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ON THE FINITENESS OF THE DISCRETE SPECTRUM OF A 3×3 OPERATOR MATRIX

TULKIN H. RASULOV

ABSTRACT. An operator matrix H associated with a lattice system describing three particles in interactions, without conservation of the number of particles, is considered. The structure of the essential spectrum of H is described by the spectra of two families of the generalized Friedrichs models. A symmetric version of the Weinberg equation for eigenvectors of H is obtained. The conditions which guarantee the finiteness of the number of discrete eigenvalues located below the bottom of the three-particle branch of the essential spectrum of H is found.

1. INTRODUCTION

One of important problems in the spectral theory of Schrödinger operators and Hamiltonians (operator matrices) in a Fock space is to study the number of eigenvalues (bound states) located outside the essential spectrum. The first mathematical result on the finiteness of the discrete spectrum of Schrödinger operators for general interactions was obtained by Uchiyama in [20]. Under natural assumptions on the potential, the essential spectrum of the continuous Schrödinger operator H_c of a system of three pair-wise interacting particles coincides with the half-axis $[\kappa; \infty)$, $\kappa \leq 0$. In independent investigations of Yafaev [21] and Zhislin [24], it was shown that for $\kappa < 0$ and a sufficiently rapid decrease of the interactions in the coordinate space representation the discrete spectrum of H_c is actually finite. In the case $\kappa = 0$ the finiteness of the discrete spectrum of H_c with certain decreasing interactions was established by Yafaev [22]. Yafaev's results are based on the investigation of the Faddeev and Weinberg type system of integral equations for the resolvent.

The problem of finiteness of the number of eigenvalues of the three-particle discrete Schrödinger operators H_d was studied by many authors, see for example, [1, 7, 13]. The authors of [1] used the Faddeev and Weinberg type equations and an expansion of the Fredholm determinant to prove finiteness of the discrete spectrum of H_d with pair contact interactions when the corresponding two-particle discrete Schrödinger operators have no virtual levels. The Birman-Schwinger principle was used in [7] to prove that the discrete spectrum of the operator H_d describing systems of three particles (two bosons and a third particle of a different nature) is finite. In [8], applying the methods developed in [22] to the Hamiltonian H_d of a system of three arbitrary particles on a lattice, finiteness of the discrete spectrum of H_d is proved if either only one or none of the two-particle subsystems has a virtual level. In [13], the finiteness of the number of eigenvalues of H_d with a specific class of potentials is proved where one of the particles has an infinite mass.

In all of the above mentioned papers devoted to the finiteness of the discrete spectrum, it was considered systems with a fixed number of quasi-particles. It is worth to mention that there are important problems in the theory of solid-state physics [12], quantum field theory [6], statistical physics [10, 11], fluid mechanics [5], magnetohydrodynamics [9] and quantum mechanics [19] where the number of quasi-particles is finite but not fixed. Recall that the study of systems describing n particles in interaction without conservation of the number of particles can be reduced to the investigation of the spectral properties of self-adjoint operators acting in the n-particle cut subspace of the Fock space [6, 11, 12, 18]. In [18], geometric and commutator techniques were developed in order to find the location of the spectrum and to prove absence of singular continuous spectrum for Hamiltonians without the conservation of particle number.

In the present paper we consider an operator matrix H associated with the lattice system describing three particles in interactions without conservation of the number of particles. This operator acts in a three-particle subspace $\mathcal{H}$ of the bosonic Fock space and it is a lattice analogue of the spin-boson Hamiltonian [11]. We find sufficient conditions for the finiteness of the discrete spectrum of H. Note that the operator matrix H has been considered before in [14, 15, 16, 23] where only its essential spectrum was investigated.

The organization of the present paper is as follows. Section 1 is an introduction to the whole work. In Section 2, the operator matrix H is described as a bounded self-adjoint operator in $\mathcal{H}$ and the main results are formulated. In Section 3, we prove some auxiliary lemmas. In Section 4, we obtain a symmetric version of the Weinberg equation for eigenvectors of H. Section 5 is devoted to the proof of the main results.

2. The operator matrix and main results

2.1. **The operator matrix.** Let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ be the set of all complex, real and integer numbers, respectively. We denote by $\mathbb{T}^3$ the three-dimensional torus (the first Brillouin zone, i.e., the dual group of $\mathbb{Z}^3$), the cube $(-\pi, \pi]^3$ with appropriately identified sides is equipped with its Haar measure. The torus $\mathbb{T}^3$ will always be considered as an Abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space $\mathbb{R}^3$ modulo $(2\pi\mathbb{Z})^3$.

Let $L_2(\mathbb{T}^3)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^s((\mathbb{T}^3)^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^3)^2$. Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_1 = \mathbb{C}$, $\mathcal{H}_1 = L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 = L_2^s((\mathbb{T}^3)^2)$, that is, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

Let us consider the operator matrix (Hamiltonian) H acting in the Hilbert space $\mathcal{H}$ as

$$H = \begin{pmatrix} H_{00} & H_{01} & 0 \\ H_{01}^* & H_{11} & H_{12} \\ 0 & H_{12}^* & H_{22} \end{pmatrix},$$

where the entries $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i \le j$, $i, j = 0, 1, 2$ are defined by

$$H_{00} f_0 = w_0 f_0, \quad H_{01} f_1 = \int_{\mathbb{T}^3} v_0(s) f_1(s) \, ds, \quad (H_{11} f_1)(p) = w_1(p) f_1(p),$$

$$(H_{12} f_2)(p) = \int_{\mathbb{T}^3} v_1(s) f_2(p, s) \, ds, \quad H_{22} = H_{22}^0 - V, \quad (H_{22}^0 f_2)(p, q) = w_2(p, q) f_2(p, q),$$

$$(V f_2)(p, q) = v_2(q) \int_{\mathbb{T}^3} v_2(s) f_2(p, s) \, ds + v_2(p) \int_{\mathbb{T}^3} v_2(s) f_2(s, q) \, ds.$$

Here $f_i \in \mathcal{H}_i$, $i = 0, 1, 2$; w_0 is a fixed real number, $w_1(\cdot)$ and $v_i(\cdot)$, $i = 0, 1, 2$ are real-valued continuous functions on $\mathbb{T}^3$, the function $w_2(\cdot, \cdot)$ is a real-valued continuous

symmetric function on $(\mathbb{T}^2)^2$. The operator H_{ij}^* $(i < j)$ denotes the adjoint to H_{ij} and

$$(H_{01}^* f_0)(p) = v_0(p) f_0, \quad (H_{12}^* f_1)(p, q) = \frac{v_1(p) f_1(q) + v_1(q) f_1(p)}{2}, \quad f_i \in \mathcal{H}_i, \quad i = 0, 1.$$

It follows that under these assumptions H is bounded and self-adjoint.

We recall that the operators H_{01} and H_{12} (resp. H_{01}^* and H_{12}^*) are called annihilation (resp. creation) operators, respectively. In the present paper we consider the case where the number of annihilations and creations of the particles of the system is equal to 1, that is, $H_{ij} \equiv 0$ for all $|i - j| > 1$.

It is known that the three-particle discrete Schrödinger operator $\widehat{H}$ in the momentum representation acts on the Hilbert space $L_2((\mathbb{T}^3)^3)$. Introducing the total quasi-momentum $K \in \mathbb{T}^3$ and choosing relative coordinate system, we decompose $\widehat{H}$ into the von Neumann direct integral (see for example [1, 7, 8, 13])

$$\widehat{H} = \int_{\mathbb{T}^3} \widehat{H}(K) \, dK,$$

where the bounded self-adjoint operator $\widehat{H}(K)$, $K \in \mathbb{T}^3$, acts on the Hilbert space $L_2(\Gamma_K)$. Here $\Gamma_K \subset (\mathbb{T}^3)^2$ being some manifold.

Notice that the operator matrix H satisfies the main spectral properties of the three-particle discrete Schrödinger operator $\widehat{H}(0)$, where the role of two-particle discrete Schrödinger operators is played by the family of the generalized Friedrichs models [3, 4]. For this reason the Hilbert space $\mathcal{H}$ is called the *three-particle cut subspace* of the bosonic Fock space $\mathcal{F}_s(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$ and the operator matrix H is associated to a system describing three particles in interaction without conservation of the number of particles. The operator H_{22} is associated to a system of three quantum particles on a lattice.

To formulate the main results of the paper we introduce the operators H_1 and H_2 acting in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_2$, respectively, as

$$H_1 := \begin{pmatrix} H_{00} & H_{01} & 0 \\ H_{01}^* & H_{11} & H_{12} \\ 0 & H_{12}^* & H_{22}^0 \end{pmatrix}, \quad H_2 := H_{22},$$

and the family of bounded self-adjoint operators (generalized Friedrichs models) $h(p)$, $p \in \mathbb{T}^3$, acting in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$h(p) = \begin{pmatrix} h_{00}(p) & h_{01} \\ h_{01}^* & h_{11}(p) \end{pmatrix},$$

where

$$h_{00}(p) f_0 = w_1(p) f_0, \quad h_{01} f_1 = \frac{1}{\sqrt{2}} \int_{\mathbb{T}^3} v_1(s) f_1(s) \, ds,$$

$$h_{11}(p) = h_{11}^0(p) - v, \quad (h_{11}^0(p) f_1)(q) = w_2(p, q) f_1(q), \quad (v f_1)(q) = v_2(q) \int_{\mathbb{T}^3} v_2(s) f_1(s) \, ds.$$

We recall that the operator $h(p)$ is also called molecular-resonance model and it is associated with the Hamiltonian of the system consisting of at most two particles on the three-dimensional lattice, interacting via both a nonlocal potential and creation and annihilation operators.

In [16] it was shown that for any $p \in \mathbb{T}^3$ the operator $h(p)$ has at most three eigenvalues.

The spectrum, the essential spectrum, the discrete and point spectrum of a bounded self-adjoint operator will be denoted by $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$, $\sigma_{\mathrm{disc}}(\cdot)$ and $\sigma_{\mathrm{p}}(\cdot)$ respectively.

Set

$$m := \min_{p,q \in \mathbb{T}^3} w_2(p, q), \quad M := \max_{p,q \in \mathbb{T}^3} w_2(p, q).$$

The following theorem describes the location of the essential spectrum of the operator H by the spectrum of the family $h(p)$ of the generalized Friedrichs models [16].

Theorem 2.1. *For the essential spectrum of H the following equality holds:*

(2.1) $$\sigma_{\mathrm{ess}}(H) = \sigma \cup [m; M], \quad \sigma := \bigcup_{p \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(h(p)).$$

Moreover, the set $\sigma_{\mathrm{ess}}(H)$ is a union of at most four bounded closed intervals.

The subsets σ and $[m; M]$ are called two-particle and three-particle branches of the essential spectrum of H, respectively.

2.2. **Main assumptions.** From now on we always assume that $\{\alpha, \beta\} = \{1, 2\}$ and $\alpha \neq \beta$. Denote $\bar{\pi} := (\pi, \pi, \pi)$.

Assumption 2.2. *The function $v_\alpha(\cdot)$ is $2\bar{\pi}$ periodic and $v_\beta(\cdot)$ satisfies the condition*

(2.2) $$\int_{\mathbb{T}^3} v_\beta(s) g(s) \, ds = 0$$

for any $2\bar{\pi}$ periodic function $g \in L_2(\mathbb{T}^3)$.

Assumption 2.3. (i) *The function $w_2(\cdot, \cdot)$ is $2\bar{\pi}$ periodic on each variable p and q, that is, $w_2(p + 2\bar{\pi}, q) = w_2(p, q + 2\bar{\pi}) = w_2(p, q)$ for all $p, q \in \mathbb{T}^3$;*
(ii) *The function $w_2(\cdot, \cdot)$ has a unique non-degenerate minimum at the point $(p_0, p_0) \in (\mathbb{T}^3)^2$. All third order partial derivatives of the functions $w_1(\cdot)$ and $w_2(\cdot, \cdot)$ are continuous on $\mathbb{T}^3$ and $(\mathbb{T}^3)^2$, respectively.*

Under the Assumption 2.2 and the part (i) of Assumption 2.3 the discrete spectrum of $h(p)$ coincides (see Lemma 3.1 below) with the union of discrete spectra of the operators

$$h_1(p) := \begin{pmatrix} h_{00}(p) & h_{01} \\ h_{01}^* & h_{11}^0(p) \end{pmatrix} \quad \text{and} \quad h_2(p) := h_{11}(p).$$

It follows from the definition of the operator $h_\alpha(p)$ that its structure is simpler than that of $h(p)$. Using the Weyl theorem one can easily show that

$$\sigma_{\mathrm{ess}}(h(p)) = \sigma_{\mathrm{ess}}(h_1(p)) = \sigma_{\mathrm{ess}}(h_2(p)) = [m(p); M(p)],$$

where the numbers $m(p)$ and $M(p)$ are defined by

$$m(p) := \min_{q \in \mathbb{T}^3} w_2(p, q), \quad M(p) := \max_{q \in \mathbb{T}^3} w_2(p, q).$$

For any fixed $p \in \mathbb{T}^3$, we define the analytic functions in $\mathbb{C} \setminus [m(p); M(p)]$ by

$$\Delta_1(p\,; z) := w_1(p) - z - \frac{1}{2} \int_{\mathbb{T}^3} \frac{v_1^2(s) \, ds}{w_2(p, s) - z}, \quad \Delta_2(p\,; z) := 1 - \int_{\mathbb{T}^3} \frac{v_2^2(s) \, ds}{w_2(p, s) - z},$$

which are Fredholm determinants associated with the operators $h_1(p)$ and $h_2(p)$, respectively.

Since the function $w_2(\cdot, \cdot)$ has a unique non-degenerate minimum at $(p_0, p_0) \in (\mathbb{T}^3)^2$ and the function $v_\alpha(\cdot)$ is a continuous on $\mathbb{T}^3$, for any $p \in \mathbb{T}^3$ the integral

$$\int_{\mathbb{T}^3} \frac{v_\alpha^2(s) \, ds}{w_2(p, s) - m}$$

is positive and finite. Then the Lebesgue dominated convergence theorem yields

$$\Delta_\alpha(p_0\,; m) = \lim_{p \to p_0} \Delta_\alpha(p\,; m),$$

and hence the function $\Delta_\alpha(\cdot\,; m)$ is a continuous on $\mathbb{T}^3$.

Note that using the fact [2, 3]

$$\sigma_{\mathrm{ess}}(H_\alpha) = \sigma_\alpha \cup [m; M], \quad \sigma_\alpha := \bigcup_{p \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(h_\alpha(p))$$

together with Assumption 2.2 and part (i) of Assumption 2.3 the equality (2.1) can be written as

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_1) \cup \sigma_{\text{ess}}(H_2).$$

(2.3)

It was shown in [2, 3] that if $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$, then $\sigma_\alpha \cap (-\infty;m] \neq \emptyset$. Assuming $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$, we introduce the following numbers:

$$E_{\min}^{(\alpha)} := \min\left\{\sigma_\alpha \cap (-\infty;m]\right\}, \quad E_{\max}^{(\alpha)} := \max\left\{\sigma_\alpha \cap (-\infty;m]\right\}.$$

The following theorem [2, 16] describes the structure of the part of the essential spectrum of H_α located in $(-\infty;M]$.

Theorem 2.4. *Let part* (ii) *of Assumption* 2.3 *be fulfilled. Then the following assertions hold.*
(i) *If* $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) \geq 0$, *then*

$$(-\infty;M] \cap \sigma_{\text{ess}}(H_\alpha) = [m;M].$$

(ii) *If* $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$ *and* $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) \geq 0$, *then*

$$(-\infty;M] \cap \sigma_{\text{ess}}(H_\alpha) = [E_{\min}^{(\alpha)};M], \quad E_{\min}^{(\alpha)} < m.$$

(iii) *If* $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$, *then*

$$(-\infty;M] \cap \sigma_{\text{ess}}(H_\alpha) = [E_{\min}^{(\alpha)};E_{\max}^{(\alpha)}] \cup [m;M], \quad E_{\max}^{(\alpha)} < m.$$

We notice that if Assumption 2.2 and part (i) of Assumption 2.3 hold, then Theorem 2.4 together with the equality (2.3) describes the structure of the part of the essential spectrum of H located in $(-\infty;M]$.

If $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$, then from $E_{\min}^{(\alpha)}, E_{\max}^{(\alpha)} \in \sigma_\alpha$ it follows that there exist positive integers n_α, k_α and points $\{p_{\alpha i}\}_{i=1}^{n_\alpha}, \{q_{\alpha j}\}_{j=1}^{k_\alpha} \subset \mathbb{T}^3$ such that

$$\{p \in \mathbb{T}^3 : \Delta_\alpha(p\,;E_{\min}^{(\alpha)}) = 0\} = \{p_{\alpha 1}, \ldots, p_{\alpha n_\alpha}\},$$

$$\{p \in \mathbb{T}^3 : \Delta_\alpha(p\,;E_{\max}^{(\alpha)}) = 0\} = \{q_{\alpha 1}, \ldots, q_{\alpha k_\alpha}\}.$$

Assumption 2.5. *There exist positive numbers* C, δ *and* $\beta_{\alpha i} \in (0;2]$, $i = 1, \ldots, n_\alpha$ *such that*

$$|\Delta_\alpha(p\,;E_{\min}^{(\alpha)})| \geq C|p - p_{\alpha i}|^{\beta_{\alpha i}}, \quad p \in U_\delta(p_{\alpha i}), \quad i = 1, \ldots, n_\alpha,$$

and the inequality $\Delta_\alpha(p\,;E_{\min}^{(\alpha)}) > 0$ *holds for all* $p \in \mathbb{T}^3 \setminus \{p_{\alpha 1}, \ldots, p_{\alpha n_\alpha}\}$.

Assumption 2.6. *There exist positive numbers* K, ρ *and* $\gamma_{\alpha j} \in (0;2]$, $j = 1, \ldots, k_\alpha$ *such that*

$$|\Delta_\alpha(p\,;E_{\max}^{(\alpha)})| \geq K|p - q_{\alpha j}|^{\gamma_{\alpha j}}, \quad p \in U_\rho(q_{\alpha j}), \quad j = 1, \ldots, k_\alpha,$$

and the inequality $\Delta_\alpha(p\,;E_{\max}^{(\alpha)}) < 0$ *holds for all* $p \in \mathbb{T}^3 \setminus \{q_{\alpha 1}, \ldots, q_{\alpha k_\alpha}\}$.

2.3. **Statement of the main results.** Here we formulate main results of the paper.

Theorem 2.7. *Let part* (i) *of Assumption* 2.3 *be fulfilled.*
(i) *If Assumption* 2.2 *holds with* $\alpha = 1$ *and in addition, the functions* $v_0(\cdot)$, $w_1(\cdot)$ *are* $2\bar{\pi}$ *periodic, then* $\sigma_{\text{disc}}(H_1) \subset \sigma_{\text{p}}(H)$.
(i) *If Assumption* 2.2 *holds with* $\alpha = 2$, *then* $\sigma_{\text{disc}}(H_2) \subset \sigma_{\text{p}}(H)$.

Theorem 2.8. *Let Assumptions* 2.2 *and* 2.3 *be fulfilled. Assume*

$(\alpha.1)$ $\min_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) > 0;$

$(\alpha.2)$ $\min_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0,$ $\max_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) \geq 0$ *and Assumption* 2.5 *holds;*

$(\alpha.3)$ $\max_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$ *and Assumptions* 2.5, 2.6 *hold.*

If for some $i, j \in \{1, 2, 3\}$ *the conditions* (1.i) *and* (2.j) *hold, then the operator matrix* H *has a finite number of discrete eigenvalues lying on the left of* m.

Remark 2.9. *The class of functions* $w_1(\cdot)$, $v_i(\cdot)$, $i = 1, 2$ *and* $w_2(\cdot, \cdot)$ *satisfying the conditions in Theorem* 2.8 *is nonempty (see Lemma* 5.1).

Remark 2.10. *Note that comparing Theorems* 2.7 *and* 2.8 *we have that if the condition* $(\alpha.j)$ *in Theorem* 2.8 *holds for some* $j \in \{1, 2, 3\}$, *then the operator* H_α *has a finite number of discrete eigenvalues lying on the left of* m. *If* $\min_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) = \Delta_\alpha(p_0\,;m) = 0$ *and* $v_\alpha(p_0) \neq 0$, *then* $\min \sigma_{\mathrm{ess}}(H_\alpha) = m$ *and it was shown in* [3] *for* $\alpha = 1$ *and in* [2] *for* $\alpha = 2$ *that the operator* H_α *has infinitely many eigenvalues lying on the left of* m. *Hence, in this case by Theorem* 2.7 *the operator* H *also has infinitely many eigenvalues lying on the left of* m.

3. Some auxiliary statements

The following lemma describes the relation between the eigenvalues of the operators $h(p)$ and $h_\alpha(p)$.

Lemma 3.1. *Let Assumption* 2.2 *and part* (i) *of Assumption* 2.3 *be fulfilled. For any fixed* $p \in \mathbb{T}^3$ *the number* $z(p) \in \mathbb{C} \setminus [m(p); M(p)]$ *is an eigenvalue for* $h(p)$ *if and only if* $z(p)$ *is an eigenvalue for at least one of the operators* $h_1(p)$ *and* $h_2(p)$.

Proof. Let $p \in \mathbb{T}^3$ be fixed. Suppose $(f_0, f_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1$ is an eigenvector of the operator $h(p)$ associated with the eigenvalue $z(p) \in \mathbb{C} \setminus [m(p); M(p)]$. Then f_0 and f_1 satisfy the following system of equations:

$$(w_1(p) - z(p))f_0 + \frac{1}{\sqrt{2}} \int_{\mathbb{T}^3} v_1(s) f_1(s)\, ds = 0,$$

(3.1)

$$\frac{1}{\sqrt{2}} v_1(q) f_0 + (w_2(p, q) - z(p)) f_1(q) - v_2(q) \int_{\mathbb{T}^3} v_2(s) f_1(s)\, ds = 0.$$

Since for any $q \in \mathbb{T}^3$ the relation $w_2(p, q) - z(p) \neq 0$ holds, from the second equation in the system (3.1) for f_1 we have

(3.2)
$$f_1(q) = \frac{C_{f_1} v_2(q)}{w_2(p, q) - z(p)} - \frac{1}{\sqrt{2}} \frac{v_1(q) f_0}{w_2(p, q) - z(p)},$$

where

(3.3)
$$C_{f_1} = \int_{\mathbb{T}^3} v_2(s) f_1(s)\, ds.$$

Substituting the expression (3.2) for f_1 into the first equation of the system (3.1) and the equality (3.3), we conclude that the system of equations (3.1) has a nontrivial solution if and only if the system of equations

$$\Delta_1(p\,;z(p)) f_0 + \frac{1}{\sqrt{2}} \int_{\mathbb{T}^3} \frac{v_1(s) v_2(s)\, ds}{w_2(p, s) - z(p)} C_{f_1} = 0,$$

$$\frac{1}{\sqrt{2}} \int_{\mathbb{T}^3} \frac{v_1(s) v_2(s)\, ds}{w_2(p, s) - z(p)} f_0 + \Delta_2(p\,;z(p)) C_{f_1} = 0$$

has a nontrivial solution $(f_0, C_{f_1}) \in \mathbb{C}^2$, i.e. if the condition

$$\Delta_1(p\,;z(p))\Delta_2(p\,;z(p)) - \frac{1}{2}\left(\int_{\mathbb{T}^3} \frac{v_1(s)v_2(s)\,ds}{w_2(p,s) - z(p)}\right)^2 = 0$$

is satisfied.

By part (i) of Assumption 2.3 for any fixed $p \in \mathbb{T}^3$ the function $(w_2(p,\cdot) - z(p))^{-1} \in L_2(\mathbb{T}^3)$ is $2\bar{\pi}$ periodic. Applying Assumption 2.2 we obtain

$$\int_{\mathbb{T}^3} \frac{v_1(s)v_2(s)\,ds}{w_2(p,s) - z(p)} = 0.$$

If we set $v_2(q) \equiv 0$ in the operator $h(p)$, then $h(p) = h_1(p)$; in this case the number $z(p) \in \mathbb{C} \setminus [m(p); M(p)]$ is an eigenvalue of $h_1(p)$ if and only if $\Delta_1(p\,;z(p)) = 0$. Similarly one can show that the number $z(p) \in \mathbb{C} \setminus [m(p); M(p)]$ is an eigenvalue of $h_2(p)$ if and only if $\Delta_2(p\,;z(p)) = 0$. The lemma is proved. $\qquad\square$

Lemma 3.2. *Let* $\min\limits_{p\in\mathbb{T}^3} \Delta_\alpha(p\,;m) > 0$. *Then there exists a positive number* C_1 *such that the inequality* $\Delta_\alpha(p\,;z) \geq C_1$ *holds for all* $p \in \mathbb{T}^3$ *and* $z \leq m$.

Proof. Since for any $p \in \mathbb{T}^3$ the function $\Delta_\alpha(p\,;\cdot)$ is monotonically decreasing in $(-\infty; m]$, we have

$$\Delta_\alpha(p\,;z) \geq \Delta_\alpha(p\,;m) \geq \min_{p\in\mathbb{T}^3} \Delta_\alpha(p\,;m) > 0$$

for all $p \in \mathbb{T}^3$ and $z \leq m$. Now setting $C_1 := \min\limits_{p\in\mathbb{T}^3} \Delta_\alpha(p\,;m)$ we complete the proof of lemma. $\qquad\square$

For some $\delta > 0$ we set

$$U_\delta(p_0) := \{p \in \mathbb{T}^3 : |p - p_0| < \delta\}.$$

Lemma 3.3. *If Assumption* 2.5 *resp.* 2.6 *holds, then for any* $\delta > 0$ *there exist the positive numbers* $C_1(\delta)$ *and* $C_2(\delta)$ *such that*
(i) $\Delta_\alpha(p\,;E_{\min}^{(\alpha)}) \geq C_1(\delta)$ *for any* $p \in \mathbb{T}^3 \setminus \bigcup\limits_{i=1}^{n_\alpha} U_\delta(p_{\alpha i})$;
resp.
(ii) $|\Delta_\alpha(p\,;E_{\max}^{(\alpha)})| \geq C_2(\delta)$ *for any* $p \in \mathbb{T}^3 \setminus \bigcup\limits_{j=1}^{k_\alpha} U_\delta(q_{\alpha j})$.

Proof. Let Assumption 2.5 be fulfilled. Then the inequality $\Delta_\alpha(p\,;E_{\min}^{(\alpha)}) > 0$ holds for any $\mathbb{T}^3 \setminus \{p_{\alpha 1}, \ldots, p_{\alpha n_\alpha}\}$. Since for any $\delta > 0$ the set $\mathbb{T}^3 \setminus \bigcup\limits_{i=1}^{n_\alpha} U_\delta(p_{\alpha i})$ is compact and $\Delta_\alpha(\cdot\,;E_{\min}^{(\alpha)})$ is the positive continuous function on this set, there exists the number $C_1(\delta) > 0$ such that the assertion (i) of lemma holds. Proof of assertion (ii) is similar. $\quad\square$

Lemma 3.4. *Let part* (ii) *of Assumption* 2.3 *be fulfilled. Then there exist positive numbers* C_1, C_2, C_3 *and* δ *such that the following inequalities hold:*
(i) $C_1(|p - p_0|^2 + |q - p_0|^2) \leq w_2(p,q) - m \leq C_2(|p - p_0|^2 + |q - p_0|^2)$, $p, q \in U_\delta(p_0)$;
(ii) $w_2(p,q) - m \geq C_3$, $(p,q) \notin U_\delta(p_0) \times U_\delta(p_0)$.

Proof. By part (ii) of Assumption 2.3 the all third order partial derivatives of $w_2(\cdot,\cdot)$ are continuous on $(\mathbb{T}^3)^2$ and it has a unique non-degenerate minimum at the point $(p_0, p_0) \in (\mathbb{T}^3)^2$. Then by the Hadamard lemma [25] there exists a δ-neighborhood of the point

$p_0 \in \mathbb{T}^3$ such that the following decomposition holds:

$$w_2(p,q) = m + \frac{1}{2}\left((W_1(p-p_0), p-p_0) + 2(W_2(p-p_0), q-p_0) + (W_1(q-p_0), q-p_0)\right)$$

$$+ \sum_{|s|+|l|=3} H_{sl}(p,q) \prod_{i=1}^{3} (p^{(i)} - p_0^{(i)})^{s_i} (q^{(i)} - p_0^{(i)})^{l_i}, \quad p, q \in U_\delta(p_0),$$

where

$$W_1 := \left(\frac{\partial^2 w_2(p_0, p_0)}{\partial p^{(i)} \partial p^{(j)}}\right)_{i,j=1}^{3}, \quad W_2 := \left(\frac{\partial^2 w_2(p_0, p_0)}{\partial p^{(i)} \partial q^{(j)}}\right)_{i,j=1}^{3},$$

$$s = (s_1, s_2, s_3), \quad l = (l_1, l_2, l_3), \quad |s| = s_1 + s_2 + s_3, \quad s_i, l_i \in \{0, 1, 2, 3\}, \quad i = 1, 2, 3,$$

and $H_{sl}(\cdot, \cdot)$ with $|s| + |l| = 3$ are continuous functions in $U_\delta(p_0) \times U_\delta(p_0)$. Therefore, there exist positive numbers C_1, C_2, C_3 such that (i) and (ii) hold true. $\qquad \square$

4. The Weinberg type system of integral equations

In this section we derive an analogue of the Weinberg type system of integral equations for the eigenvectors, corresponding to the eigenvalues of H, lying on the left of m.

Let $\tau_{\mathrm{ess}}(H)$ be the lower bound of the essential spectrum of H. It is clear that $\Delta_\alpha(p; z) > 0$ for all $p \in \mathbb{T}^3$ and $z \in (-\infty; \tau_{\mathrm{ess}}(H))$; if $\max_{p \in \mathbb{T}^3} \Delta_\alpha(p; m) < 0$, then $\Delta_\alpha(p; z) < 0$ for all $p \in \mathbb{T}^3$ and $z \in (E_{\max}^{(\alpha)}; m)$. So $\mathrm{sign}(\Delta_\alpha(p; z))$ depends on the location of $z \in (-\infty; m) \setminus \sigma_{\mathrm{ess}}(H)$ and does not depend on $p \in \mathbb{T}^3$. For $z \in (-\infty; m) \setminus \sigma_{\mathrm{ess}}(H)$ we set $\xi_\alpha(z) := \mathrm{sign}(\Delta_\alpha(p; z))$.

Let for any $z \in (-\infty; m) \setminus \sigma_{\mathrm{ess}}(H)$ the operator $W(z)$ act in the Hilbert space $\mathcal{H}$ as a 3×3 operator matrix with entries $W_{ij}(z) : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0, 1, 2$ defined by

$$W_{00}(z)g_0 = (1 + z - w_0)g_0, \quad W_{01}(z)g_1 = -\int_{\mathbb{T}^3} \frac{v_0(s)g_1(s)\,ds}{\sqrt{\xi_1(z)\Delta_1(s; z)}},$$

$$W_{02}(z) \equiv 0, \quad (W_{10}(z)g_0)(p) = -\frac{\xi_1(z)v_0(p)g_0}{\sqrt{\xi_1(z)\Delta_1(p; z)}},$$

$$(W_{11}(z)g_1)(p) = \frac{\xi_1(z)v_1(p)}{2\sqrt{\xi_1(z)\Delta_1(p; z)}} \int_{\mathbb{T}^3} \frac{v_1(s)g_1(s)\,ds}{\sqrt{\xi_1(z)\Delta_1(s; z)}(w_2(p,s) - z)},$$

$$(W_{12}(z)g_2)(p) = -\frac{\xi_1(z)v_2(p)}{\sqrt{\xi_1(z)\Delta_1(p; z)}} \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \frac{v_1(s)v_2(t)g_2(s,t)\,ds\,dt}{\sqrt{\xi_2(z)\Delta_2(s; z)}(w_2(p,s) - z)},$$

$$(W_{20}(z)g_0)(p,q) = -\frac{v_1(p)(W_{10}(z)g_0)(q) + v_1(q)(W_{10}(z)g_0)(p)}{2(w_2(p,q) - z)},$$

$$(W_{21}(z)g_1)(p,q) = -\frac{\xi_2(z)v_1(p)v_2(q)}{2(w_2(p,q) - z)\sqrt{\xi_2(z)\Delta_2(p; z)}} \int_{\mathbb{T}^3} \frac{v_2(s)g_1(s)\,ds}{\sqrt{\xi_1(z)\Delta_1(s; z)}(w_2(p,s) - z)}$$

$$- \frac{\xi_2(z)v_1(q)v_2(p)}{2(w_2(p,q) - z)\sqrt{\xi_2(z)\Delta_2(q; z)}} \int_{\mathbb{T}^3} \frac{v_2(s)g_1(s)\,ds}{\sqrt{\xi_1(z)\Delta_1(s; z)}(w_2(q,s) - z)}$$

$$- \frac{v_1(p)(W_{11}(z)g_1)(q) + v_1(q)(W_{11}(z)g_1)(p)}{2(w_2(p,q) - z)},$$

$$(W_{22}(z)g_2)(p,q) = \frac{\xi_2(z)v_2(p)v_2(q)}{(w_2(p,q)-z)\sqrt{\xi_2(z)\Delta_2(p;z)}}\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\frac{v_2(s)v_2(t)g_2(s,t)\,ds\,dt}{\sqrt{\xi_2(z)\Delta_2(s;z)}(w_2(p,s)-z)}$$

$$+\frac{\xi_2(z)v_2(p)v_2(q)}{(w_2(p,q)-z)\sqrt{\xi_2(z)\Delta_2(q;z)}}\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\frac{v_2(s)v_2(t)g_2(s,t)\,ds\,dt}{\sqrt{\xi_2(z)\Delta_2(s;z)}(w_2(q,s)-z)}$$

$$-\frac{v_1(p)(W_{12}(z)g_2)(q)+v_1(q)(W_{12}(z)g_2)(p)}{2(w_2(p,q)-z)},$$

where $g_i \in \mathcal{H}_i$, $i=0,1,2$.

We have the following lemma.

Lemma 4.1. *Let Assumption 2.2 and part* (i) *of Assumption 2.3 be fulfilled. If $f \in \mathcal{H}$ is an eigenvector corresponding to the eigenvalue $z \in (-\infty;m)\setminus\sigma_{\mathrm{ess}}(H)$ of H, then f satisfies the Weinberg equation $W(z)f = f$.*

Proof. Let $z \in (-\infty;m)\setminus\sigma_{\mathrm{ess}}(H)$ be an eigenvalue of the operator H and

$$f = (f_0, f_1, f_2) \in \mathcal{H}$$

be the corresponding eigenvector. Then f_0, f_1 and f_2 satisfy the system of equations

$$(H_{00}-z)f_0 + H_{01}f_1 = 0,$$

(4.1) $$(H_{10}f_0)(p) + ((H_{11}-z)f_1)(p) + (H_{12}f_2)(p) = 0,$$

$$(H_{21}f_1)(p,q) + ((H_{22}^0 - z)f_2)(p,q) - (Vf_2)(p,q) = 0.$$

Since $z < m$, from the third equation of the system (4.1) for f_2 we have

(4.2) $$f_2(p,q) = \frac{v_2(q)\overline{f}_2(p) + v_2(p)\overline{f}_2(q)}{w_2(p,q)-z} - \frac{v_1(q)f_1(p)+v_1(p)f_1(q)}{2(w_2(p,q)-z)},$$

where

(4.3) $$\overline{f}_2(p) = \int_{\mathbb{T}^3} v_2(s)f_2(p,s)\,ds.$$

Substituting the expression (4.2) for f_2 into the second equation in the system (4.1) and the equality (4.3) and using Assumptions 2.2 and 2.3, we obtain

$$f_0 = (1 + z - w_0)f_0 - \int_{\mathbb{T}^3} v_0(s)f_1(s)\,ds = 0,$$

(4.4) $$\Delta_1(p;z)f_1(p) = -v_0(p)f_0 + \frac{v_1(p)}{2}\int_{\mathbb{T}^3}\frac{v_1(s)f_1(s)\,ds}{w_2(p,s)-z} - v_2(p)\int_{\mathbb{T}^3}\frac{v_1(s)\overline{f}_2(s)\,ds}{w_2(p,s)-z},$$

$$\Delta_2(p;z)\overline{f}_2(p) = -\frac{v_1(p)}{2}\int_{\mathbb{T}^3}\frac{v_2(s)f_1(s)\,ds}{w_2(p,s)-z} + v_2(p)\int_{\mathbb{T}^3}\frac{v_2(s)\overline{f}_2(s)\,ds}{w_2(p,s)-z}.$$

It is clear that the inequality $\xi_\alpha(z)\Delta_\alpha(p;z) > 0$ holds for all $z \in (-\infty;m)\setminus\sigma_{\mathrm{ess}}(H)$ and $p \in \mathbb{T}^3$. Therefore, the system of equations (4.4) has a nontrivial solution if and only if the following system of equations:

$$f_0 = W_{00}(z)f_0 + W_{01}(z)f_1 = 0,$$

$$f_1(p) = (W_{10}(z)f_0)(p) + (W_{11}(z)f_1)(p)$$

$$-\frac{\xi_1(z)v_2(p)}{\sqrt{\xi_1(z)\Delta_1(p;z)}}\int_{\mathbb{T}^3}\frac{v_1(s)\overline{f}_2(s)\,ds}{\sqrt{\xi_2(z)\Delta_2(s;z)}(w_2(p,s)-z)},$$

$$\overline{f}_2(p) = -\frac{\xi_2(z)v_1(p)}{2\sqrt{\xi_2(z)\Delta_2(p;z)}}\int_{\mathbb{T}^3}\frac{v_2(s)f_1(s)\,ds}{\sqrt{\xi_1(z)\Delta_1(s;z)}(w_2(p,s)-z)}$$

$$+\frac{\xi_2(z)v_2(p)}{\sqrt{\xi_2(z)\Delta_2(p;z)}}\int_{\mathbb{T}^3}\frac{v_2(s)\overline{f}_2(s)\,ds}{\sqrt{\xi_2(z)\Delta_2(s;z)}(w_2(p,s)-z)}$$

has a nontrivial solution.

Substituting the last expressions for f_1 and $\overline{f}_2$ into the formula (4.2) and using the equality (4.3), we obtain the Weinberg equation $W(z)f = f$. $\qquad\square$

Set
$$\Sigma := \overline{[\tau_{\mathrm{ess}}(H) - 1; m] \setminus \sigma_{\mathrm{ess}}(H)}.$$

Lemma 4.2. *Let assumptions in Theorem 2.8 be fulfilled. Then the operator $W(z)$ is compact for $z \in \Sigma$ and the operator-valued function $W(z)$ is continuous in the uniform operator topology for $z \in \Sigma$.*

Proof. First for the convenience using Theorem 2.7 we describe the structure of the set Σ :

(i) if $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) \geq 0$, then $\Sigma = [m - 1; m]$;

(ii) if $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$ and $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) \geq 0$, then $\Sigma = [E_{\min} - 1; E_{\min}]$, where $E_{\min} = \min\{E_{\min}^{(1)}, E_{\min}^{(2)}\}$ and $E_{\min} < m$;

(iii) if $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$, $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) \geq 0$ and $\min\limits_{p \in \mathbb{T}^3} \Delta_\beta(p\,;m) \geq 0$, then $\Sigma = [E_{\min}^{(\alpha)} - 1; E_{\min}^{(\alpha)}]$ with $E_{\min}^{(\alpha)} < m$;

(iv) if $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$ and $\min\limits_{p \in \mathbb{T}^3} \Delta_\beta(p\,;m) \geq 0$, then $\Sigma = [E_{\min}^{(\alpha)} - 1; E_{\min}^{(\alpha)}] \cup [E_{\max}^{(\alpha)}; m]$ with $E_{\max}^{(\alpha)} < m$;

(v) if $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$, $\min\limits_{p \in \mathbb{T}^3} \Delta_\beta(p\,;m) < 0$ and $\max\limits_{p \in \mathbb{T}^3} \Delta_\beta(p\,;m) \geq 0$, then

$$\Sigma = \begin{cases} [E_{\min} - 1; E_{\min}], & \text{if } E_{\max}^{(\alpha)} \geq E_{\min}^{(\beta)}, \\ [E_{\min}^{(\alpha)} - 1; E_{\min}^{(\alpha)}] \cup [E_{\max}^{(\alpha)}; E_{\min}^{(\beta)}], & \text{if } E_{\max}^{(\alpha)} < E_{\min}^{(\beta)} \end{cases}$$

with $E_{\max}^{(\alpha)}, E_{\min}^{(\beta)} < m$;

(vi) if $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$, then $\Sigma = [\tau_{\mathrm{ess}}(H) - 1; m] \setminus \{(E_{\min}^{(1)}; E_{\max}^{(1)}) \cup (E_{\min}^{(2)}; E_{\max}^{(2)})\}$ with $E_{\max}^{(\alpha)} < m$.

We will prove the statement of the lemma for the case (vi) with $E_{\min} := E_{\min}^{(1)} = E_{\min}^{(2)}$ and $E_{\max} := E_{\max}^{(1)} = E_{\max}^{(2)}$. Other cases can be proven in a similar.

Let $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$ and Assumptions 2.5, 2.6 be fulfilled. For $z \in (-\infty; m) \setminus \sigma_{\mathrm{ess}}(H)$ denote by $W(p, q, s, t; z)$ the kernel of the operator $W_{22}(z)$.

We have the following inequalities:

$$w_2(p, q) - z \geq m - E_{\min} > 0 \quad \text{for all} \quad p, q \in \mathbb{T}^3, \quad z \leq E_{\min};$$

$$w_2(p, q) - z \geq (m - E_{\max})/2 > 0 \quad \text{for all} \quad p, q \in \mathbb{T}^3, \quad z \in [E_{\max}; (m + E_{\max})/2].$$

Then by Assumptions 2.5, 2.6 and Lemma 3.3 the function $|W(\cdot, \cdot, \cdot, \cdot; z)|$ can be estimated by

$$C_1 \times \left(1 + \sum_{i=1}^{n_2} \frac{\chi_\delta(s - p_{2i})}{|s - p_{2i}|^{\beta_{2i}/2}}\right)$$

$$\times \left(1 + \sum_{i=1}^{n_1} \frac{\chi_\delta(p - p_{1i})}{|p - p_{1i}|^{\beta_{1i}/2}} + \sum_{i=1}^{n_1} \frac{\chi_\delta(q - p_{1i})}{|q - p_{1i}|^{\beta_{1i}/2}} + \sum_{i=1}^{n_2} \frac{\chi_\delta(p - p_{2i})}{|p - p_{2i}|^{\beta_{2i}/2}} + \sum_{i=1}^{n_2} \frac{\chi_\delta(q - p_{2i})}{|q - p_{2i}|^{\beta_{2i}/2}}\right)$$

for $z \leq E_{\min}$ and by

$$C_2 \times \left(1 + \sum_{j=1}^{k_2} \frac{\chi_\rho(s - q_{2j})}{|s - q_{2j}|^{\gamma_{2j}/2}}\right)$$

$$\times \left(1 + \sum_{j=1}^{k_1} \frac{\chi_\rho(p - q_{1j})}{|p - q_{1j}|^{\gamma_{1j}/2}} + \sum_{j=1}^{k_1} \frac{\chi_\rho(q - q_{1j})}{|q - q_{1j}|^{\gamma_{1j}/2}} + \sum_{j=1}^{k_2} \frac{\chi_\rho(p - q_{2j})}{|p - q_{2j}|^{\gamma_{2j}/2}} + \sum_{j=1}^{k_2} \frac{\chi_\rho(q - q_{2j})}{|q - q_{2j}|^{\gamma_{2j}/2}}\right)$$

for $z \in [E_{\max}, (m + E_{\max})/2]$, where $\chi_\delta(\cdot)$ is the characteristic function of $U_\delta(\overline{0})$.

Since $\xi_\alpha(z) = 1$ for any $z \in (E_{\max}; m)$ and $\max_{p \in \mathbb{T}^3} \Delta_\alpha(p; m) < 0$, for any $z \in (E_{\max}; m)$ we have $\max_{p \in \mathbb{T}^3}(\xi_\alpha(z)\Delta_\alpha(p; m)) > 0$. Therefore, Lemmas 3.2 and 3.4 imply that the function $|W(\cdot, \cdot, \cdot, \cdot; z)|$ can be estimated by

$$C_3 \left(1 + \frac{\chi_\delta(p - p_0)\chi_\delta(q - p_0)}{|p - p_0|^2 + |q - p_0|^2}\right) \left(1 + \frac{\chi_\delta(p - p_0)\chi_\delta(s - p_0)}{|p - p_0|^2 + |s - p_0|^2} + \frac{\chi_\delta(q - p_0)\chi_\delta(s - p_0)}{|q - p_0|^2 + |s - p_0|^2}\right)$$

for $z \in [(m + E_{\max})/2; m]$.

The latter three functions are square integrable on $(\mathbb{T}^3)^4$ and hence the operator $W_{22}(z)$ is Hilbert Schmidt for any $z \in (-\infty; E_{\min}] \cup [E_{\max}; m]$.

A similar argument shows that the operators $W_{11}(z)$, $W_{12}(z)$ and $W_{21}(z)$ are also Hilbert Schmidt for any $z \in \Sigma$.

For any $z \in (-\infty; m) \backslash \sigma_{\mathrm{ess}}(H)$ the kernel function of $W_{ij}(z)$, $i, j = 1, 2$ is continuous on its domain. Therefore the continuity of the operator-valued functions $W_{ij}(z)$, $i, j = 1, 2$ in the uniform operator topology for $z \in \Sigma$ follows from Lebesgue's dominated convergence theorem.

Since for all $z \in \Sigma$ the operators $W_{00}(z)$, $W_{01}(z)$, $W_{10}(z)$ and $W_{20}(z)$ are of rank 1 and continuous in the uniform operator topology for $z \in \Sigma$, one concludes that $W(z)$ is compact for $z \in \Sigma$ and the operator-valued function $W(z)$ is continuous in the uniform operator topology for $z \in \Sigma$. $\qquad \square$

5. Proof of the main results

In this section we prove Theorems 2.7 and 2.8.

Proof of Theorem 2.7. Let $\alpha = 1$. If $z_1 \in \mathbb{C} \backslash \sigma_{\mathrm{ess}}(H_1)$ is an eigenvalue of the operator H_1 and $f = (f_0, f_1, f_2) \in \mathcal{H}$ is the corresponding eigenvector, then f_0, f_1 and f_2 are satisfy the following system of equations:

$$(w_0 - z_1)f_0 + \int_{\mathbb{T}^3} v_0(s)f_1(s)\,ds = 0,$$

(5.1)
$$v_0(p)f_0 + (w_1(p) - z_1)f_1(p) + \int_{\mathbb{T}^3} v_1(s)f_2(p, s)\,ds = 0,$$

$$\frac{1}{2}(v_1(p)f_1(q) + v_1(q)f_1(p)) + (w_2(p, q) - z_1)f_2(p, q) = 0.$$

Since $z_1 \notin [m; M]$, from the third equation of the system (5.1) for f_2 we have

(5.2)
$$f_2(p, q) = -\frac{v_1(p)f_1(q) + v_1(q)f_1(p)}{2(w_2(p, q) - z_1)}.$$

Substituting the expression (5.2) for f_2 into the second equation of the system (5.1), we obtain

$$\Delta_1(p; z_1)f_1(p) = \frac{v_1(p)}{2} \int_{\mathbb{T}^3} \frac{v_1(s)f_1(s)\,ds}{w_2(p, s) - z_1} - v_0(p)f_0.$$

Since $z_1 \notin \sigma_{\mathrm{ess}}(H_1)$ the inequality $\Delta_1(p \, ; z_1) \neq 0$ holds for all $p \in \mathbb{T}^3$. From the last equation we have

$$f_1(p) = \frac{v_1(p)}{2\Delta_1(p \, ; z_1)} \int_{\mathbb{T}^3} \frac{v_1(s) f_1(s) \, ds}{w_2(p, s) - z_1} - \frac{v_0(p) f_0}{\Delta_1(p \, ; z_1)}.$$

The functions $v_0(\cdot)$, $v_1(\cdot)$, $w_1(\cdot)$ and $w_2(\cdot, q)$, $q \in \mathbb{T}^3$ are $2\bar{\pi}$ periodic and hence the function $f_1(\cdot)$ is also $2\bar{\pi}$ periodic. Therefore, for any fixed $p \in \mathbb{T}^3$ the function $f_2(p, \cdot)$ defined by (5.2), is $2\bar{\pi}$ periodic. Hence this function satisfies the condition (2.2), that is, $V f_2 = 0$. So the number $z_1 \in \sigma_{\mathrm{disc}}(H_1)$ is an eigenvalue of H with the same eigenvector $f = (f_0, f_1, f_2) \in \mathcal{H}$. Therefore, $\sigma_{\mathrm{disc}}(H_1) \subset \sigma_{\mathrm{p}}(H)$.

Let now $z_2 \in \sigma_{\mathrm{disc}}(H_2)$ and $g_2 \in \mathcal{H}_2$ be the eigenfunction corresponding to the discrete eigenvalue z_2. Then similar analysis shows that $H_{12} g_2 = 0$, which guarantee that the number $z_2 \in \sigma_{\mathrm{disc}}(H_2)$ is an eigenvalue of H and corresponding eigenvector g has form $g = (0, 0, g_2) \in \mathcal{H}$, that is, $\sigma_{\mathrm{disc}}(H_2) \subset \sigma_{\mathrm{p}}(H)$. Theorem 2.7 is proved. $\qquad \square$

Proof of Theorem 2.8. We prove the finiteness of the number of discrete eigenvalues located on the left of m for the case when $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p \, ; m) < 0$. Other cases can be proven similarly. Suppose that the operator H has an infinite number of discrete eigenvalues $(E_k)_{k \in \mathbb{N}} \subset (E_{\max}; m)$. Then three cases are possible
(i) $\lim_{k \to \infty} E_k = m$;
(ii) $\lim_{k \to \infty} E_k = E_{\max}$;
(iii) there exist $(E_k')_{k \in \mathbb{N}}, (E_k'')_{k \in \mathbb{N}} \subset (E_k)_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} E_k' = m$ and

$$\lim_{k \to \infty} E_k'' = E_{\max}.$$

Let us consider the case (iii). For each $k \in \mathbb{N}$ we denote by $\varphi_k \in \mathcal{H}$ and $\psi_k \in \mathcal{H}$ the orthonormal eigenvectors corresponding to the eigenvalues E_k' and E_k'', respectively. Then it follows from Lemma 4.1 that $\varphi_k = W(E_k')\varphi_k$ and $\psi_k = W(E_k'')\psi_k$ for any $k \in \mathbb{N}$. By virtue of Lemma 4.2 the operators $W(E_{\max}), W(m)$ are compact and $\|W(z) - W(E_{\max})\| \to 0$ and $\|W(z) - W(m)\| \to 0$ as $z \to E_{\max} + 0$ and $z \to m - 0$, respectively. Therefore,

$$1 = \|\varphi_k\| = \|W(E_k')\varphi_k\| \leq \|(W(E_k') - W(E_{\max}))\varphi_k\| + \|W(E_{\max})\varphi_k\| \to 0,$$
$$1 = \|\psi_k\| = \|W(E_k'')\psi_k\| \leq \|(W(E_k'') - W(m))\psi_k\| + \|W(m)\psi_k\| \to 0$$

as $k \to \infty$. This contradiction implies that the points $z = E_{\max}$ and $z = m$ can not be limit points of the set of discrete eigenvalues of H belonging to the interval $(E_{\max}; m)$. Similar arguments show that other edges of Σ are also cannot be accumulation point for the set of discrete eigenvalues of H smaller than m. $\qquad \square$

The following example shows that the class of functions $w_1(\cdot)$, $v_i(\cdot)$, $i = 1, 2$ and $w_2(\cdot, \cdot)$ satisfying the conditions of Theorem 2.8 is nonempty.

Lemma 5.1. *Let*

$$\widehat{v}_1(p) := \sum_{i=1}^{3} c_i \cos p^{(i)}, \quad \widehat{v}_2(p) := \sum_{i=1}^{3} d_i \cos(p^{(i)}/2),$$

$$v_\alpha(p) := \sqrt{2^{2-\alpha} \mu_\alpha} \, \widehat{v}_\alpha(p), \quad \alpha = 1, 2,$$

$$w_1(p) \equiv 1, \quad w_2(p, q) = \varepsilon(p) + \varepsilon(q), \quad \varepsilon(p) = \sum_{i=1}^{3} (1 - \cos p^{(i)}),$$

where $\mu_\alpha > 0$; c_i, d_i, $i = 1, 2, 3$ are arbitrary real numbers.

Set

$$\mu_\alpha^{(0)} := \left(\int_{\mathbb{T}^3} \frac{\widehat{v}_\alpha^2(s)\,ds}{\varepsilon(s)} \right)^{-1}, \quad \mu_\alpha^{(1)} := \left(\int_{\mathbb{T}^3} \frac{\widehat{v}_\alpha^2(s)\,ds}{6 + \varepsilon(s)} \right)^{-1}.$$

Then the functions $w_1(\cdot)$, $v_\alpha(\cdot)$, $\alpha = 1, 2$ and $w_2(\cdot, \cdot)$ are satisfy Assumptions 2.2, 2.3, 2.5, 2.6. Moreover,

(i) *if $0 < \mu_\alpha < \mu_\alpha^{(0)}$, then $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) > 0$;*

(ii) *if $\mu_\alpha^{(0)} < \mu_\alpha \leq \mu_\alpha^{(1)}$, then $\min\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$ and $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) \geq 0$;*

(iii) *if $\mu_\alpha > \mu_\alpha^{(1)}$, then $\max\limits_{p \in \mathbb{T}^3} \Delta_\alpha(p\,;m) < 0$.*

Proof. Let $g \in L_2(\mathbb{T}^3)$ be as in Assumption 2.2. Then we have

$$\int_{\mathbb{T}^3} v_2(s)g(s)\,ds = \int_{\mathbb{T}^3} v_2(s + 2\overline{\pi})g(s + 2\overline{\pi})\,ds = -\int_{\mathbb{T}^3} v_2(s)g(s)\,ds,$$

which yields the equality (2.2), that is, Assumption 2.2 holds with $\alpha = 1$ and $\beta = 2$.

From the definition of $w_2(\cdot, \cdot)$ it follows that this function has a unique zero non-degenerate minimum at $(\overline{0}, \overline{0}) \in (\mathbb{T}^3)^2$ and it satisfies all conditions of Assumption 2.3.

The assertions (i)–(iii) directly follow from the definition of the numbers $\mu_\alpha^{(0)}$ and $\mu_\alpha^{(1)}$.

Let $\mu_\alpha^{(0)} < \mu_\alpha \leq \mu_\alpha^{(1)}$. We prove that the function $\Delta_\alpha(\cdot; E_{\min}^{(\alpha)})$ has a unique non-degenerate minimum at $\overline{0} \in \mathbb{T}^3$. Simple calculations show that $\Delta_\alpha(p; E_{\min}^{(\alpha)}) > \Delta_\alpha(\overline{0}; E_{\min}^{(\alpha)})$ for all $p \neq \overline{0}$.

Since $E_{\min}^{(\alpha)} \in (-\infty, 0)$, it is clear that the function $\Delta_\alpha(\cdot\,; E_{\min}^{(\alpha)})$ is twice continuously differentiable in $\mathbf{T}^3$. Moreover, from the equalities

$$\frac{\partial^2 \Delta_\alpha(p\,; E_{\min}^{(\alpha)})}{\partial p^{(i)} \partial p^{(i)}} = \mu_\alpha \cos p^{(i)} \int_{\mathbb{T}^3} \frac{\widetilde{v}_\alpha^2(s)\,ds}{(\varepsilon(p) + \varepsilon(s) - E_{\min}^{(\alpha)})^2}$$

$$- 2\mu_\alpha (\sin p^{(i)})^2 \int_{\mathbb{T}^3} \frac{\widetilde{v}_\alpha^2(s)\,ds}{(\varepsilon(p) + \varepsilon(s) - E_{\min}^{(\alpha)})^3}, \quad i = 1, 2, 3,$$

$$\frac{\partial^2 \Delta_\alpha(p\,; E_{\min}^{(\alpha)})}{\partial p^{(i)} \partial p^{(j)}} = -2\mu_\alpha \sin p^{(i)} \sin p^{(j)} \int_{\mathbb{T}^3} \frac{\widetilde{v}_\alpha^2(s)\,ds}{(\varepsilon(p) + \varepsilon(s) - E_{\min}^{(\alpha)})^3}, \quad i \neq j, \quad i, j = 1, 2, 3$$

we get

$$\frac{\partial^2 \Delta_\alpha(\overline{0}\,; E_{\min}^{(\alpha)})}{\partial p^{(i)} \partial p^{(i)}} > 0, \quad \frac{\partial^2 \Delta_\alpha(\overline{0}\,; E_{\min}^{(\alpha)})}{\partial p^{(i)} \partial p^{(j)}} = 0, \quad i \neq j, \quad i, j = 1, 2, 3.$$

Using these facts, one may verify that the matrix of the second order partial derivatives of the function $\Delta_\alpha(\cdot; E_{\min}^{(\alpha)})$ at the point $p = \overline{0}$ is positive definite. Thus the function $\Delta_\alpha(\cdot; E_{\min}^{(\alpha)})$ has a non-degenerate minimum at the point $p = \overline{0}$. Then the equality $\Delta_\alpha(\overline{0}; E_{\min}^{(\alpha)}) = 0$ implies that there exist the numbers $\delta > 0$ and $C > 0$ such that

$$|\Delta_\alpha(p\,; E_{\min}^{(\alpha)})| \geq C|p|^2, \quad p \in U_\delta(\overline{0}),$$

that is, Assumption 2.5 holds with $n_\alpha = 1$, $p_{\alpha 1} = \overline{0}$ and $\beta_{\alpha 1} = 2$.

In the case $\mu_\alpha > \mu_\alpha^{(1)}$ one can similarly show that there exist $\rho > 0$ and $K > 0$ such that

$$|\Delta_\alpha(p\,; E_{\max}^{(\alpha)})| \geq K|p - \overline{\pi}|^2, \quad p \in U_\rho(\overline{\pi}),$$

that is, Assumption 2.6 holds with $k_\alpha = 1$, $q_{\alpha 1} = \overline{\pi}$ and $\gamma_{\alpha 1} = 2$. $\square$

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