

Clearly, B and C are contractions, and hence the system matrix $M(\Sigma)$ is a contraction too. It follows that Σ is passive. Note that $\operatorname{Im} B = \mathcal{D}(R)$, and hence $\overline{\operatorname{Im} B} = \overline{\mathcal{D}(R)} = \mathcal{X}$. Furthermore, $\operatorname{Im} C^* = \operatorname{Im} S$, and thus $\overline{\operatorname{Im} C^*} = \mathcal{X}$. The latter implies that C is one-to-one. We conclude that the system Σ is minimal. Finally, the transfer function of Σ is the Schur class function θ given by $\theta(\lambda) = \lambda CB$.

Next we consider a second system

$$\widehat{\Sigma} = (0, \widehat{B}, \widehat{C}, 0; \mathcal{X}, \mathcal{U}, \mathcal{Y}), \text{ where}$$

$$\widehat{B} = \Pi_2 \tau_{\mathcal{U}} : \mathcal{U} \to \mathcal{X} \quad \text{and} \quad \widehat{C} = (\Pi_1 \tau_{\mathcal{Y}})^* : \mathcal{X} \to \mathcal{Y}. \tag{6.13}$$

Note that $\operatorname{Im} \widehat{B} = \operatorname{Im} R$. Thus $\operatorname{Im} R$ is not dense in $\mathcal{X}$, and hence the system $\widehat{\Sigma}$ is not minimal.

Proposition 6.5. *The systems Σ and $\widehat{\Sigma}$ defined by* (6.12) *and* (6.13)*, respectively, have the same transfer function, and the operator S defined by* (6.11) *is a pseudo-similarity from Σ to $\widehat{\Sigma}$.*

Now put $H = (S^*S)^{1/2}$. Then we know from [6, Proposition 4.5] that $H \in \mathrm{KYP}_\Sigma$, and thus $H \in \mathrm{RI}_\Sigma$, by Theorem 3.1. Moreover, Σ_H is unitarily equivalent to $\widehat{\Sigma}$. In particular, Σ_H is not minimal, and thus $H \notin \mathrm{RI}_\Sigma^\circ$.

The above proposition can be obtained by applying the result of [5, Section 2.3.1]. For sake of completeness we present the proof. In order to do this it will be convenient first to prove the following lemma.

Lemma 6.6. *Let $a \in \mathcal{D}(S)$ and let $x \in \mathcal{X}$. Then the following three statements are equivalent:*

$$\text{(a) } \tau_{\mathcal{Y}}^* \Pi_2^* a = x, \quad \text{(b) } \tau_{\mathcal{Y}}^* \Pi_1^* S a = x, \tag{6.14}$$

$$\text{(c) } x \in \mathcal{D}(SS^*) \quad \textit{and} \quad (I + SS^*)x = Sa. \tag{6.15}$$

In particular, $\tau_{\mathcal{Y}}^* \Pi_2^* a = \tau_{\mathcal{Y}}^* \Pi_1^* S a$ for each $a \in \mathcal{D}(S)$.

Proof. We split the proof into two parts.

Part 1. We prove the equivalence of items (a) and (c). To do this we use the fact (see formula (5.9) in [19, page 168]) that there exist (unique) vectors $x_1 \in \mathcal{D}(S^*)$ and $x_2 \in \mathcal{D}(S)$ such that

$$\Pi_2^* a = \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} x_1 \\ S^* x_1 \end{bmatrix} + \begin{bmatrix} -S x_2 \\ x_2 \end{bmatrix}. \tag{6.16}$$

Note that $\tau_{\mathcal{Y}}^* \Pi_2^* a = x_1$. The identity (6.16) is equivalent to

$$x_1 = S x_2 \quad \text{and} \quad a = S^* x_1 + x_2. \tag{6.17}$$

Since $a \in \mathcal{D}(S)$ and $x_2 \in \mathcal{D}(S)$, the second identity in (6.17) shows that $S^* x_1 = a - x_2 \in \mathcal{D}(S)$. Thus $x_1 \in \mathcal{D}(SS^*)$ and using the first identity in (6.17) we obtain

$$Sa = SS^* x_1 + S x_2 = (I + SS^*) x_1.$$

We conclude that $(I + SS^*)x = Sa$ with $x = x_1$.

Conversely, assume $x \in \mathcal{D}(SS^*)$ satisfies $(I + SS^*)x = Sa$. Put $x_1 = x$ and define $x_2 = a - S^*x_1$. Then $x_2 \in \mathcal{D}(S)$ and

$$Sx_2 = Sa - SS^*x_1 = Sa - (I + SS^*)\,x_1 + x_1 = x_1.$$

Thus the two identities in (6.17) are satisfied which implies that (6.16) holds. Hence $\tau_{\mathcal{Y}}^*\Pi_2^*a = x_1 = x$. □

PART 2. We prove $\tau_{\mathcal{Y}}^*\Pi_1^*Sa = \tau_{\mathcal{Y}}^*\Pi_2^*a$. Again using formula (5.9) in [19, page 168], there exist (unique) vectors $x_1 \in \mathcal{D}(S^*)$ and $x_2 \in \mathcal{D}(S)$ such that

$$\Pi_1^*Sa = \begin{bmatrix} Sa \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ S^*x_1 \end{bmatrix} + \begin{bmatrix} -Sx_2 \\ x_2 \end{bmatrix}. \tag{6.18}$$

The latter identity is equivalent to

$$Sa = x_1 - Sx_2 \quad \text{and} \quad x_2 = -S^*x_1. \tag{6.19}$$

Since $x_2 \in \mathcal{D}(S)$, the second identity in (6.19) shows that $x_1 \in \mathcal{D}(SS^*)$ and $Sx_2 = -SS^*x_1$. Using this fact the first identity in (6.19) yields

$$Sa = x_1 + SS^*x_1 = (I + SS^*)x_1.$$

But then we can apply the result of the previous part to show that $\tau_{\mathcal{Y}}^*\Pi_2^*a = x_1$. On the other hand, from (6.18) it follows that $\tau_{\mathcal{Y}}^*\Pi_1^*Sa$ is also equal to x_1. Hence we have $\tau_{\mathcal{Y}}^*\Pi_1^*Sa = \tau_{\mathcal{Y}}^*\Pi_2^*a$ as desired. Together the two parts prove the lemma. □

Proof of Proposition 6.5. Recall that S is one-to-one and has a dense range. Therefore, since Σ and $\widehat{\Sigma}$ are given by (6.12) and (6.13), respectively, it suffices to show that

$$B\mathcal{U} \subset \mathcal{D}(S), \quad \widehat{B} = SB \quad \text{and} \quad \widehat{C}Sa = Ca \quad (a \in \mathcal{D}(S))\,. \tag{6.20}$$

Take $u \in \mathcal{U}\,(= \mathcal{D}(R))$. Then

$$Bu = \Pi_1\tau_{\mathcal{U}}u = \Pi_1 \begin{bmatrix} u \\ Ru \end{bmatrix} = u \in \mathcal{D}(R) \subset \mathcal{D}(S) \quad \text{and}$$
$$SBu = Su = Ru = \Pi_2 \begin{bmatrix} u \\ Ru \end{bmatrix} = \Pi_2\tau_{\mathcal{U}}u = \widehat{B}u.$$

This proves the first part of (6.20). To prove the second part, let $a \in \mathcal{D}(S)$. Using Lemma 6.6 we have

$$\widehat{C}Sa = (\Pi_1\tau_{\mathcal{Y}})^*Sa = \tau_{\mathcal{Y}}^*\Pi_1^*Sa = \tau_{\mathcal{Y}}^*\Pi_2^*a = (\Pi_2\tau_{\mathcal{Y}})^*a = Ca.$$

Hence S is a pseudo-similarity from Σ to $\widehat{\Sigma}$. In particular, the two systems have the same transfer function, i.e., $CB = \widehat{C}\widehat{B}$. □

APPENDIX A

In this appendix we review a number of results regarding 2×2 nonnegative operator matrices that are used in the present paper. In particular, we shall consider Schur complements for such operators. Throughout we assume that $\alpha : \mathcal{X} \to \mathcal{X}$, $\beta : \mathcal{U} \to \mathcal{X}$, $\delta : \mathcal{U} \to \mathcal{U}$ are bounded Hilbert space operators and T is the bounded operator defined by

$$T = \begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}. \tag{A.1}$$

Proposition A.1. *The operator T is nonnegative if and only if α and δ are nonnegative and there exists a contraction $\Gamma : \mathcal{X} \to \mathcal{U}$ such that*

(a) $\operatorname{Ker}\Gamma \supset \operatorname{Ker}\alpha$ *and* $\operatorname{Im}\Gamma \subset \overline{\operatorname{Im}\delta}$,
(b) $\beta^* = \delta^{1/2}\Gamma\alpha^{1/2}$.

Moreover, in that case Γ *is uniquely determined by conditions* (a) *and* (b)*.*

If T is nonnegative and Γ is the contraction satisfying the two conditions in the above proposition, then we call Γ the *minimal contraction* determined by T. For the proof of the proposition see the proof of [17, Theorem XVI.1.1], of [11, Lemma 2.4.4] or of [15, Lemma A.1].

Assume T is nonnegative, and let Γ be the minimal contraction determined by T. Then the operator Δ on $\mathcal{X}$ given by

$$\Delta = \alpha^{1/2}(I - \Gamma^*\Gamma)\alpha^{1/2} \tag{A.2}$$

is called the *Schur complement* of T supported by $\mathcal{X}$. If δ is invertible, then $\Delta = \alpha - \beta\delta^{-1}\beta^*$, which is the classical Schur complement formula (see, e.g., [18, Lemma A.1.2]). From formula (A.2) it follows that the Schur complement $\Delta = 0$ if and only if Γ is a partial isometry with initial space equal to $\overline{\operatorname{Im}\alpha}$.

Proposition A.2. *Let T be nonnegative. The Schur complement of T supported by $\mathcal{X}$ is also given by*

$$\langle \Delta x, x\rangle = \inf\left\{\langle T\begin{bmatrix} x \\ u\end{bmatrix}, \begin{bmatrix} x \\ u\end{bmatrix}\rangle \mid u \in \mathcal{U}\right\}, \quad x \in \mathcal{X}. \tag{A.3}$$

Proof. By direct checking one proves that

$$T = \begin{bmatrix} I_{\mathcal{X}} & \alpha^{1/2}\Gamma^* \\ 0 & \delta^{1/2}\end{bmatrix}\begin{bmatrix} \Delta & 0 \\ 0 & I_{\mathcal{U}}\end{bmatrix}\begin{bmatrix} I_{\mathcal{X}} & 0 \\ \Gamma\alpha^{1/2} & \delta^{1/2}\end{bmatrix}.$$

Using this identity we see that

$$\begin{aligned}\langle T\begin{bmatrix} x \\ u\end{bmatrix}, \begin{bmatrix} x \\ u\end{bmatrix}\rangle &= \langle \begin{bmatrix} \Delta & 0 \\ 0 & I_{\mathcal{U}}\end{bmatrix}\begin{bmatrix} I_{\mathcal{X}} & 0 \\ \Gamma\alpha^{1/2} & \delta^{1/2}\end{bmatrix}\begin{bmatrix} x \\ u\end{bmatrix}, \begin{bmatrix} I_{\mathcal{X}} & 0 \\ \Gamma\alpha^{1/2} & \delta^{1/2}\end{bmatrix}\begin{bmatrix} x \\ u\end{bmatrix}\rangle \\ &= \langle \begin{bmatrix} \Delta & 0 \\ 0 & I_{\mathcal{U}}\end{bmatrix}\begin{bmatrix} x \\ \Gamma\alpha^{1/2}x + \delta^{1/2}u\end{bmatrix}, \begin{bmatrix} x \\ \Gamma\alpha^{1/2}x + \delta^{1/2}u\end{bmatrix}\rangle \\ &= \langle \Delta x, x\rangle + \|\Gamma\alpha^{1/2}x + \delta^{1/2}u\|^2.\end{aligned}$$

Thus for $x \in \mathcal{X}$ and $u \in \mathcal{U}$ we have

$$\langle \Delta x, x\rangle \le \langle T\begin{bmatrix} x \\ u\end{bmatrix}, \begin{bmatrix} x \\ u\end{bmatrix}\rangle \le \langle \Delta x, x\rangle + \|\Gamma\alpha^{1/2}x + \delta^{1/2}u\|^2. \tag{A.4}$$

Now fix $x \in \mathcal{X}$. Recall that $\operatorname{Im}\Gamma \subset \overline{\operatorname{Im}\delta} = \overline{\operatorname{Im}\delta^{1/2}}$. Thus $\Gamma\alpha^{1/2}x \in \overline{\operatorname{Im}\delta^{1/2}}$. It follows that there exist a sequence $u_1, u_2, \ldots$ in $\mathcal{U}$ such that

$$\lim_{n\to\infty} \|\Gamma\alpha^{1/2}x + \delta^{1/2}u_n\| = 0.$$

But then (A.4) shows that (A.3) holds. □

The notion of a Schur complement is closely related that of a *shorted operator* as defined by M. G. Kreĭn in [20]. In fact, if T is nonnegative, then Δ is the Schur complement of T supported by $\mathcal{X}$ if and only if

$$\begin{bmatrix} \Delta & 0 \\ 0 & 0\end{bmatrix}$$

is the shorted operator corresponding to T and $\mathcal{X}$. This follows from formula (A.3); see Section 2 in [3] for further details.

Acknowledgments. The authors thank the referee for his/her careful reading of the paper. The remarks of the referee are incorporated in the paragraph directly after Lemma 4.2 and in Remarks 5.5 and 5.7.

References

1. T. Ando, *De Branges Spaces and Analytic Operator Functions*, Lecture notes of the division of Applied Mathematics Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan, 1990.
2. Y. Arlinskiĭ, S. Hassi, and H.S.V. de Snoo, *Parametrization of contractive block operator matrices and passive discrete-time systems*, Complex Anal. Oper. Theory **1** (2007), 211–233.
3. Y. Arlinskiĭ, *The Kalman–Yakubovich–Popov inequality for passive discrete time-invariant systems*, Operators and Matrices **2** (2008), no. 1, 15–51.
4. D. Z. Arov, M. A. Kaashoek, D. R. Pik, *Minimal and optimal linear discrete time-invariant dissipative scattering systems*, Integr. Equ. Oper. Theory **29** (1997), 127–154.
5. D. Z. Arov, M. A. Kaashoek, D. R. Pik, *Minimal representations of a contractive operator as a product of two bounded operators*, Acta Sci. Math. (Szeged) **71** (2005), 313–336.
6. D. Z. Arov, M. A. Kaashoek, D. R. Pik, *The Kalman–Yakubovich–Popov inequality and infinite dimensional discrete time dissipative systems*, J. Operator Theory **55** (2006), no. 2, 393–438.
7. D. Z. Arov and M. A. Nudel'man, *Passive linear stationary dynamical scattering systems with continuous time*, Integr. Equ. Oper. Theory **24** (1996), 1–45.
8. D. Z. Arov and M. A. Nudel'man, *Criterion of unitarily similarity of minimal passive scattering systems with a given transfer function*, Ukrain. Mat. Zh. **52** (2000), no. 2, 147-156. (Russian); English transl. Ukrainian Math. J. **52** (2000), no. 2, 161–172.
9. D. Z. Arov and M. A. Nudel'man, *Conditions for the similarity of all minimal passive realizations of a given transfer function (scattering and resistance matrices)*, Mat. Sb. **193** (2002), no. 6, 3–24. (Russian); English transl. Sb. Math. **193** (2002), no. 5–6, 791–810.
10. D. Z. Arov and O. J. Staffans, *The infinite-dimensional continuous time Kalman–Yakubovich–Popov inequality*, Oper. Theory Adv. Appl. **171** (2007), 37–72.
11. M. Bakonyi and H. Woerdeman, *Matrix Completions, Moments, and Sums of Hermitian Squares*, Princeton University Press, Princeton, NJ, 2011.
12. J. A. Ball and V. Bolotnikov, *De Branges-Rovnyak Spaces: Basics and Theory*, Operator Theory, D. Alpay (ed.), Springer, Basel, 2015, pp. 631–680.
13. S. S. Boiko and V. K. Dubovoi, *On some extremal problems connected with the suboperator of scattering through inner channels of the system*, Dopov. Nats. Akad. Nauk Ukr. **4** (1997), 7–11.
14. A. Devinatz, *The factorization of operator-valued functions*, Annals of Math. **73** (1961), no. 3, 458–495.
15. M. A. Dritschel and J. Rovnyak, *The operator Fejér–Riesz theorem*, Oper. Theory Adv. Appl. **207** (2010), 223–254.
16. N. Dunford and J. T. Schwartz, *Linear Operators*, Part II, Interscience Publishers, New York, 1963.
17. C. Foias and A. Frazho, *The Commutant Lifting Approach to Interpolaton Problems*, Birkhäuser Verlag, Basel, 1990.
18. C. Foias, A. Frazho, I. Gohberg, and M. A. Kaashoek, *Metric Constrained Interpolation, Commutant Lifting and Systems*, Birkhäuser Verlag, Basel, 1998.
19. T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren Der Mathematischen Wissenschaften, Band 132, Springer-Verlag, Berlin, 1966.
20. M. G. Kreĭn, *Theory of selfadjoint extensions of semibounded operators and its applications.* I, Mat. Sb. **20** (1947), no. 3, 431–498. (Russian)
21. B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, Revised and enlarged edition, Universitext, Springer, New York, 2010.

Division of Applied Mathematics and Informatics, Institute of Physics and Mathematics, South-Ukrainian Pedagogical University, Odessa, 65020, Ukraine
E-mail address: arov_damir@mail.ru

Department of Mathematics, Vrije Universiteit, Amsterdam, The Netherlands
E-mail address: m.a.kaashoek@vu.nl

Faculty of Social and Behavioural Sciences, University of Amsterdam, Amsterdam, The Netherlands
E-mail address: drpik2@uva.nl

Received 07/02/2016; Revised 03/03/2016