Methods of Functional Analysis and Topology
Vol. 21 (2015), no. 4, pp. 336–360

OPERATORS OF STOCHASTIC DIFFERENTIATION ON SPACES OF NONREGULAR TEST FUNCTIONS OF LÉVY WHITE NOISE ANALYSIS

N. A. KACHANOVSKY

The paper is dedicated to one of my dear mentors Professor Yu. M. Berezansky on his ninetieth birthday

Abstract. The operators of stochastic differentiation, which are closely related with the extended Skorohod stochastic integral and with the Hida stochastic derivative, play an important role in the classical (Gaussian) white noise analysis. In particular, these operators can be used in order to study properties of the extended stochastic integral and of solutions of stochastic equations with Wick-type nonlinearities.

During recent years the operators of stochastic differentiation were introduced and studied, in particular, in the framework of the Meixner white noise analysis, and on spaces of regular test and generalized functions of the Lévy white noise analysis. In this paper we make the next step: introduce and study operators of stochastic differentiation on spaces of test functions that belong to the so-called nonregular rigging of the space of square integrable with respect to the measure of a Lévy white noise functions, using Lytvynov's generalization of the chaotic representation property. This can be considered as a contribution in a further development of the Lévy white noise analysis.

Introduction

Let $L = (L_t)_{t\in[0,+\infty)}$ be a Lévy process (i.e., a random process on $[0,+\infty)$ with stationary independent increments and such that $L_0 = 0$, see, e.g., [5, 28, 29] for details) without Gaussian part and drift (it is comparatively simple to consider such processes from technical point of view). In [23] the extended Skorohod stochastic integral with respect to L and the corresponding Hida stochastic derivative on the space of square integrable random variables (L^2) were constructed in terms of Lytvynov's generalization of the *chaotic representation property* (CRP) (see [25] and Subsection 1.2), some properties of these operators were established; and it was shown that the above-mentioned integral coincides with the well-known (constructed in terms of Itô's generalization of the CRP [14]) extended stochastic integral with respect to a Lévy process (e.g., [7, 6]). In [21, 10] the stochastic integral and derivative were extended to spaces of test and generalized functions that belong to riggings of (L^2), this gives a possibility to extend an area of their possible applications (in particular, now it is possible to define the stochastic integral and derivative as linear *continuous* operators). Together with the mentioned operators, it is natural to introduce and to study so-called *operators of stochastic differentiation* in the Lévy white noise analysis, by analogy with the Gaussian analysis [35, 1], the Gamma-analysis [17, 18], and the Meixner analysis [19, 20]. These operators are closely related with the extended Skorohod stochastic integral with respect to a Lévy process and with the corresponding Hida stochastic derivative and, by analogy with the

2010 *Mathematics Subject Classification.* 60H05, 60G51, 60H40, 46F05, 46F25.

Key words and phrases. Operator of stochastic differentiation, stochastic derivative, extended stochastic integral, Lévy process.

"classical case", can be used, in particular, in order to study properties of the extended stochastic integral and properties of solutions of normally ordered stochastic equations (stochastic equations with Wick-type nonlinearities in another terminology). In [9, 8] the operators of stochastic differentiation on spaces of a so-called *regular parametrized* rigging of (L^2) ([21]) were introduced and studied. But, in connection with some problems of the stochastic analysis (in particular, of the theory of normally ordered stochastic equations), sometimes it can be convenient to consider another, a so-called *nonregular* rigging of (L^2) (see [21] and Subsection 1.3) and operators (e.g., the extended stochastic integral, the Hida stochastic derivative) on spaces that belong to this rigging. Therefore it is natural to introduce and to study operators of stochastic differentiation on the above-mentioned spaces.

In this paper we introduce and study the operators of stochastic differentiation on the spaces of test functions that belong to the nonregular rigging of (L^2). In a forthcoming paper we'll consider the operators of stochastic differentiation on the spaces of nonregular generalized functions. Then we'll consider elements of the so-called Wick calculus in the Lévy white noise analysis, this will give us the possibility to continue the study of properties and to consider some applications of the above mentioned operators.

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a convenient for our considerations probability triplet connected with L; then, following [25, 23, 21], we describe in detail Lytvynov's generalization of the CRP, the nonregular rigging of (L^2), and stochastic integrals and derivatives on the spaces that belong to this rigging. In the second section we deal with the operators of stochastic differentiation.

1. Preliminaries

In this paper we denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the scalar product in a space H; and by $\langle\cdot,\cdot\rangle_H$ or $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ the dual pairing generated by the scalar product in a space H. Another notation for norms, scalar products and dual pairings will be introduced when it will be necessary.

1.1. **Lévy processes.** Denote $\mathbb{R}_+ := [0,+\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_t)_{t\in\mathbb{R}_+}$ (a random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift. As is well known (e.g., [7]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_t}] = \exp\Big[t\int_{\mathbb{R}}(e^{i\theta x} - 1 - i\theta x)\nu(dx)\Big], \tag{1.1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, here and below $\mathcal{B}$ denotes the Borel σ-algebra, $\mathbb{E}$ denotes the expectation. We assume that ν *is a Radon measure whose support contains an infinite number of points,* $\nu(\{0\}) = 0$*, there exists* $\varepsilon > 0$ *such that*

$$\int_{\mathbb{R}} x^2 e^{\varepsilon|x|}\nu(dx) < \infty$$

and

$$\int_{\mathbb{R}} x^2\nu(dx) = 1. \tag{1.2}$$

Let us define a measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [4]). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle\omega,\varphi\rangle$; note that one can understand $\langle\cdot,\cdot\rangle$ as the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on $\mathbb{R}_+$, see Subsection 1.3 for details. The notation $\langle\cdot,\cdot\rangle$ will be preserved for dual pairings in tensor powers of spaces.

Definition. A probability measure μ on $(\mathcal{D}',\mathcal{C}(\mathcal{D}'))$, where $\mathcal{C}$ denotes the cylindrical σ-algebra, with the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+\times\mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x)\, du\nu(dx)\Big], \quad \varphi\in\mathcal{D}, \tag{1.3}$$

is called the *measure of a Lévy white noise*.

The existence of μ follows from the Bochner–Minlos theorem (e.g., [13]), see [25]. Below we assume that the *σ-algebra $\mathcal{C}(\mathcal{D}')$ is complete with respect to* μ, i.e., $\mathcal{C}(\mathcal{D}')$ contains all subsets of all measurable sets O such that $\mu(O)=0$.

Denote $(L^2) := L^2(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$ the space of (classes of) real-valued square integrable with respect to μ functions on $\mathcal{D}'$; let also $\mathcal{H} := L^2(\mathbb{R}_+)$. Substituting in (1.3) $\varphi = t\psi$, $t\in\mathbb{R}$, $\psi\in\mathcal{D}$, and using the Taylor decomposition by t and (1.2), one can show that

$$\int_{\mathcal{D}'}\langle\omega,\psi\rangle^2\mu(d\omega) = \int_{\mathbb{R}_+}\big(\psi(u)\big)^2 du \tag{1.4}$$

(this statement follows also from results of [25] and [7]). Let $f\in\mathcal{H}$ and $\mathcal{D}\ni\varphi_k\to f$ in $\mathcal{H}$ as $k\to\infty$. It follows from (1.4) that $\{\langle\circ,\varphi_k\rangle\}_{k\geq 1}$ is a Cauchy sequence in (L^2), therefore one can define $\langle\circ,f\rangle := (L^2)-\lim_{k\to\infty}\langle\circ,\varphi_k\rangle$. It is easy to show (by the method of "mixed sequences") that $\langle\circ,f\rangle$ does not depend on the choice of an approximating sequence for f and therefore is well defined in (L^2).

Let us consider $\langle\circ,1_{[0,t)}\rangle\in(L^2)$, $t\in\mathbb{R}_+$ (here and below 1_A denotes the indicator of a set A). It follows from (1.1) and (1.3) that $\big(\langle\circ,1_{[0,t)}\rangle\big)_{t\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$, i.e., one can write $L_t = \langle\circ,1_{[0,t)}\rangle\in(L^2)$.

Remark. Note that one can understand the Lévy white noise as a generalized random process (in the sense of [11]) with trajectories from $\mathcal{D}'$: *formally* $L_t'(\omega) = \langle\omega,1_{[0,t)}\rangle' = \langle\omega,\delta_t\rangle = \omega(t)$, where δ_t is the Dirac delta-function concentrated at t. Therefore μ is the measure of L' in the classical sense of this notion [12].

Remark. A Lévy process L without Gaussian part and drift is a *Poisson process* if its Lévy measure $\nu(\Delta)=\delta_1(\Delta)$, i.e., if ν is a point mass at 1. This measure does not satisfy the conditions accepted above (the support of δ_1 does not contain an infinite number of points); nevertheless, all results of the present paper have natural (and often strong) analogs in the Poissonian analysis. The reader can find more information about peculiarities of the Poissonian case in [23], Subsection 1.2.

1.2. **Lytvynov's generalization of the CRP.** As is well known, some random processes L have the so-called *chaotic representation property* (CRP) that consists in the following: any square integrable random variable can be decomposed in a series of repeated stochastic integrals from nonrandom functions with respect to L (see, e.g., [26] for a detailed presentation). The CRP plays a very important role in the stochastic analysis (in particular, it can be used in order to construct extended stochastic integrals [16, 32, 15], stochastic derivatives and operators of stochastic differentiation, e.g., [35, 1]), but, unfortunately, the only Lévy processes that satisfy this property are Wiener and Poisson processes (e.g., [34]).

There are different generalizations of the CRP for Lévy processes: Itô's approach [14], Nualart-Schoutens' approach [27, 30], Lytvynov's approach [25], Oksendal's approach [7, 6] etc. The interconnections between these generalizations of the CRP are described in, e.g., [25, 2, 7, 33, 6, 23]. In the present paper we deal with Lytvynov's generalization of the CRP that will be described now in detail.

Denote by $\widehat{\otimes}$ a symmetric tensor product and set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Let $\mathcal{P} \equiv \mathcal{P}(\mathcal{D}')$ be the set of continuous polynomials on $\mathcal{D}'$, i.e., $\mathcal{P}$ consists of zero and elements of the form

$$f(\omega) = \sum_{n=0}^{N_f} \langle \omega^{\otimes n}, f^{(n)} \rangle, \quad \omega \in \mathcal{D}', \quad N_f \in \mathbb{Z}_+, \quad f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}, \quad f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle \omega^{\otimes 0}, f^{(0)} \rangle := f^{(0)} \in \mathcal{D}^{\widehat{\otimes} 0} := \mathbb{R}$. Since the measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (1.3) and properties of the measure ν, see also [25]), $\mathcal{P}$ is a dense set in (L^2) [31]. Denote by $\mathcal{P}_n$ the set of continuous polynomials of power $\leq n$, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear now that

$$(L^2) = \mathop{\oplus}_{n=0}^{\infty} \mathbf{P}_n.$$

Let $f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\!\langle \circ^{\otimes n}, f^{(n)} \rangle\!:$ the orthogonal projection in (L^2) of a monomial $\langle \circ^{\otimes n}, f^{(n)} \rangle$ onto $\mathbf{P}_n$. Let us define scalar products $(\cdot,\cdot)_{\mathrm{ext}}$ on $\mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$

$$(f^{(n)}, g^{(n)})_{\mathrm{ext}} := \frac{1}{n!} \int_{\mathcal{D}'} :\!\langle \omega^{\otimes n}, f^{(n)} \rangle\!::\!\langle \omega^{\otimes n}, g^{(n)} \rangle\!: \mu(d\omega),$$

and let $|\cdot|_{\mathrm{ext}}$ be the corresponding norms, i.e., $|f^{(n)}|_{\mathrm{ext}} = \sqrt{(f^{(n)}, f^{(n)})_{\mathrm{ext}}}$. Denote by $\mathcal{H}_{\mathrm{ext}}^{(n)}$, $n \in \mathbb{Z}_+$, the completions of $\mathcal{D}^{\widehat{\otimes} n}$ with respect to the norms $|\cdot|_{\mathrm{ext}}$. For $F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$ define a Wick monomial $:\!\langle \circ^{\otimes n}, F^{(n)} \rangle\!: \stackrel{\mathrm{def}}{=} (L^2) - \lim_{k\to\infty} :\!\langle \circ^{\otimes n}, f_k^{(n)} \rangle\!:$, where $\mathcal{D}^{\widehat{\otimes} n} \ni f_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{\mathrm{ext}}^{(n)}$ (well-posedness of this definition can be proved by the method of "mixed sequences"). Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\!\langle \circ^{\otimes n}, f^{(n)} \rangle\!: | f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}\}$ is a dense one in $\mathbf{P}_n$, we have the next statement (which describes Lytvynov's generalization of the CRP).

Theorem. ([25]). *A random variable $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} :\!\langle \circ^{\otimes n}, F^{(n)} \rangle\!: \tag{1.5}$$

(the series converges in (L^2)) and

$$\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{\mathrm{ext}}^2 < \infty. \tag{1.6}$$

So, for $F, G \in (L^2)$ the scalar product has the form

$$(F, G)_{(L^2)} = \int_{\mathcal{D}'} F(\omega) G(\omega) \mu(d\omega) = \mathbb{E}[FG] = \sum_{n=0}^{\infty} n! (F^{(n)}, G^{(n)})_{\mathrm{ext}},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$ are the kernels from decompositions (1.5) for F and G respectively. In particular, for $F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$ and $G^{(m)} \in \mathcal{H}_{\mathrm{ext}}^{(m)}$, $n, m \in \mathbb{Z}_+$,

$$\begin{aligned} \big(:\!\langle \circ^{\otimes n}, F^{(n)} \rangle\!:, :\!\langle \circ^{\otimes m}, G^{(m)} \rangle\!:\big)_{(L^2)} &= \int_{\mathcal{D}'} :\!\langle \omega^{\otimes n}, F^{(n)} \rangle\!::\!\langle \omega^{\otimes m}, G^{(m)} \rangle\!: \mu(d\omega) \\ &= \mathbb{E}\big[:\!\langle \circ^{\otimes n}, F^{(n)} \rangle\!::\!\langle \circ^{\otimes m}, G^{(m)} \rangle\!:\big] = \delta_{n,m} n! (F^{(n)}, G^{(n)})_{\mathrm{ext}}. \end{aligned}$$

Note that in the space (L^2) we have $:\!\langle \circ^{\otimes 0}, F^{(0)} \rangle\!: = \langle \circ^{\otimes 0}, F^{(0)} \rangle = F^{(0)}$ and $:\!\langle \circ, F^{(1)} \rangle\!: = \langle \circ, F^{(1)} \rangle$ [25].

In order to work in the spaces $\mathcal{H}_{\text{ext}}^{(n)}$, we need to know the explicit formulas for the scalar products $(\cdot,\cdot)_{\text{ext}}$. Let us write out these formulas. Denote by $\|\cdot\|_\nu$ the norm in the space $L^2(\mathbb{R},\nu)$ of (classes of) square integrable with respect to ν real-valued functions on $\mathbb{R}$. Let

(1.7) $p_n(x) := x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x, \quad a_{n,j} \in \mathbb{R}, \quad j \in \{1,\dots,n-1\}, \quad n \in \mathbb{N},$

be orthogonal in $L^2(\mathbb{R},\nu)$ polynomials, i.e., for natural numbers n,m such that $n \neq m$, $\int_{\mathbb{R}} p_n(x)p_m(x)\nu(dx) = 0$. Then for $F^{(n)}, G^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{N}$, we have [25]

(1.8)
$$(F^{(n)},G^{(n)})_{\text{ext}} = \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\cdots>l_k,\\ l_1s_1+\cdots+l_ks_k=n}} \frac{n!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k}$$
$$\times \int_{\mathbb{R}_+^{s_1+\cdots+s_k}} F^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\cdots+s_k},\dots,u_{s_1+\cdots+s_k}}_{l_k})$$
$$\times G^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\cdots+s_k},\dots,u_{s_1+\cdots+s_k}}_{l_k})\,du_1\cdots du_{s_1+\cdots+s_k}.$$

In particular, for $n=1$ $(F^{(1)},G^{(1)})_{\text{ext}} = \|p_1\|_\nu^2\int_{\mathbb{R}_+}F^{(1)}(u)G^{(1)}(u)\,du$; if $n=2$ then $(F^{(2)},G^{(2)})_{\text{ext}} = \|p_1\|_\nu^4\int_{\mathbb{R}_+^2}F^{(2)}(u,v)G^{(2)}(u,v)\,dudv + \frac{\|p_2\|_\nu^2}{2}\int_{\mathbb{R}_+}F^{(2)}(u,u)G^{(2)}(u,u)\,du$, etc.

It follows from (1.8) that $\mathcal{H}_{\text{ext}}^{(1)} = \mathcal{H} \equiv L^2(\mathbb{R}_+)$: by (1.7) $p_1(x) = x$ and therefore by (1.2) $\|p_1\|_\nu = 1$; and for $n \in \mathbb{N}\backslash\{1\}$ one can identify $\mathcal{H}^{\widehat{\otimes}n}$ with the proper subspace of $\mathcal{H}_{\text{ext}}^{(n)}$ that consists of "vanishing on diagonals" elements (i.e., $F^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k,j \in \{1,\dots,n\}$ such that $k \neq j$ but $u_k = u_j$). In this sense the *space* $\mathcal{H}_{\text{ext}}^{(n)}$ *is an extension of* $\mathcal{H}^{\widehat{\otimes}n}$ (this explains why we use the subscript ext in the notations $\mathcal{H}_{\text{ext}}^{(n)}$, $(\cdot,\cdot)_{\text{ext}}$ and $|\cdot|_{\text{ext}}$).

1.3. **A nonregular rigging of** (L^2)**.** Denote by T the set of indexes $\tau = (\tau_1,\tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Let $\mathcal{H}_\tau$ be the Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., $\mathcal{H}_\tau$ is a completion of the set $C_0^\infty(\mathbb{R}_+)$ of infinite differentiable functions on $\mathbb{R}_+$ with compact supports with respect to the norm generated by the scalar product

$$(\varphi,\psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+}\Big(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1}\varphi^{[k]}(u)\psi^{[k]}(u)\Big)\tau_2(u)\,du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ respectively. It is well known (e.g., [4]) that $\mathcal{D} = \text{pr}\lim_{\tau\in T}\mathcal{H}_\tau$ (moreover, $\mathcal{D}^{\widehat{\otimes}n} = \text{pr}\lim_{\tau\in T}\mathcal{H}_\tau^{\widehat{\otimes}n}$, see, e.g., [3] for details) and for each $\tau \in T$ $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$, therefore one can consider the chain

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D},$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Note that $\mathcal{D}' = \text{ind}\lim_{\tau\in T}\mathcal{H}_{-\tau}$ (it is convenient for us to consider $\mathcal{D}'$ as a topological space). By analogy with [22] one can easily show that the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$ with some $\widetilde{\tau} \in T$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$. Excepting from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$, we will assume, in what follows, that *for each* $\tau \in T$ $\mu(\mathcal{H}_{-\tau}) = 1$.

Denote the norms in $\mathcal{H}_\tau$ and its tensor powers by $|\cdot|_\tau$, i.e., for $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$, $n \in \mathbb{N}$, $|f^{(n)}|_\tau = \sqrt{(f^{(n)},f^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes}n}}}$.

Lemma. ([21]). *There exists $\tau' \in T$ such that for each $n \in \mathbb{N}$ the space $\mathcal{H}_{\tau'}^{\widehat{\otimes}n}$ is densely and continuously embedded into the space $\mathcal{H}_{\mathrm{ext}}^{(n)}$. Moreover, for all $f^{(n)} \in \mathcal{H}_{\tau'}^{\widehat{\otimes}n}$*

$$|f^{(n)}|_{\mathrm{ext}}^2 \le n!c^n |f^{(n)}|_{\tau'}^2,$$

where $c > 0$ is some constant.

Corollary. *If for some $\tau \in T$ the space $\mathcal{H}_\tau$ is continuously embedded into the space $\mathcal{H}_{\tau'}$ then for each $n \in \mathbb{N}$ the space $\mathcal{H}_\tau^{\widehat{\otimes}n}$ is densely and continuously embedded into the space $\mathcal{H}_{\mathrm{ext}}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$*

$$|f^{(n)}|_{\mathrm{ext}}^2 \le n!c(\tau)^n |f^{(n)}|_\tau^2. \tag{1.9}$$

In what follows, it will be convenient to assume that the *indexes τ such that $\mathcal{H}_\tau$ is not continuously embedded into $\mathcal{H}_{\tau'}$, are removed from T.*

Denote $\mathcal{P}_W := \left\{ f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, f^{(n)} \in \mathcal{D}^{\widehat{\otimes}n}, N_f \in \mathbb{Z}_+ \right\} \subset (L^2)$. Accept on default $q \in \mathbb{Z}_+$, $\tau \in T$; set $\mathcal{H}_\tau^{\widehat{\otimes}0} := \mathbb{R}$; and define scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \quad g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle: \in \mathcal{P}_W,$$

$$(f,g)_{\tau,q} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes}n}}. \tag{1.10}$$

Let $\|\cdot\|_{\tau,q}$ be the corresponding norms, i.e., $\|f\|_{\tau,q} = \sqrt{(f,f)_{\tau,q}}$. In order to verify the well-posedness of this definition, i.e., that formula (1.10) defines *scalar*, and not just quasiscalar products, we note that if for $f \in \mathcal{P}_W$ $\|f\|_{\tau,q} = 0$ then by (1.10) for each coefficient $f^{(n)}$ of f $|f^{(n)}|_\tau = 0$ and therefore by (1.9) $|f^{(n)}|_{\mathrm{ext}} = 0$. So, in this case $f = 0$ in (L^2).

Definition. We define Kondratiev spaces of nonregular test functions $(\mathcal{H}_\tau)_q$ as completions of $\mathcal{P}_W$ with respect to the norms $\|\cdot\|_{\tau,q}$ and set $(\mathcal{H}_\tau) := \mathrm{pr}\lim_{q\in\mathbb{Z}_+}(\mathcal{H}_\tau)_q$, $(\mathcal{D}) := \mathrm{pr}\lim_{q\in\mathbb{Z}_+,\tau\in T}(\mathcal{H}_\tau)_q$.

As is easy to see, $f \in (\mathcal{H}_\tau)_q$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \quad f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n} \tag{1.11}$$

(the series converges in $(\mathcal{H}_\tau)_q$), with

$$\|f\|_{\tau,q}^2 := \|f\|_{(\mathcal{H}_\tau)_q}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_\tau^2 < \infty, \tag{1.12}$$

and for $f, g \in (\mathcal{H}_\tau)_q$

$$(f,g)_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes}n}},$$

where $f^{(n)}, g^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$ are the kernels from decompositions (1.11) for f and g correspondingly (since for each $n \in \mathbb{Z}_+$ $\mathcal{H}_\tau^{\widehat{\otimes}n} \subseteq \mathcal{H}_{\mathrm{ext}}^{(n)}$, for $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$ $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ is a well-defined Wick monomial, see Subsection 1.2). Further, $f \in (\mathcal{H}_\tau)$ ($f \in (\mathcal{D})$) if and only if f can be presented in form (1.11) and norm (1.12) is finite for each $q \in \mathbb{Z}_+$ (for each $q \in \mathbb{Z}_+$ and each $\tau \in T$).

Remark. One can give a more general definition of the Kondratiev spaces, writing in (1.10) K^{qn}, $K > 1$, instead of 2^{qn}. But such a generalization is not essential for our considerations, so, for simplification of calculations we shall restrict ourselves, in this paper, to the case $K = 2$.

Proposition. ([21]). *For each $\tau \in T$ there exists $q_0 = q_0(\tau) \in \mathbb{Z}_+$ such that for each $q \in \mathbb{N}_{q_0} := \{q_0, q_0 + 1, \cdots\}$ the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into (L^2).*

In view of this proposition for $\tau \in T$ and $q \geq q_0(\tau)$ one can consider a chain

$$(\mathcal{D}') \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}), \tag{1.13}$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \text{ind} \lim_{q\to\infty}(\mathcal{H}_{-\tau})_{-q}$ and $(\mathcal{D}') = \text{ind} \lim_{q\to\infty,\tau\in T}(\mathcal{H}_{-\tau})_{-q}$ are the spaces dual of $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ with respect to (L^2).

Definition. Chain (1.13) is called a nonregular rigging of the space (L^2). The negative spaces of this chain $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$, $(\mathcal{D}')$ are called Kondratiev spaces of nonregular generalized functions.

Remark. Let $q \in \mathbb{Z}_+$, $\tau \in T$ and $\beta \in [0, 1]$. By analogy with the classical Gaussian and Poissonian analysis, one can introduce on $\mathcal{P}_W$ scalar products $(\cdot, \cdot)_{\tau,q,\beta}$ by setting for $f, g \in \mathcal{P}_W$

$$(f, g)_{\tau,q,\beta} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^{1+\beta} 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes} n}},$$

and define "parametrized Kondratiev spaces of nonregular test functions" $(\mathcal{H}_\tau)_q^\beta$ as completions of $\mathcal{P}_W$ with respect to the norms generated by these scalar products. It is possible to study properties of the spaces $(\mathcal{H}_\tau)_q^\beta$ and its projective limits, to introduce and to study operators on them, in particular, stochastic derivatives, operators of stochastic differentiation, etc.; such considerations are interesting by itself and can be useful for applications. But $(\mathcal{H}_\tau)_q^\beta \not\subset (L^2)$ if $\beta < 1$, generally speaking, so, we can not consider $(\mathcal{H}_\tau)_q^\beta$ with $\beta < 1$ as spaces of test functions in the framework of the Lévy white noise analysis.

Finally, we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider chains

$$\mathcal{D}'^{(m)} \supset \mathcal{H}_{-\tau}^{(m)} \supset \mathcal{H}_{\text{ext}}^{(m)} \supset \mathcal{H}_\tau^{\widehat{\otimes} m} \supset \mathcal{D}^{\widehat{\otimes} m}, \tag{1.14}$$

$m \in \mathbb{Z}_+$ (for $m = 0$ $\mathcal{D}^{\widehat{\otimes} 0} = \mathcal{H}_\tau^{\widehat{\otimes} 0} = \mathcal{H}_{\text{ext}}^{(0)} = \mathcal{H}_{-\tau}^{(0)} = \mathcal{D}'^{(0)} = \mathbb{R}$), where $\mathcal{H}_{-\tau}^{(m)}$ and $\mathcal{D}'^{(m)} = \text{ind} \lim_{\tau\in T} \mathcal{H}_{-\tau}^{(m)}$ are the spaces dual of $\mathcal{H}_\tau^{\widehat{\otimes} m}$ and $\mathcal{D}^{\widehat{\otimes} m}$ with respect to $\mathcal{H}_{\text{ext}}^{(m)}$. The next statement follows from the definition of the spaces $(\mathcal{H}_{-\tau})_{-q}$ and the general duality theory (cf. [22]).

Proposition. ([21]). *There exists a system of generalized functions*

$$\left\{ :\langle \circ^{\otimes m}, F_{\text{ext}}^{(m)} \rangle: \in (\mathcal{H}_{-\tau})_{-q} \mid F_{\text{ext}}^{(m)} \in \mathcal{H}_{-\tau}^{(m)},\ m \in \mathbb{Z}_+ \right\}$$

such that

1) for $F_{\text{ext}}^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)} \subset \mathcal{H}_{-\tau}^{(m)}$ $:\langle \circ^{\otimes m}, F_{\text{ext}}^{(m)} \rangle:$ is a Wick monomial that was defined in Subsection 1.2;

2) any generalized function $F \in (\mathcal{H}_{-\tau})_{-q}$ can be presented as a formal series

$$F = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F_{\text{ext}}^{(m)} \rangle:, \quad F_{\text{ext}}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}, \tag{1.15}$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$*, i.e.,*

$$\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm} |F^{(m)}_{\text{ext}}|^2_{\mathcal{H}^{(m)}_{-\tau}} < \infty, \tag{1.16}$$

and, vice versa, any formal series (1.15) *with finite norm* (1.16) *is a generalized function from* $(\mathcal{H}_{-\tau})_{-q}$*;*

3) for $F, G \in (\mathcal{H}_{-\tau})_{-q}$ *the scalar product has a form*

$$(F, G)_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm} (F^{(m)}_{\text{ext}}, G^{(m)}_{\text{ext}})_{\mathcal{H}^{(m)}_{-\tau}},$$

where $F^{(m)}_{\text{ext}}, G^{(m)}_{\text{ext}} \in \mathcal{H}^{(m)}_{-\tau}$ *are the kernels from decompositions* (1.15) *for* F *and* G *respectively;*

4) the dual pairing between $F \in (\mathcal{H}_{-\tau})_{-q}$ *and* $f \in (\mathcal{H}_{\tau})_q$ *that is generated by the scalar product in* (L^2)*, has the form*

$$\langle\!\langle F, f\rangle\!\rangle_{(L^2)} = \sum_{m=0}^{\infty} m! \langle F^{(m)}_{\text{ext}}, f^{(m)}\rangle_{\text{ext}}, \tag{1.17}$$

where $F^{(m)}_{\text{ext}} \in \mathcal{H}^{(m)}_{-\tau}$ *and* $f^{(m)} \in \mathcal{H}^{\widehat{\otimes} m}_{\tau}$ *are the kernels from decompositions* (1.15) *and* (1.11) *for* F *and* f *respectively,* $\langle \cdot, \cdot \rangle_{\text{ext}}$ *denotes the dual pairings between elements of negative and positive spaces from chains* (1.14)*, these pairings are generated by the scalar products in* $\mathcal{H}^{(m)}_{\text{ext}}$*.*

Corollary. $F \in (\mathcal{H}_{-\tau})$ $(F \in (\mathcal{D}'))$ *if and only if* F *can be presented in the form* (1.15) *and norm* (1.16) *is finite for some* $q \in \mathbb{N}_{q_0(\tau)}$ *(for some* $\tau \in T$ *and some* $q \in \mathbb{N}_{q_0(\tau)}$*).*

1.4. **Stochastic integrals and derivatives.** Decomposition (1.5) defines an isometric isomorphism (a generalized Wiener-Itô-Sigal isomorphism) $\mathbf{I} : (L^2) \to \bigoplus_{n=0}^{\infty} n!\mathcal{H}^{(n)}_{\text{ext}}$, where $\bigoplus_{n=0}^{\infty} n!\mathcal{H}^{(n)}_{\text{ext}}$ is a weighted extended Fock space (cf. [24]): for $F \in (L^2)$ of form (1.5) $\mathbf{I}F = (F^{(0)}, F^{(1)}, \dots, F^{(n)}, \dots) \in \bigoplus_{n=0}^{\infty} n!\mathcal{H}^{(n)}_{\text{ext}}$. Let $\mathbf{1} : \mathcal{H} \to \mathcal{H}$ be the identity operator. Then the operator $\mathbf{I} \otimes \mathbf{1} : (L^2) \otimes \mathcal{H} \to \bigoplus_{n=0}^{\infty} n!(\mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H})$ is an isometric isomorphism between the spaces $(L^2) \otimes \mathcal{H}$ and $\bigoplus_{n=0}^{\infty} n!(\mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H})$. It is clear that for arbitrary $n \in \mathbb{Z}_+$ and $F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H}$ a vector $(\underbrace{0, \dots, 0}_{n}, F^{(n)}_{\cdot}, 0, \dots)$ belongs to $\bigoplus_{n=0}^{\infty} n!(\mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H})$. Set

$$:\langle \circ^{\otimes n}, F^{(n)}_{\cdot}\rangle: \stackrel{def}{=} (\mathbf{I} \otimes \mathbf{1})^{-1}(\underbrace{0, \dots, 0}_{n}, F^{(n)}_{\cdot}, 0, \dots) \in (L^2) \otimes \mathcal{H}. \tag{1.18}$$

By the construction elements $:\langle \circ^{\otimes n}, F^{(n)}_{\cdot}\rangle:$, $n \in \mathbb{Z}_+$, form an orthogonal basis in the space $(L^2) \otimes \mathcal{H}$: any $F \in (L^2) \otimes \mathcal{H}$ can be presented as

$$F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)}_{\cdot}\rangle:, \quad F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H} \tag{1.19}$$

(the series converges in $(L^2) \otimes \mathcal{H}$), with $\|F\|^2_{(L^2)\otimes\mathcal{H}} = \sum_{n=0}^{\infty} n! |F^{(n)}_{\cdot}|^2_{\mathcal{H}^{(n)}_{\text{ext}}\otimes\mathcal{H}} < \infty$.

In order to help the reader to gain a better insight of our constructions, we'll describe briefly the structure of an extended stochastic integral on $(L^2) \otimes \mathcal{H}$ that is based on this decomposition, and of the corresponding Hida stochastic derivative on (L^2) (the detailed presentation is given in [23]).

Let $F_\cdot^{(n)} \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{N}$. We select a representative (a function) $\dot{f}_\cdot^{(n)} \in F_\cdot^{(n)}$ such that

$$\dot{f}_u^{(n)}(u_1, \dots, u_n) = 0 \quad \text{if for some} \quad \mathrm{k} \in \{1, \dots, \mathrm{n}\} \quad u = u_k. \tag{1.20}$$

Let $\widehat{f}^{(n)}$ be the symmetrization of $\dot{f}_\cdot^{(n)}$ by $n+1$ variables. Define $\widehat{F}^{(n)} \in \mathcal{H}_{\rm ext}^{(n+1)}$ as the equivalence class in $\mathcal{H}_{\rm ext}^{(n+1)}$ generated by $\widehat{f}^{(n)}$ (i.e., $\widehat{f}^{(n)} \in \widehat{F}^{(n)}$).

Lemma. ([23]). *For each $F_\cdot^{(n)} \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{N}$, the element $\widehat{F}^{(n)} \in \mathcal{H}_{\rm ext}^{(n+1)}$ is well-defined (in particular, $\widehat{F}^{(n)}$ does not depend on the choice of a representative $\dot{f}_\cdot^{(n)} \in F_\cdot^{(n)}$ satisfying* (1.20)*) and $|\widehat{F}^{(n)}|_{\rm ext} \le |F_\cdot^{(n)}|_{\mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}}$.*

Definition. For $F \in (L^2) \otimes \mathcal{H}$ we define an extended stochastic integral $\int F(u)\, \widehat{d}L_u \in (L^2)$ by setting

$$\int F(u)\, \widehat{d}L_u := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \widehat{F}^{(n)} \rangle:\,, \tag{1.21}$$

where $\widehat{F}^{(0)} := F_\cdot^{(0)} \in \mathcal{H} = \mathcal{H}_{\rm ext}^{(1)}$, and $\widehat{F}^{(n)} \in \mathcal{H}_{\rm ext}^{(n+1)}$, $n \in \mathbb{N}$, are constructed by the kernels $F_\cdot^{(n)} \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}$ from decomposition (1.19) for F, if the series in the right hand side of (1.21) converges in (L^2).

The domain of this integral, i.e., of the operator

$$\int \circ(u)\, \widehat{d}L_u : (L^2) \otimes \mathcal{H} \to (L^2), \tag{1.22}$$

consists of $F \in (L^2) \otimes \mathcal{H}$ such that (see (1.6))

$$\Big\| \int F(u)\, \widehat{d}L_u \Big\|_{(L^2)}^2 = \sum_{n=0}^{\infty} (n+1)! |\widehat{F}^{(n)}|_{\rm ext}^2 < \infty.$$

It is shown in [23] that extended stochastic integral (1.22) is a natural generalization of the Itô stochastic integral.

Remark. Let $\Delta \subseteq \mathbb{R}_+$ be a measurable set, i.e., $\Delta \in \mathcal{B}(\mathbb{R}_+)$ (for example, $\Delta = [t_1, t_2)$, $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$). One can define an extended stochastic integral on Δ

$$\int_\Delta \circ(u)\, \widehat{d}L_u : (L^2) \otimes \mathcal{H} \to (L^2)$$

by the formula

$$\int_\Delta F(u)\, \widehat{d}L_u := \int F(u) 1_\Delta(u)\, \widehat{d}L_u$$

with the corresponding domain. Integrals $\int_\Delta \circ(u)\, \widehat{d}L_u$ are useful for applications, but in the framework of the present paper we do not need such a generalization. The reader can find more information about such integrals in, e.g., [23, 21].

Describe now the structure of the Hida stochastic derivative on (L^2). Let $G^{(n)} \in \mathcal{H}_{\rm ext}^{(n)}$, $n \in \mathbb{N}$, $\dot{g}^{(n)} \in G^{(n)}$ be a representative of $G^{(n)}$. We consider $\dot{g}^{(n)}(\cdot)$, i.e., separate one argument of $\dot{g}^{(n)}$, and define $G^{(n)}(\cdot) \in \mathcal{H}_{\rm ext}^{(n-1)} \otimes \mathcal{H}$ as the equivalence class in $\mathcal{H}_{\rm ext}^{(n-1)} \otimes \mathcal{H}$ generated by $\dot{g}^{(n)}(\cdot)$ (i.e., $\dot{g}^{(n)}(\cdot) \in G^{(n)}(\cdot)$).

Lemma. ([23]). *For each $G^{(n)} \in \mathcal{H}_{\rm ext}^{(n)}$, $n \in \mathbb{N}$, the element $G^{(n)}(\cdot) \in \mathcal{H}_{\rm ext}^{(n-1)} \otimes \mathcal{H}$ is well-defined (in particular, $G^{(n)}(\cdot)$ does not depend on the choice of a representative $\dot{g}^{(n)} \in G^{(n)}$) and*

$$|G^{(n)}(\cdot)|_{\mathcal{H}_{\rm ext}^{(n-1)} \otimes \mathcal{H}} \le |G^{(n)}|_{\rm ext}. \tag{1.23}$$

Note that, in spite of estimate (1.23), the space $\mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{N}\backslash\{1\}$, is not a subspace of $\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$ because different elements of $\mathcal{H}_{\text{ext}}^{(n)}$ can coincide as elements of $\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$.

Definition. For $G \in (L^2)$ we define a Hida stochastic derivative $\partial.G \in (L^2) \otimes \mathcal{H}$ by setting

$$\partial.G := \sum_{n=0}^{\infty}(n+1){:}\langle \circ^{\otimes n}, G^{(n+1)}(\cdot)\rangle{:}\,, \tag{1.24}$$

where $G^{(n+1)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{Z}_+$, are constructed as described above by the kernels $G^{(n+1)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$ from decomposition (1.5) for G, if the series in the right hand side of (1.24) converges in $(L^2) \otimes \mathcal{H}$.

The domain of this derivative, i.e., of the operator

$$\partial. : (L^2) \to (L^2) \otimes \mathcal{H}, \tag{1.25}$$

consists of $G \in (L^2)$ such that $\|\partial.G\|^2_{(L^2)\otimes\mathcal{H}} = \sum_{n=0}^{\infty}(n+1)!(n+1)|G^{(n+1)}(\cdot)|^2_{\mathcal{H}_{\text{ext}}^{(n)}\otimes\mathcal{H}} < \infty$.

The interconnection between the extended stochastic integral and the Hida stochastic derivative is given by the next statement.

Theorem. ([23]). *Extended stochastic integral* (1.22) *and Hida stochastic derivative* (1.25) *are mutually adjoint operators:*

$$\int \circ(u)\,\widehat{d}L_u = (\partial.)^*\circ, \quad \partial. = \Big(\int \circ\,\widehat{d}L\Big)^*_{\cdot}. \tag{1.26}$$

In particular, integral (1.22) *and derivative* (1.25) *are closed operators.*

Note that equalities (1.26) can be used as alternative definitions of the extended stochastic integral and of the Hida stochastic derivative on the space of square integrable random variables.

Let us consider the Hida stochastic derivative on the spaces of test functions. Since, as is easily seen, the restriction of a generalized Wiener-Itô-Sigal isomorphism $\mathbf{I}$ to the space $(\mathcal{H}_\tau)_q$ is an isometric isomorphism between $(\mathcal{H}_\tau)_q$ and a weighted Fock space $\mathop{\oplus}\limits_{n=0}^{\infty}(n!)^2 2^{qn}\mathcal{H}_\tau^{\widehat{\otimes}n}$ (cf. [24]), and, of course, the restriction of the identity operator on $\mathcal{H}$ to the space $\mathcal{H}_\tau$ is the identity operator on $\mathcal{H}_\tau$, for arbitrary $n \in \mathbb{Z}_+$ and $f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n} \otimes \mathcal{H}_\tau \subset \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$ we have ${:}\langle \circ^{\otimes n}, f_\cdot^{(n)}\rangle{:} \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$. Moreover, elements ${:}\langle \circ^{\otimes n}, f_\cdot^{(n)}\rangle{:}$, $f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n} \otimes \mathcal{H}_\tau$, $n \in \mathbb{Z}_+$, form an orthogonal basis in the spaces $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$.

Definition. For $g \in (\mathcal{H}_\tau)_q$ we define a Hida stochastic derivative $\partial.g \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ by the formula

$$\partial.g := \sum_{n=0}^{\infty}(n+1){:}\langle \circ^{\otimes n}, g^{(n+1)}(\cdot)\rangle{:} \tag{1.27}$$

(cf. (1.24)), where $g^{(n+1)} \in \mathcal{H}_\tau^{\widehat{\otimes}n+1}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (1.11) for g considered as elements of $\mathcal{H}_\tau^{\widehat{\otimes}n} \otimes \mathcal{H}_\tau$.

Since (see (1.12))

$$\begin{aligned}\|\partial.g\|^2_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau} &= \sum_{n=0}^{\infty}((n+1)!)^2 2^{qn}|g^{(n+1)}(\cdot)|^2_{\mathcal{H}_\tau^{\widehat{\otimes}n}\otimes\mathcal{H}_\tau}\\ &= 2^{-q}\sum_{n=0}^{\infty}((n+1)!)^2 2^{q(n+1)}|g^{(n+1)}|^2_\tau \le 2^{-q}\|g\|^2_{\tau,q},\end{aligned}$$

this definition is well posed and, moreover, the Hida stochastic derivative

$$\partial_{\cdot} : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \tag{1.28}$$

is a linear *continuous* operator. Moreover, as it follows from construction of the kernels $G^{(n+1)}(\cdot) \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}$ from (1.24), this derivative is (generated by) the restriction of derivative (1.25) to $(\mathcal{H}_\tau)_q$. We note also that the restrictions of derivative (1.28) to $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ generate linear continuous operators $\partial_{\cdot} : (\mathcal{H}_\tau) \to (\mathcal{H}_\tau) \otimes \mathcal{H}_\tau := \text{pr} \lim_{q\in\mathbb{Z}_+} (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ and $\partial_{\cdot} : (\mathcal{D}) \to (\mathcal{D}) \otimes \mathcal{D} := \text{pr} \lim_{q\in\mathbb{Z}_+,\tau\in T} (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ respectively.

Using the notion of the Hida stochastic derivative, one can easily widen the extended stochastic integral to the spaces of nonregular generalized functions.

Definition. We define an extended stochastic integral

$$\int \circ(u)\, \widehat{d}L_u : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau} \to (\mathcal{H}_{-\tau})_{-q} \tag{1.29}$$

as a linear continuous operator adjoint to Hida stochastic derivative (1.28), i.e., for $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$

$$\int F(u)\, \widehat{d}L_u := (\partial_{\cdot})^* F \in (\mathcal{H}_{-\tau})_{-q}. \tag{1.30}$$

Since derivative (1.28) is the restriction of derivative (1.25), by (1.26) and (1.30) integral (1.29) is an extension of integral (1.22).

By analogy one can define linear continuous operators $\int \circ(u)\, \widehat{d}L_u : (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau} \to (\mathcal{H}_{-\tau})$ and $\int \circ(u)\, \widehat{d}L_u : (\mathcal{D}') \otimes \mathcal{D}' \to (\mathcal{D}')$, where $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau} := \text{ind} \lim_{q\to\infty} (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$, $(\mathcal{D}') \otimes \mathcal{D}' := \text{ind} \lim_{q\to\infty,\tau\in T} (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$.

In contrast to formula (1.21) for integral (1.22), formula (1.30) for integrals (1.29) is inconvenient for calculations. Therefore let us obtain representations for these integrals in terms of the orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$.

First we note that, as in the case of the spaces $(\mathcal{H}_{-\tau})_{-q}$, it follows from the general duality theory that there exists a system of orthogonal in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ generalized functions $\big\{ :\!\langle \circ^{\otimes m}, F_{\text{ext},\cdot}^{(m)} \rangle\!: \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau} \mid F_{\text{ext},\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau},\ m \in \mathbb{Z}_+ \big\}$ such that for $F_{\text{ext},\cdot}^{(m)} \in \mathcal{H}_{\rm ext}^{(m)} \otimes \mathcal{H} \subset \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$ $:\!\langle \circ^{\otimes m}, F_{\text{ext},\cdot}^{(m)} \rangle\!:$ is given by (1.18); and any generalized function $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ can be presented as a convergent in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ series

$$F(\cdot) = \sum_{m=0}^{\infty} :\!\langle \circ^{\otimes m}, F_{\text{ext},\cdot}^{(m)} \rangle\!:, \quad F_{\text{ext},\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}, \tag{1.31}$$

now $\|F\|^2_{(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}} = \sum_{m=0}^\infty 2^{-qm} |F_{\text{ext},\cdot}^{(m)}|^2_{\mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}} < \infty$.

Consider a family of chains

$$\mathcal{D}'^{\widehat{\otimes} m} \supset \mathcal{H}_{-\tau}^{\widehat{\otimes} m} \supset \mathcal{H}^{\widehat{\otimes} m} \supset \mathcal{H}_{\tau}^{\widehat{\otimes} m} \supset \mathcal{D}^{\widehat{\otimes} m}, \quad m \in \mathbb{Z}_+ \tag{1.32}$$

(as is well known, $\mathcal{H}_{-\tau}^{\widehat{\otimes} m}$ and $\mathcal{D}'^{\widehat{\otimes} m} = \text{ind} \lim_{\tau\in T} \mathcal{H}_{-\tau}^{\widehat{\otimes} m}$ are the spaces dual of $\mathcal{H}_{\tau}^{\widehat{\otimes} m}$ and $\mathcal{D}^{\widehat{\otimes} m}$ with respect to $\mathcal{H}^{\widehat{\otimes} m}$; in the case $m = 0$ all spaces from chain (1.32) are equal to $\mathbb{R}$). Since the spaces of test functions in chains (1.32) and (1.14) coincide, there exists a family of natural isomorphisms

$$U_m : \mathcal{D}'^{(m)} \to \mathcal{D}'^{\widehat{\otimes} m} \tag{1.33}$$

such that for all $F_{\rm ext}^{(m)} \in \mathcal{D}'^{(m)}$ and $f^{(m)} \in \mathcal{D}^{\widehat{\otimes} m}$

$$\langle F_{\rm ext}^{(m)}, f^{(m)} \rangle_{\rm ext} = \langle U_m F_{\rm ext}^{(m)}, f^{(m)} \rangle. \tag{1.34}$$

It is easy to see that the restrictions of U_m to $\mathcal{H}_{-\tau}^{(m)}$ are isometric isomorphisms between the spaces $\mathcal{H}_{-\tau}^{(m)}$ and $\mathcal{H}_{-\tau}^{\widehat{\otimes} m}$.

Remark. As we saw above, $\mathcal{H}_{\mathrm{ext}}^{(1)} = \mathcal{H}$, and therefore in the case $m = 1$ chains (1.32) and (1.14) coincide. Thus $U_1 = \mathbf{1}$ is the identity operator on $\mathcal{D}'^{(1)} = \mathcal{D}'$. In the case $m = 0$ U_0 is, obviously, the identity operator on $\mathbb{R}$.

Proposition. *Let $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$. The extended stochastic integral can be presented in the form*

$$\int F(u)\,\widehat{d}L_u = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}_{\mathrm{ext}}^{(m)} \rangle : , \tag{1.35}$$

where

$$\widehat{F}_{\mathrm{ext}}^{(m)} := U_{m+1}^{-1}\{\mathrm{Pr}[(U_m \otimes \mathbf{1})F_{\mathrm{ext},\cdot}^{(m)}]\} \in \mathcal{H}_{-\tau}^{(m+1)}, \tag{1.36}$$

Pr *is the symmetrization operator (more exactly, the orthoprojector acting for each $m \in \mathbb{Z}_+$ from $\mathcal{H}_{-\tau}^{\widehat{\otimes} m} \otimes \mathcal{H}_{-\tau}$ to $\mathcal{H}_{-\tau}^{\widehat{\otimes} m+1}$), $F_{\mathrm{ext},\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$, $m \in \mathbb{Z}_+$, are the kernels from decomposition* (1.31) *for F.*

Proof. By direct calculation one can easily show that the series in the right hand side of (1.35) converges in $(\mathcal{H}_{-\tau})_{-q}$. Further, using (1.11), (1.17), (1.36), (1.34), (1.27), (1.31), (1.30) and the continuity of operators (1.28) and (1.29), for all $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ and $g \in (\mathcal{H}_\tau)_q$ we obtain

$$\begin{aligned}
&\langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}_{\mathrm{ext}}^{(m)} \rangle :, g \rangle\!\rangle_{(L^2)} = \langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}_{\mathrm{ext}}^{(m)} \rangle :, \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, g^{(n)} \rangle : \rangle\!\rangle_{(L^2)} \\
&\quad = \sum_{m=0}^{\infty} (m+1)! \langle \widehat{F}_{\mathrm{ext}}^{(m)}, g^{(m+1)} \rangle_{\mathrm{ext}} = \sum_{m=0}^{\infty} (m+1)! \langle (U_m \otimes \mathbf{1}) F_{\mathrm{ext},\cdot}^{(m)}, g^{(m+1)} \rangle \\
&\quad = \sum_{m=0}^{\infty} m!(m+1) \langle F_{\mathrm{ext},\cdot}^{(m)}, g^{(m+1)}(\cdot) \rangle_{\mathcal{H}_{\mathrm{ext}}^{(m)} \otimes \mathcal{H}} \\
&\quad = \langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F_{\mathrm{ext},\cdot}^{(m)} \rangle :, \sum_{n=0}^{\infty} (n+1) :\langle \circ^{\otimes n}, g^{(n+1)}(\cdot) \rangle : \rangle\!\rangle_{(L^2)\otimes\mathcal{H}} \\
&\quad = \langle\!\langle F(\cdot), \partial_\cdot g \rangle\!\rangle_{(L^2)\otimes\mathcal{H}} = \langle\!\langle \int F(u)\,\widehat{d}L_u, g \rangle\!\rangle_{(L^2)}.
\end{aligned}$$

The result of the proposition from this calculation follows. □

Remark. Sometimes it can be convenient to introduce the Hida stochastic derivative and the extended stochastic integral as linear continuous operators acting from $(\mathcal{H}_\tau)_q$ to $(\mathcal{H}_\tau)_q \otimes \mathcal{H}$ and from $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}$ to $(\mathcal{H}_{-\tau})_{-q}$ correspondingly, this case is described in detail in [21].

Unfortunately, in contrast to the Hida stochastic derivative, the extended stochastic integral with respect to a Lévy process can not be naturally restricted to the spaces of nonregular test functions, in general. More precisely, for $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ $\int f(u)\,\widehat{d}L_u$ not necessary a nonregular test function (one can show that for $\tau \in T$ and $q \in \mathbb{Z}_+$ such that $q > \log_2 c(\tau)$, where $c(\tau) > 0$ from estimate (1.9), if $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ then $\int f(u)\,\widehat{d}L_u \in (L^2)$; and for q sufficiently large this integral is a *regular* test function [21]). Nevertheless, one can introduce on each space of nonregular test functions a linear operator that has some important properties of the extended stochastic integral.

Let $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$. Using the above described orthogonal basis in this space, we can write

$$f(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f_\cdot^{(n)} \rangle :, \quad f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau \tag{1.37}$$

(the series converges in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$), and also

$$\|f\|^2_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau} = \sum_{n=0}^{\infty}(n!)^2 2^{qn}|f_\cdot^{(n)}|^2_{\mathcal{H}_\tau^{\widehat{\otimes}n}\otimes\mathcal{H}_\tau} < \infty. \tag{1.38}$$

Definition. We define a linear continuous operator $\mathbb{I} : (\mathcal{H}_\tau)_{q+1}\otimes\mathcal{H}_\tau \to (\mathcal{H}_\tau)_q$ by setting for $f \in (\mathcal{H}_\tau)_{q+1}\otimes\mathcal{H}_\tau$

$$\mathbb{I}(f) := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \hat{f}^{(n)}\rangle: \tag{1.39}$$

(cf. (1.21), (1.35)), where $\hat{f}^{(n)} := \mathrm{Pr} f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n+1}$ are the orthoprojections onto $\mathcal{H}_\tau^{\widehat{\otimes}n+1}$ (the symmetrizations by all variables) of the kernels $f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}\otimes\mathcal{H}_\tau$ from decomposition (1.37) for f.

Since (see (1.12), (1.39) and (1.38))

$$\begin{aligned}\|\mathbb{I}(f)\|^2_{\tau,q} = \sum_{n=0}^{\infty}((n+1)!)^2 2^{q(n+1)}|\hat{f}^{(n)}|^2_\tau \le 2^q \sum_{n=0}^{\infty}(n!)^2 2^{(q+1)n}[(n+1)^2 2^{-n}]|f_\cdot^{(n)}|^2_{\mathcal{H}_\tau^{\widehat{\otimes}n}\otimes\mathcal{H}_\tau}\\ \le 9\cdot 2^{q-2}\|f\|^2_{(\mathcal{H}_\tau)_{q+1}\otimes\mathcal{H}_\tau},\end{aligned}$$

this definition is well posed. It is clear that the restriction of the operator $\mathbb{I}$ to the space $(\mathcal{H}_\tau)\otimes\mathcal{H}_\tau$ (correspondingly to the space $(\mathcal{D})\otimes\mathcal{D}$) is a linear continuous operator acting from $(\mathcal{H}_\tau)\otimes\mathcal{H}_\tau$ to $(\mathcal{H}_\tau)$ (correspondingly from $(\mathcal{D})\otimes\mathcal{D}$ to $(\mathcal{D})$).

Sometimes it can be convenient to define $\mathbb{I}$ by formula (1.39) as a linear unbounded operator acting from $(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau$ to $(\mathcal{H}_\tau)_q$ with the domain

$$\mathrm{dom}(\mathbb{I}) := \Big\{f \in (\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau : \|\mathbb{I}(f)\|^2_{\tau,q} = \sum_{n=0}^{\infty}((n+1)!)^2 2^{q(n+1)}|\hat{f}^{(n)}|^2_\tau < \infty\Big\},$$

in this case $\mathbb{I}$ is a *closed* operator (this will be proved in a forthcoming paper).

Remark. If L is a Wiener or Poisson random process then the corresponding operator $\mathbb{I}$ is the restriction to the corresponding space of nonregular test functions of the extended stochastic integral. Respectively, the adjoint to this integral operator is an extension of the Hida stochastic derivative to the corresponding space of nonregular generalized functions. For processes L that we consider in the present paper, the Hida stochastic derivative can not be naturally extended to the spaces of nonregular generalized functions, but the adjoint to $\mathbb{I}$ operator plays the role of the mentioned derivative (the detailed presentation will be given in a forthcoming paper). We note also that all atypical for a classical analysis difficulties with determination of stochastic integrals and derivatives (in particular, the necessity to introduce the spaces $\mathcal{H}^{(m)}_{\mathrm{ext}}$, $\mathcal{H}^{(m)}_{-\tau}$, $\mathcal{D}'^{(m)}$, the isomorphisms U_m) are related to the fact that considered here Lévy processes have no the CRP.

2. Operators of stochastic differentiation

2.1. **Stochastic differentiation on spaces of test functions.** In order to define operators of stochastic differentiation on the spaces of nonregular test functions, we need some preparation. Let $F^{(n)}_{\mathrm{ext}} \in \mathcal{H}^{(n)}_{-\tau}$, $G^{(m)}_{\mathrm{ext}} \in \mathcal{H}^{(m)}_{-\tau}$, $n,m \in \mathbb{Z}_+$. We define

$$F^{(n)}_{\mathrm{ext}} \diamond G^{(m)}_{\mathrm{ext}} := U^{-1}_{n+m}[(U_n F^{(n)}_{\mathrm{ext}})\widehat{\otimes}(U_m G^{(m)}_{\mathrm{ext}})] \in \mathcal{H}^{(n+m)}_{-\tau}, \tag{2.1}$$

where $U_m : \mathcal{H}^{(m)}_{-\tau} \to \mathcal{H}^{\widehat{\otimes}m}_{-\tau}$, $m \in \mathbb{Z}_+$, are the restrictions to $\mathcal{H}^{(m)}_{-\tau}$ of operators (1.33). It follows from the linearity of operators U_m and from properties of a symmetric tensor product that a product $\diamond$ is commutative, associative and distributive. Further, by (2.1)

$$\begin{aligned}|F_{\rm ext}^{(n)}\diamond G_{\rm ext}^{(m)}|_{\mathcal{H}_{-\tau}^{(n+m)}} &= |(U_nF_{\rm ext}^{(n)})\widehat{\otimes}(U_mG_{\rm ext}^{(m)})|_{\mathcal{H}_{-\tau}^{\widehat{\otimes}n+m}}\\ &\le |U_nF_{\rm ext}^{(n)}|_{\mathcal{H}_{-\tau}^{\widehat{\otimes}n}}|U_mG_{\rm ext}^{(m)}|_{\mathcal{H}_{-\tau}^{\widehat{\otimes}m}} = |F_{\rm ext}^{(n)}|_{\mathcal{H}_{-\tau}^{(n)}}|G_{\rm ext}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)}}.\end{aligned}\tag{2.2}$$

Let $F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau}^{(n)}$, $f^{(m)}\in\mathcal{H}_{\tau}^{\widehat{\otimes}m}$, $m>n$. We define a generalized partial pairing $\langle F_{\rm ext}^{(n)},f^{(m)}\rangle_{\rm ext}\in\mathcal{H}_{\tau}^{\widehat{\otimes}m-n}$ by setting for arbitrary $G_{\rm ext}^{(m-n)}\in\mathcal{H}_{-\tau}^{(m-n)}$

$$\langle G_{\rm ext}^{(m-n)},\langle F_{\rm ext}^{(n)},f^{(m)}\rangle_{\rm ext}\rangle_{\rm ext} = \langle F_{\rm ext}^{(n)}\diamond G_{\rm ext}^{(m-n)},f^{(m)}\rangle_{\rm ext}.\tag{2.3}$$

By (2.2)

$$\begin{aligned}|\langle F_{\rm ext}^{(n)}\diamond G_{\rm ext}^{(m-n)},f^{(m)}\rangle_{\rm ext}| &\le |F_{\rm ext}^{(n)}\diamond G_{\rm ext}^{(m-n)}|_{\mathcal{H}_{-\tau}^{(m)}}|f^{(m)}|_\tau\\ &\le |F_{\rm ext}^{(n)}|_{\mathcal{H}_{-\tau}^{(n)}}|G_{\rm ext}^{(m-n)}|_{\mathcal{H}_{-\tau}^{(m-n)}}|f^{(m)}|_\tau,\end{aligned}$$

which implies that this definition is well posed and

$$|\langle F_{\rm ext}^{(n)},f^{(m)}\rangle_{\rm ext}|_{\mathcal{H}_{\tau}^{\widehat{\otimes}m-n}} \equiv |\langle F_{\rm ext}^{(n)},f^{(m)}\rangle_{\rm ext}|_\tau \le |F_{\rm ext}^{(n)}|_{\mathcal{H}_{-\tau}^{(n)}}|f^{(m)}|_\tau.\tag{2.4}$$

Definition. Let $n\in\mathbb{N}$, $F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau}^{(n)}$. We define a linear continuous operator

$$(D^n\circ)(F_{\rm ext}^{(n)}):(\mathcal{H}_\tau)_q\to(\mathcal{H}_\tau)_q\tag{2.5}$$

by setting for $f\in(\mathcal{H}_\tau)_q$

$$\begin{aligned}(D^nf)(F_{\rm ext}^{(n)}) &:= \sum_{m=n}^{\infty}\frac{m!}{(m-n)!}{:}\langle\circ^{\otimes m-n},\langle F_{\rm ext}^{(n)},f^{(m)}\rangle_{\rm ext}\rangle{:}\\ &\equiv \sum_{m=0}^{\infty}\frac{(m+n)!}{m!}{:}\langle\circ^{\otimes m},\langle F_{\rm ext}^{(n)},f^{(m+n)}\rangle_{\rm ext}\rangle{:}\,,\end{aligned}\tag{2.6}$$

where $f^{(m)}\in\mathcal{H}_{\tau}^{\widehat{\otimes}m}$ are the kernels from decomposition (1.11) for f.

Since (see (1.12), (2.6) and (2.4))

$$\begin{aligned}\|(D^nf)(F_{\rm ext}^{(n)})\|_{\tau,q}^2 &= \sum_{m=0}^{\infty}(m!)^2 2^{qm}\frac{((m+n)!)^2}{(m!)^2}|\langle F_{\rm ext}^{(n)},f^{(m+n)}\rangle_{\rm ext}|_\tau^2\\ &\le |F_{\rm ext}^{(n)}|_{\mathcal{H}_{-\tau}^{(n)}}^2 2^{-qn}\sum_{m=0}^{\infty}((m+n)!)^2 2^{q(m+n)}|f^{(m+n)}|_\tau^2 \le |F_{\rm ext}^{(n)}|_{\mathcal{H}_{-\tau}^{(n)}}^2 2^{-qn}\|f\|_{\tau,q}^2,\end{aligned}$$

this definition is well posed. It is clear that the restriction of the operator $(D^n\circ)(F_{\rm ext}^{(n)})$ to the space $(\mathcal{H}_\tau)$ or to the space $(\mathcal{D})$ is a linear continuous operator on the corresponding space.

Let us consider main properties of the operator D^n.

Theorem 2.1. *1) For $k_1,\dots,k_m\in\mathbb{N}$, $F_j^{(k_j)}\in\mathcal{H}_{-\tau}^{(k_j)}$, $j\in\{1,\dots,m\}$,*

$$(D^{k_m}(\cdots(D^{k_2}((D^{k_1}\circ)(F_1^{(k_1)})))(F_2^{(k_2)})\cdots))(F_m^{(k_m)}) = (D^{k_1+\cdots+k_m}\circ)(F_1^{(k_1)}\diamond\cdots\diamond F_m^{(k_m)}).$$

2) For each $f\in(\mathcal{H}_\tau)_q$ the kernels $f^{(n)}\in\mathcal{H}_{\tau}^{\widehat{\otimes}n}$, $n\in\mathbb{N}$, from decomposition (1.11) *can be presented in the form*

$$f^{(n)} = \frac{1}{n!}\mathbb{E}(D^nf),$$

i.e., for each $F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau}^{(n)}$ $\langle F_{\rm ext}^{(n)},f^{(n)}\rangle_{\rm ext} = \frac{1}{n!}\mathbb{E}((D^nf)(F_{\rm ext}^{(n)}))$, here $\mathbb{E}\circ := \langle\!\langle 1,\circ\rangle\!\rangle_{(L^2)} = \int_{\mathcal{D}'}\circ(\omega)\mu(d\omega)$ is an expectation.

3) The adjoint to D^n operator has the form

$$(D^nG)(F_{\rm ext}^{(n)})^* = \sum_{m=0}^{\infty} :\langle \circ^{m+n}, G_{\rm ext}^{(m)} \diamond F_{\rm ext}^{(n)}\rangle : \in (\mathcal{H}_{-\tau})_{-q}, \tag{2.7}$$

where $F_{\rm ext}^{(n)} \in \mathcal{H}_{-\tau}^{(n)}$, $G \in (\mathcal{H}_{-\tau})_{-q}$, and $G_{\rm ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$ are the kernels from decomposition (1.15) *for* G.

Proof. 1) The proof consists in the application of the mathematical induction method.
2) Using (2.6) and (1.17) we obtain

$$\mathbb{E}((D^nf)(F_{\rm ext}^{(n)})) = \langle\!\langle 1, (D^nf)(F_{\rm ext}^{(n)})\rangle\!\rangle_{(L^2)} = n!\langle F_{\rm ext}^{(n)}, f^{(n)}\rangle_{\rm ext}.$$

3) Let $f \in (\mathcal{H}_\tau)_q$, $F_{\rm ext}^{(n)} \in \mathcal{H}_{-\tau}^{(n)}$, $G \in (\mathcal{H}_{-\tau})_{-q}$. By direct calculation one can easily show that the series in the right hand side of (2.7) converges in $(\mathcal{H}_{-\tau})_{-q}$. Further, using (1.15), (2.6), (1.17), (2.3), (1.11) and the commutativity of the product $\diamond$, we obtain

$$\begin{aligned}
&\langle\!\langle (D^nG)(F_{\rm ext}^{(n)})^*, f\rangle\!\rangle_{(L^2)} = \langle\!\langle G, (D^nf)(F_{\rm ext}^{(n)})\rangle\!\rangle_{(L^2)}\\
&= \langle\!\langle \sum_{k=0}^{\infty} :\langle \circ^{\otimes k}, G_{\rm ext}^{(k)}\rangle :, \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} :\langle \circ^{\otimes m}, \langle F_{\rm ext}^{(n)}, f^{(m+n)}\rangle_{\rm ext}\rangle :\rangle\!\rangle_{(L^2)}\\
&= \sum_{m=0}^{\infty} (m+n)!\langle G_{\rm ext}^{(m)}, \langle F_{\rm ext}^{(n)}, f^{(m+n)}\rangle_{\rm ext}\rangle_{\rm ext}\\
&= \sum_{m=0}^{\infty} (m+n)!\langle F_{\rm ext}^{(n)} \diamond G_{\rm ext}^{(m)}, f^{(m+n)}\rangle_{\rm ext}\\
&= \langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+n}, G_{\rm ext}^{(m)} \diamond F_{\rm ext}^{(n)}\rangle :, \sum_{k=0}^{\infty} :\langle \circ^{\otimes k}, f^{(k)}\rangle :\rangle\!\rangle_{(L^2)}\\
&= \langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+n}, G_{\rm ext}^{(m)} \diamond F_{\rm ext}^{(n)}\rangle :, f\rangle\!\rangle_{(L^2)},
\end{aligned}$$

whence the result follows. □

Now we consider in more detail the case $n = 1$. Denote $D := D^1$.

Theorem 2.2. *1) For all $G \in (\mathcal{H}_{-\tau})_{-q}$ and $F^{(1)} \in \mathcal{H}_{-\tau}$*

$$(DG)(F^{(1)})^* = \int (G \otimes F^{(1)})(u)\, \widehat{d}L_u \in (\mathcal{H}_{-\tau})_{-q}. \tag{2.8}$$

2) For all $f \in (\mathcal{H}_\tau)_q$ and $F^{(1)} \in \mathcal{H}_{-\tau}$

$$(Df)(F^{(1)}) = \langle F^{(1)}(\cdot), \partial_\cdot f\rangle \in (\mathcal{H}_\tau)_q, \tag{2.9}$$

where $\langle F^{(1)}(\cdot), \partial_\cdot f\rangle$ is a partial pairing, i.e., the unique element of $(\mathcal{H}_\tau)_q$ such that for arbitrary $G \in (\mathcal{H}_{-\tau})_{-q}$ $\langle\!\langle G, \langle F^{(1)}(\cdot), \partial_\cdot f\rangle\rangle\!\rangle_{(L^2)} = \langle\!\langle G \otimes F^{(1)}, \partial_\cdot f\rangle\!\rangle_{(L^2)\otimes\mathcal{H}}$.

Remark. Similarly to the proof of the fact that the generalized partial pairing $\langle\cdot,\cdot\rangle_{\rm ext}$ is well posed and satisfies estimate (2.4), one can easily show that a partial pairing is well posed and satisfies a generalized Cauchy-Bunyakovsky inequality (in our case this inequality has the form $|\langle F^{(1)}(\cdot), \partial_\cdot f\rangle|_{\tau,q} \le |F^{(1)}|_{\mathcal{H}_{-\tau}} |\partial_\cdot f|_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau}$).

Proof. 1) The result follows from representation (2.7) with $n = 1$: it is necessary to compare the construction of kernels of extended stochastic integral (1.29) (see (1.36)) with the construction of a product $\diamond$ (see (2.1)).

2) Taking into account (2.8) and (1.30), for all $G \in (\mathcal{H}_{-\tau})_{-q}$ we obtain

$$\begin{aligned}\langle\!\langle G, (Df)(F^{(1)})\rangle\!\rangle_{(L^2)} &= \langle\!\langle \int (G \otimes F^{(1)})(u)\, \widehat{d}L_u, f\rangle\!\rangle_{(L^2)} \\ &= \langle\!\langle G \otimes F^{(1)}, \partial_\cdot f\rangle\!\rangle_{(L^2)\otimes\mathcal{H}} = \langle\!\langle G, \langle F^{(1)}(\cdot), \partial_\cdot f\rangle\rangle\!\rangle_{(L^2)},\end{aligned}$$

whence the result follows. □

Remark. Substituting in (2.9) $F^{(1)} = \delta_t \in \mathcal{H}_{-\tau}$ (δ_t is the Dirac delta-function concentrated at $t \in \mathbb{R}_+$; the inclusion is proved in, e.g., [4]), we obtain for $f \in (\mathcal{H}_\tau)_q$ $(Df)(\delta_t) = \langle \delta_t(\cdot), \partial_\cdot f\rangle = \partial_t f \in (\mathcal{H}_\tau)_q$. So, the Hida stochastic derivative can be presented in the form $\partial_\cdot \circ = (D\circ)(\delta_\cdot)$.

In some applications of the Gaussian analysis (in particular, in the Malliavin calculus) an important role belongs to the commutator between the extended stochastic integral and the operator of stochastic differentiation (see, e.g., [1]). An analog of this commutator is calculated in the Meixner analysis [19, 20] and on the spaces of regular test and generalized functions of the Lévy analysis [9, 8]. Unfortunately, there is no natural extension of this construction to the spaces of nonregular test functions of the Lévy analysis: as we saw above, the extended stochastic integral can not be naturally restricted to these spaces. Nevertheless, an analog of this integral on the mentioned spaces is the operator $\mathbb{I}$. So, it is natural to calculate the commutator between $\mathbb{I}$ and D. In order to do this, let us introduce operators of stochastic differentiation on the spaces $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ (this notion is intuitively clear and can be used without an additional explanation, but we prefer to give an exact definition).

As above, we begin with a preparation. Let $F^{(n)}_{\text{ext}} \in \mathcal{H}^{(n)}_{-\tau}$, $G^{(m)}_{\text{ext},\cdot} \in \mathcal{H}^{(m)}_{-\tau} \otimes \mathcal{H}_{-\tau}$, $n, m \in \mathbb{Z}_+$. We define

(2.10)
$$F^{(n)}_{\text{ext}} \Diamond G^{(m)}_{\text{ext},\cdot} := (U^{-1}_{n+m} \otimes \mathbf{1})\{(\Pr \otimes \mathbf{1})[(U_n F^{(n)}_{\text{ext}}) \otimes ((U_m \otimes \mathbf{1}) G^{(m)}_{\text{ext},\cdot})]\} \in \mathcal{H}^{(n+m)}_{-\tau} \otimes \mathcal{H}_{-\tau},$$

where, as above, U_m, $m \in \mathbb{Z}_+$, are the restrictions to $\mathcal{H}^{(m)}_{-\tau}$ of operators (1.33), $\Pr \otimes \mathbf{1}$ is the operator of symmetrization "by $n+m$ variables, except the variable $\cdot$" or, more exactly, the orthoprojector acting from $\mathcal{H}^{\widehat{\otimes} n}_{-\tau} \otimes \mathcal{H}^{\widehat{\otimes} m}_{-\tau} \otimes \mathcal{H}_{-\tau}$ to $\mathcal{H}^{\widehat{\otimes} n+m}_{-\tau} \otimes \mathcal{H}_{-\tau}$ (of course, this operator depends on n and m, but we simplify the notation). As is easy to see,

$$\begin{aligned}|F^{(n)}_{\text{ext}} \Diamond G^{(m)}_{\text{ext},\cdot}|_{\mathcal{H}^{(n+m)}_{-\tau}\otimes\mathcal{H}_{-\tau}} &= |(\Pr \otimes \mathbf{1})[(U_n F^{(n)}_{\text{ext}}) \otimes ((U_m \otimes \mathbf{1}) G^{(m)}_{\text{ext},\cdot})]|_{\mathcal{H}^{\widehat{\otimes} n+m}_{-\tau}\otimes\mathcal{H}_{-\tau}} \\ &\le |U_n F^{(n)}_{\text{ext}}|_{\mathcal{H}^{\widehat{\otimes} n}_{-\tau}} |(U_m \otimes \mathbf{1}) G^{(m)}_{\text{ext},\cdot}|_{\mathcal{H}^{\widehat{\otimes} m}_{-\tau}\otimes\mathcal{H}_{-\tau}} = |F^{(n)}_{\text{ext}}|_{\mathcal{H}^{(n)}_{-\tau}} |G^{(m)}_{\text{ext},\cdot}|_{\mathcal{H}^{(m)}_{-\tau}\otimes\mathcal{H}_{-\tau}}.\end{aligned} \tag{2.11}$$

Remark. Let $G^{(m)}_{\text{ext},\cdot} = G^{(m)}_{\text{ext}} \otimes H^{(1)}_{\text{ext}}$, $G^{(m)}_{\text{ext}} \in \mathcal{H}^{(m)}_{-\tau}$, $H^{(1)}_{\text{ext}} \in \mathcal{H}_{-\tau}$; and $F^{(n)}_{\text{ext}} \in \mathcal{H}^{(n)}_{-\tau}$. By (2.10) and (2.1)

$$F^{(n)}_{\text{ext}} \Diamond (G^{(m)}_{\text{ext}} \otimes H^{(1)}_{\text{ext}}) = (F^{(n)}_{\text{ext}} \diamond G^{(m)}_{\text{ext}}) \otimes H^{(1)}_{\text{ext}}. \tag{2.12}$$

Let $F^{(n)}_{\text{ext}} \in \mathcal{H}^{(n)}_{-\tau}$, $f^{(m)}_{\cdot} \in \mathcal{H}^{\widehat{\otimes} m}_{\tau} \otimes \mathcal{H}_\tau$, $m \ge n$. We define a generalized partial pairing $\langle F^{(n)}_{\text{ext}}, f^{(m)}_{\cdot}\rangle_{\text{EXT}} \in \mathcal{H}^{\widehat{\otimes} m-n}_{\tau} \otimes \mathcal{H}_\tau$ by setting for arbitrary $G^{(m-n)}_{\text{ext},\cdot} \in \mathcal{H}^{(m-n)}_{-\tau} \otimes \mathcal{H}_{-\tau}$

$$\langle G^{(m-n)}_{\text{ext},\cdot}, \langle F^{(n)}_{\text{ext}}, f^{(m)}_{\cdot}\rangle_{\text{EXT}}\rangle_{\mathcal{H}^{(m-n)}_{\text{ext}}\otimes\mathcal{H}} = \langle F^{(n)}_{\text{ext}} \Diamond G^{(m-n)}_{\text{ext},\cdot}, f^{(m)}_{\cdot}\rangle_{\mathcal{H}^{(m)}_{\text{ext}}\otimes\mathcal{H}}. \tag{2.13}$$

By (2.11)

$$\begin{aligned}|\langle F^{(n)}_{\text{ext}} \Diamond G^{(m-n)}_{\text{ext},\cdot}, f^{(m)}_{\cdot}\rangle_{\mathcal{H}^{(m)}_{\text{ext}}\otimes\mathcal{H}}| &\le |F^{(n)}_{\text{ext}} \Diamond G^{(m-n)}_{\text{ext},\cdot}|_{\mathcal{H}^{(m)}_{-\tau}\otimes\mathcal{H}_{-\tau}} |f^{(m)}_{\cdot}|_{\mathcal{H}^{\widehat{\otimes} m}_{\tau}\otimes\mathcal{H}_\tau} \\ &\le |F^{(n)}_{\text{ext}}|_{\mathcal{H}^{(n)}_{-\tau}} |G^{(m-n)}_{\text{ext},\cdot}|_{\mathcal{H}^{(m-n)}_{-\tau}\otimes\mathcal{H}_{-\tau}} |f^{(m)}_{\cdot}|_{\mathcal{H}^{\widehat{\otimes} m}_{\tau}\otimes\mathcal{H}_\tau},\end{aligned}$$

which implies that this definition is well posed and

$$|\langle F_{\rm ext}^{(n)}, f_{\cdot}^{(m)}\rangle_{\rm EXT}|_{\mathcal{H}_\tau^{\widehat{\otimes}m-n}\otimes\mathcal{H}_\tau} \le |F_{\rm ext}^{(n)}|_{\mathcal{H}_{-\tau}^{(n)}} |f_{\cdot}^{(m)}|_{\mathcal{H}_\tau^{\widehat{\otimes}m}\otimes\mathcal{H}_\tau}. \tag{2.14}$$

Remark. Let $f_{\cdot}^{(m)} = f^{(m)}\otimes h^{(1)}$, $f^{(m)}\in\mathcal{H}_\tau^{\widehat{\otimes}m}$, $h^{(1)}\in\mathcal{H}_\tau$; and $F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau}^{(n)}$. For $G_{\rm ext}^{(m-n)}\in\mathcal{H}_{-\tau}^{(m-n)}$ and $H_{\rm ext}^{(1)}\in\mathcal{H}_{-\tau}$ by (2.13), (2.12) and (2.3) we obtain

$$\begin{aligned}
&\langle G_{\rm ext}^{(m-n)}\otimes H_{\rm ext}^{(1)}, \langle F_{\rm ext}^{(n)}, f^{(m)}\otimes h^{(1)}\rangle_{\rm EXT}\rangle_{\mathcal{H}_{\rm ext}^{(m-n)}\otimes\mathcal{H}}\\
&\quad = \langle F_{\rm ext}^{(n)}\Diamond(G_{\rm ext}^{(m-n)}\otimes H_{\rm ext}^{(1)}), f^{(m)}\otimes h^{(1)}\rangle_{\mathcal{H}_{\rm ext}^{(m)}\otimes\mathcal{H}}\\
&\quad = \langle (F_{\rm ext}^{(n)}\diamond G_{\rm ext}^{(m-n)})\otimes H_{\rm ext}^{(1)}, f^{(m)}\otimes h^{(1)}\rangle_{\mathcal{H}_{\rm ext}^{(m)}\otimes\mathcal{H}}\\
&\quad = \langle F_{\rm ext}^{(n)}\diamond G_{\rm ext}^{(m-n)}, f^{(m)}\rangle_{\rm ext}\langle H_{\rm ext}^{(1)}, h^{(1)}\rangle\\
&\quad = \langle G_{\rm ext}^{(m-n)}, \langle F_{\rm ext}^{(n)}, f^{(m)}\rangle_{\rm ext}\rangle_{\rm ext}\langle H_{\rm ext}^{(1)}, h^{(1)}\rangle\\
&\quad = \langle G_{\rm ext}^{(m-n)}\otimes H_{\rm ext}^{(1)}, \langle F_{\rm ext}^{(n)}, f^{(m)}\rangle_{\rm ext}\otimes h^{(1)}\rangle_{\mathcal{H}_{\rm ext}^{(m-n)}\otimes\mathcal{H}}.
\end{aligned}$$

The set $\{G_{\rm ext}^{(m-n)}\otimes H_{\rm ext}^{(1)} : G_{\rm ext}^{(m-n)}\in\mathcal{H}_{-\tau}^{(m-n)},\ H_{\rm ext}^{(1)}\in\mathcal{H}_{-\tau}\}$ is total in the space $\mathcal{H}_{-\tau}^{(m-n)}\otimes\mathcal{H}_{-\tau}$. Thus we can conclude that

$$\langle F_{\rm ext}^{(n)}, f^{(m)}\otimes h^{(1)}\rangle_{\rm EXT} = \langle F_{\rm ext}^{(n)}, f^{(m)}\rangle_{\rm ext}\otimes h^{(1)}$$

in the space $\mathcal{H}_\tau^{\widehat{\otimes}m-n}\otimes\mathcal{H}_\tau$. As a corollary from this formula one can obtain the following intuitively clear result. Let $F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau}^{(n)}$, $f_{\cdot}^{(m)}\in\mathcal{H}_\tau^{\widehat{\otimes}m}\otimes\mathcal{H}_\tau$, $m\ge n$, $g_{\cdot}^{(m-n)} := \langle F_{\rm ext}^{(n)}, f_{\cdot}^{(m)}\rangle_{\rm EXT}\in\mathcal{H}_\tau^{\widehat{\otimes}m-n}\otimes\mathcal{H}_\tau$. Then for each $u\in\mathbb{R}_+$ $f_u^{(m)}\in\mathcal{H}_\tau^{\widehat{\otimes}m}$ and $g_u^{(m-n)} = \langle F_{\rm ext}^{(n)}, f_u^{(m)}\rangle_{\rm ext}\in\mathcal{H}_\tau^{\widehat{\otimes}m-n}$.

Definition. Let $n\in\mathbb{N}$, $F_{\rm ext}^{(n)}\in\mathcal{H}_{-\tau}^{(n)}$. We define a linear continuous operator

$$(\mathbf{D}^n\circ)(F_{\rm ext}^{(n)}) : (\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau \to (\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau$$

by setting for $f\in(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau$

$$\begin{aligned}
(\mathbf{D}^n f(\cdot))(F_{\rm ext}^{(n)}) &:= \sum_{m=n}^{\infty}\frac{m!}{(m-n)!}{:}\langle\circ^{\otimes m-n}, \langle F_{\rm ext}^{(n)}, f_{\cdot}^{(m)}\rangle_{\rm EXT}\rangle{:}\\
&\equiv \sum_{m=0}^{\infty}\frac{(m+n)!}{m!}{:}\langle\circ^{\otimes m}, \langle F_{\rm ext}^{(n)}, f_{\cdot}^{(m+n)}\rangle_{\rm EXT}\rangle{:}\,,
\end{aligned} \tag{2.15}$$

where $f_{\cdot}^{(m)}\in\mathcal{H}_\tau^{\widehat{\otimes}m}\otimes\mathcal{H}_\tau$ are the kernels from decomposition (1.37) for f.

Since (see (1.38), (2.15) and (2.14))

$$\begin{aligned}
\|(\mathbf{D}^n f(\cdot))(F_{\rm ext}^{(n)})\|^2_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau} &= \sum_{m=0}^{\infty}(m!)^2 2^{qm}\frac{((m+n)!)^2}{(m!)^2}|\langle F_{\rm ext}^{(n)}, f_{\cdot}^{(m+n)}\rangle_{\rm EXT}|^2_{\mathcal{H}_\tau^{\widehat{\otimes}m}\otimes\mathcal{H}_\tau}\\
&\le |F_{\rm ext}^{(n)}|^2_{\mathcal{H}_{-\tau}^{(n)}}2^{-qn}\sum_{m=0}^{\infty}((m+n)!)^2 2^{q(m+n)}|f_{\cdot}^{(m+n)}|^2_{\mathcal{H}_\tau^{\widehat{\otimes}m+n}\otimes\mathcal{H}_\tau}\\
&\le |F_{\rm ext}^{(n)}|^2_{\mathcal{H}_{-\tau}^{(n)}}2^{-qn}\|f\|^2_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau},
\end{aligned}$$

this definition is well posed. It is clear that the restriction of the operator $(\mathbf{D}^n\circ)(F_{\rm ext}^{(n)})$ to the space $(\mathcal{H}_\tau)\otimes\mathcal{H}_\tau$ or to the space $(\mathcal{D})\otimes\mathcal{D}$ is a linear continuous operator on the corresponding space.

Remark. Let $n \in \mathbb{N}$, $F_{\mathrm{ext}}^{(n)} \in \mathcal{H}_{-\tau}^{(n)}$, $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$, $g(\cdot) := (\mathbf{D}^n f(\cdot))(F_{\mathrm{ext}}^{(n)}) \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$, $\dot{f} \in f$ be a representative (a function) from the equivalence class f. Then, as follows from the previous remark, $\dot{g}(\cdot) := (\mathbf{D}^n \dot{f}(\cdot))(F_{\mathrm{ext}}^{(n)})$ is a representative of the equivalence class $g(\cdot)$ such that for each $u \in \mathbb{R}_+$ $\dot{g}(u) = (D^n \dot{f}(u))(F_{\mathrm{ext}}^{(n)})$ (i.e., considering f as a function on $\mathbb{R}_+$ with values in $(\mathcal{H}_\tau)_q$ and substituting in $(\mathbf{D}^n f(\cdot))(F_{\mathrm{ext}}^{(n)})$ a number u on the place of $\cdot$, we obtain $(D^n f(u))(F_{\mathrm{ext}}^{(n)})$).

Theorem 2.3. *Denote* $\mathbf{D} := \mathbf{D}^1$. *For all* $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau$ *and* $F^{(1)} \in \mathcal{H}_{-\tau}$

$$(D(\mathbb{I}(f)))(F^{(1)}) = \mathbb{I}((\mathbf{D}(f(\cdot)))(F^{(1)})) + \langle F^{(1)}(\cdot), f(\cdot)\rangle \in (\mathcal{H}_\tau)_q, \tag{2.16}$$

where the last pairing is a partial one (see Theorem 2.2 for an explanation of this term).

Proof. Using (1.39) and (2.6) we obtain

$$(D(\mathbb{I}(f)))(F^{(1)}) = \sum_{m=0}^{\infty} (m+1) {:}\langle \circ^{\otimes m}, \langle F^{(1)}, \hat{f}^{(m)}\rangle_{\mathrm{ext}}\rangle{:} \in (\mathcal{H}_\tau)_q,$$

where $\hat{f}^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes} m+1}$ are the kernels from decomposition (1.39) (which is decomposition (1.11) for $\mathbb{I}(f)$), i.e., $\hat{f}^{(m)} = \mathrm{Pr} f_{\cdot}^{(m)}$ are orthoprojections onto $\mathcal{H}_\tau^{\widehat{\otimes} m+1}$ of the kernels $f_{\cdot}^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes} m} \otimes \mathcal{H}_\tau$ from decomposition (1.37) for f. On the other hand, by (2.15) and (1.39)

$$\mathbb{I}((\mathbf{D}(f(\cdot)))(F^{(1)})) = \sum_{m=0}^{\infty} m {:}\langle \circ^{\otimes m}, \mathrm{Pr}\langle F^{(1)}, f_{\cdot}^{(m)}\rangle_{\mathrm{EXT}}\rangle{:} \in (\mathcal{H}_\tau)_q.$$

Let $G = \sum_{k=0}^{\infty} {:}\langle \circ^{\otimes k}, G_{\mathrm{ext}}^{(k)}\rangle{:} \in (\mathcal{H}_{-\tau})_{-q}$, $G_{\mathrm{ext}}^{(k)} \in \mathcal{H}_{-\tau}^{(k)}$. By (1.17) we have

$$\langle\!\langle G, (D(\mathbb{I}(f)))(F^{(1)})\rangle\!\rangle_{(L^2)} = \sum_{m=0}^{\infty} m!(m+1)\langle G_{\mathrm{ext}}^{(m)}, \langle F^{(1)}, \hat{f}^{(m)}\rangle_{\mathrm{ext}}\rangle_{\mathrm{ext}},$$

$$\langle\!\langle G, \mathbb{I}((\mathbf{D}(f(\cdot)))(F^{(1)}))\rangle\!\rangle_{(L^2)} = \sum_{m=0}^{\infty} m! m \langle G_{\mathrm{ext}}^{(m)}, \mathrm{Pr}\langle F^{(1)}, f_{\cdot}^{(m)}\rangle_{\mathrm{EXT}}\rangle_{\mathrm{ext}}.$$

Further, since for each m $U_m G_{\mathrm{ext}}^{(m)}$ belongs to a *symmetric* tensor power of $\mathcal{H}_{-\tau}$, by (1.34), (2.13) and (2.10)

$$\begin{aligned}
m\langle G_{\mathrm{ext}}^{(m)}, \mathrm{Pr}\langle F^{(1)}, f_{\cdot}^{(m)}\rangle_{\mathrm{EXT}}\rangle_{\mathrm{ext}} &= m\langle U_m G_{\mathrm{ext}}^{(m)}, \langle F^{(1)}, f_{\cdot}^{(m)}\rangle_{\mathrm{EXT}}\rangle_{\mathcal{H}^{\otimes m}} \\
&= m\langle (U_m G_{\mathrm{ext}}^{(m)})(\cdot), \langle F^{(1)}, f_{\cdot}^{(m)}\rangle_{\mathrm{EXT}}\rangle_{\mathcal{H}^{\widehat{\otimes} m-1} \otimes \mathcal{H}} \\
&= m\langle (U_{m-1}^{-1} \otimes \mathbf{1})(U_m G_{\mathrm{ext}}^{(m)})(\cdot), \langle F^{(1)}, f_{\cdot}^{(m)}\rangle_{\mathrm{EXT}}\rangle_{\mathcal{H}_{\mathrm{ext}}^{(m-1)} \otimes \mathcal{H}} \\
&= m\langle F^{(1)} \Diamond (U_{m-1}^{-1} \otimes \mathbf{1})(U_m G_{\mathrm{ext}}^{(m)})(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}_{\mathrm{ext}}^{(m)} \otimes \mathcal{H}} \\
&= m\langle (\mathrm{Pr} \otimes \mathbf{1})[F^{(1)} \otimes (U_m G_{\mathrm{ext}}^{(m)})(\cdot)], f_{\cdot}^{(m)}\rangle_{\mathcal{H}^{\otimes m+1}} \\
&= \langle F^{(1)}(\cdot_1) \otimes (U_m G_{\mathrm{ext}}^{(m)})(\cdot_2, \dots, \cdot_m, \cdot) + F^{(1)}(\cdot_2) \otimes (U_m G_{\mathrm{ext}}^{(m)})(\cdot_3, \dots, \cdot_m, \cdot_1, \cdot) \\
&\quad + \dots + F^{(1)}(\cdot_m) \otimes (U_m G_{\mathrm{ext}}^{(m)})(\cdot_1, \dots, \cdot_{m-1}, \cdot), f_{\cdot}^{(m)}(\cdot_1, \dots, \cdot_m)\rangle_{\mathcal{H}^{\otimes m+1}},
\end{aligned}$$

and by (2.3), (1.34), (2.1) and the last calculation

$$\begin{aligned}
&(m+1)\langle G_{\text{ext}}^{(m)}, \langle F^{(1)}, \hat{f}^{(m)}\rangle_{\text{ext}}\rangle_{\text{ext}} = (m+1)\langle F^{(1)} \diamond G_{\text{ext}}^{(m)}, \mathrm{Pr} f_{\cdot}^{(m)}\rangle_{\mathcal{H}_{\text{ext}}^{(m+1)}}\\
&\quad= (m+1)\langle F^{(1)}\widehat{\otimes}(U_m G_{\text{ext}}^{(m)}), f_{\cdot}^{(m)}\rangle_{\mathcal{H}^{\otimes m+1}}\\
&\quad= (m+1)\langle [F^{(1)}\widehat{\otimes}(U_m G_{\text{ext}}^{(m)})](\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}^{\otimes m+1}}\\
&\quad= \langle F^{(1)}(\cdot)\otimes (U_m G_{\text{ext}}^{(m)})(\cdot_1,\dots,\cdot_m) + F^{(1)}(\cdot_1)\otimes (U_m G_{\text{ext}}^{(m)})(\cdot_2,\dots,\cdot_m,\cdot)\\
&\qquad + F^{(1)}(\cdot_2)\otimes (U_m G_{\text{ext}}^{(m)})(\cdot_3,\dots,\cdot_m,\cdot_1,\cdot)\\
&\qquad + \dots + F^{(1)}(\cdot_m)\otimes (U_m G_{\text{ext}}^{(m)})(\cdot_1,\dots,\cdot_{m-1},\cdot), f_{\cdot}^{(m)}(\cdot_1,\dots,\cdot_m)\rangle_{\mathcal{H}^{\otimes m+1}}\\
&\quad= \langle U_m G_{\text{ext}}^{(m)}, \langle F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}}\rangle_{\mathcal{H}^{\widehat{\otimes} m}} + m\langle G_{\text{ext}}^{(m)}, \mathrm{Pr}\langle F^{(1)}, f_{\cdot}^{(m)}\rangle_{\text{EXT}}\rangle_{\text{ext}},
\end{aligned}$$

here $\langle F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}} \in \mathcal{H}_\tau^{\widehat{\otimes} m}$ is a partial pairing.

Later, by (1.34) and (1.37)

$$\begin{aligned}
&\sum_{m=0}^{\infty} m!\langle U_m G_{\text{ext}}^{(m)}, \langle F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}}\rangle_{\mathcal{H}^{\widehat{\otimes} m}}\\
&\quad= \sum_{m=0}^{\infty} m!\langle G_{\text{ext}}^{(m)}, \langle F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}}\rangle_{\mathcal{H}_{\text{ext}}^{(m)}}\\
&\quad= \sum_{m=0}^{\infty} m!\langle G_{\text{ext}}^{(m)} \otimes F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}_{\text{ext}}^{(m)}\otimes\mathcal{H}}\\
&\quad= \langle\!\langle G\otimes F^{(1)}(\cdot), \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, f_{\cdot}^{(m)}\rangle :\rangle\!\rangle_{(L^2)\otimes\mathcal{H}}\\
&\quad= \langle\!\langle G\otimes F^{(1)}, f\rangle\!\rangle_{(L^2)\otimes\mathcal{H}} = \langle\!\langle G, \langle F^{(1)}(\cdot), f(\cdot)\rangle_{\mathcal{H}}\rangle\!\rangle_{(L^2)},
\end{aligned} \tag{2.17}$$

where $\langle F^{(1)}(\cdot), f(\cdot)\rangle_{\mathcal{H}} \equiv \langle F^{(1)}(\cdot), f(\cdot)\rangle \in (\mathcal{H}_\tau)_q$ is a partial pairing.

So, for arbitrary $G \in (\mathcal{H}_{-\tau})_{-q}$

$$\langle\!\langle G, (D(\mathbb{I}(f)))(F^{(1)})\rangle\!\rangle_{(L^2)} = \langle\!\langle G, \mathbb{I}((\mathbf{D}(f(\cdot)))(F^{(1)}))\rangle\!\rangle_{(L^2)} + \langle\!\langle G, \langle F^{(1)}(\cdot), f(\cdot)\rangle\rangle\!\rangle_{(L^2)},$$

from where (2.16) follows. □

Remark. With the notation from the proof of Theorem 2.3, by (1.17)

$$\sum_{m=0}^{\infty} m!\langle G_{\text{ext}}^{(m)}, \langle F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}}\rangle_{\mathcal{H}_{\text{ext}}^{(m)}} = \langle\!\langle G, \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \langle F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}}\rangle :\rangle\!\rangle_{(L^2)},$$

therefore by (2.17) we obtain

$$\langle F^{(1)}(\cdot), f(\cdot)\rangle_{\mathcal{H}} \equiv \langle F^{(1)}(\cdot), f(\cdot)\rangle = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \langle F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}}\rangle : .$$

Using this representation one can easily verify that actually for $f \in (\mathcal{H}_\tau)_{q+1}\otimes\mathcal{H}_\tau$ $\langle F^{(1)}(\cdot), f(\cdot)\rangle \in (\mathcal{H}_\tau)_{q+1}$ (we recall that by a generalized Cauchy-Bunyakovsky inequality $|\langle F^{(1)}(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}}|_{\mathcal{H}_\tau^{\widehat{\otimes} m}} \le |F^{(1)}|_{\mathcal{H}_{-\tau}} |f_{\cdot}^{(m)}|_{\mathcal{H}_\tau^{\widehat{\otimes} m}\otimes\mathcal{H}_\tau}$).

As is easily seen, the results of Theorems 2.1, 2.2, 2.3 hold true (up to obvious modifications) if we consider the operators of stochastic differentiation on the spaces $(\mathcal{H}_\tau)$ or $(\mathcal{D})$.

2.2. **Interconnection between operators of stochastic differentiation on the spaces $(\mathcal{H}_\tau)_q$ and (L^2).** In [9, 8] the operators of stochastic differentiation were introduced and studied on the so-called spaces of regular test and generalized functions and, in particular, on the space (L^2). Since the spaces $(\mathcal{H}_\tau)_q$ are embedded into (L^2), it is natural to raise a question about interconnection between operators of stochastic differentiation on $(\mathcal{H}_\tau)_q$ and on (L^2). Actually, the answer is very simple: roughly speaking, the operator of stochastic differentiation on $(\mathcal{H}_\tau)_q$ is the restriction to $(\mathcal{H}_\tau)_q$ of the corresponding operator on (L^2). In this subsection we'll explain this fact in detail.

First, let us recall the definition of the operator of stochastic differentiation on (L^2). Let $n, m \in \mathbb{Z}_+$. Consider a function $H : \mathbb{R}_+^{n+m} \to \mathbb{R}$. Denote

$$\widetilde{H}(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m})$$
$$:= \begin{cases} H(u_1, \dots, u_{n+m}), & \text{if for all } i \in \{1, \dots, n\}, j \in \{n+1, \dots, n+m\}\ u_i \neq u_j \\ 0, & \text{in other cases} \end{cases}.$$

Let $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $G^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$. We select representatives (functions) $\dot{f}^{(n)} \in F^{(n)}$, $\dot{g}^{(m)} \in G^{(m)}$ from the equivalence classes $F^{(n)}$, $G^{(m)}$, and set $H(u_1, \dots, u_{n+m}) := \dot{f}^{(n)}(u_1, \dots, u_n) \cdot \dot{g}^{(m)}(u_{n+1}, \dots, u_{n+m})$. Denote $\widetilde{f^{(n)}g^{(m)}} := \widetilde{H}$. Let $\widehat{f^{(n)}g^{(m)}}$ be the symmetrization of $\widetilde{f^{(n)}g^{(m)}}$ by all variables, $F^{(n)}\widehat{\diamond}G^{(m)} \in \mathcal{H}_{\text{ext}}^{(n+m)}$ be the equivalence class in $\mathcal{H}_{\text{ext}}^{(n+m)}$ that is generated by $\widehat{f^{(n)}g^{(m)}}$ (i.e., $\widehat{f^{(n)}g^{(m)}} \in F^{(n)}\widehat{\diamond}G^{(m)}$).

Lemma. ([8]). *The element $F^{(n)}\widehat{\diamond}G^{(m)} \in \mathcal{H}_{\text{ext}}^{(n+m)}$ is well defined (in particular, this element does not depend on the choice of representatives from $F^{(n)}$ and $G^{(m)}$) and*

$$|F^{(n)}\widehat{\diamond}G^{(m)}|_{\text{ext}} \leq |F^{(n)}|_{\text{ext}}|G^{(m)}|_{\text{ext}}. \tag{2.18}$$

Let $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $f^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$, $m > n$. We define a "product" $(F^{(n)}, f^{(m)})_{\text{ext}} \in \mathcal{H}_{\text{ext}}^{(m-n)}$ by setting for each $G^{(m-n)} \in \mathcal{H}_{\text{ext}}^{(m-n)}$

$$(G^{(m-n)}, (F^{(n)}, f^{(m)})_{\text{ext}})_{\text{ext}} = (F^{(n)}\widehat{\diamond}G^{(m-n)}, f^{(m)})_{\text{ext}}. \tag{2.19}$$

Since by (2.18)

$$|(F^{(n)}\widehat{\diamond}G^{(m-n)}, f^{(m)})_{\text{ext}}| \leq |F^{(n)}\widehat{\diamond}G^{(m-n)}|_{\text{ext}}|f^{(m)}|_{\text{ext}} \leq |F^{(n)}|_{\text{ext}}|G^{(m-n)}|_{\text{ext}}|f^{(m)}|_{\text{ext}},$$

this definition is well posed and $|(F^{(n)}, f^{(m)})_{\text{ext}}|_{\text{ext}} \leq |F^{(n)}|_{\text{ext}}|f^{(m)}|_{\text{ext}}$.

Definition. Let $n \in \mathbb{N}$, $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$. We define a linear operator

$$(\widehat{D}^n \circ)(F^{(n)}) : (L^2) \to (L^2) \tag{2.20}$$

by setting for $F \in (L^2)$

$$\begin{aligned} (\widehat{D}^n F)(F^{(n)}) &:= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} :\langle \circ^{\otimes m-n}, (F^{(n)}, f^{(m)})_{\text{ext}} \rangle: \\ &\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} :\langle \circ^{\otimes m}, (F^{(n)}, f^{(m+n)})_{\text{ext}} \rangle:, \end{aligned} \tag{2.21}$$

where $f^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$ are the kernels from decomposition (1.5) for F. The domain of this operator is

$$\text{dom}((\widehat{D}^n \circ)(F^{(n)}))$$
$$:= \Big\{ F \in (L^2) : \|(\widehat{D}^n F)(F^{(n)})\|_{(L^2)}^2 = \sum_{m=0}^{\infty} \frac{((m+n)!)^2}{m!} |(F^{(n)}, f^{(m+n)})_{\text{ext}}|_{\text{ext}}^2 < \infty \Big\}.$$

Comparing the constructions of operators (2.5) and (2.20), one can conclude that for a study of the interconnection between these operators it is necessary to study the interconnection between products $\diamond$ and $\widehat{\diamond}$.

Proposition. *Let $n, m \in \mathbb{Z}_+$. For $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \subset \mathcal{H}_{-\tau}^{(n)}$ and $G^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)} \subset \mathcal{H}_{-\tau}^{(m)}$*

$$F^{(n)} \diamond G^{(m)} = F^{(n)} \widehat{\diamond} G^{(m)} \in \mathcal{H}_{\text{ext}}^{(n+m)} \subset \mathcal{H}_{-\tau}^{(n+m)} \tag{2.22}$$

(more exactly, $F^{(n)} \diamond G^{(m)} = OF^{(n)} \widehat{\diamond} G^{(m)}$, where $O : \mathcal{H}_{\text{ext}}^{(n+m)} \to \mathcal{H}_{-\tau}^{(n+m)}$ is the embedding operator).

Proof. For $n = 0$ or $m = 0$ (2.22) is, obviously, fulfilled, therefore we consider the case $n, m \in \mathbb{N}$. At first we establish that for each $\lambda \in \mathcal{D}$

$$\langle F^{(n)} \diamond G^{(m)}, \lambda^{\otimes n+m} \rangle_{\text{ext}} = (F^{(n)} \widehat{\diamond} G^{(m)}, \lambda^{\otimes n+m})_{\text{ext}}. \tag{2.23}$$

It follows from the definition of $F^{(n)} \diamond G^{(m)}$ (see (2.1)) that

$$\begin{aligned} \langle F^{(n)} \diamond G^{(m)}, \lambda^{\otimes n+m} \rangle_{\text{ext}} &= \langle U_{n+m} U_{n+m}^{-1} [(U_n F^{(n)}) \widehat{\otimes} (U_m G^{(m)})], \lambda^{\otimes n+m} \rangle \\ &= \langle (U_n F^{(n)}) \otimes (U_m G^{(m)}), \lambda^{\otimes n+m} \rangle = \langle U_n F^{(n)}, \lambda^{\otimes n} \rangle \langle U_m G^{(m)}, \lambda^{\otimes m} \rangle \\ &= \langle F^{(n)}, \lambda^{\otimes n} \rangle_{\text{ext}} \langle G^{(m)}, \lambda^{\otimes m} \rangle_{\text{ext}}. \end{aligned} \tag{2.24}$$

On the other hand, using the above accepted notation, by (1.8) we obtain

$$\begin{aligned} (F^{(n)} \widehat{\diamond} G^{(m)}, \lambda^{\otimes n+m})_{\text{ext}} &= (\widehat{f^{(n)} g^{(m)}}, \lambda^{\otimes n+m})_{\text{ext}} \\ &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}:\ j = 1, \dots, k,\ l_1 > l_2 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n+m}} \frac{(n+m)!}{s_1! \cdots s_k!} \Big(\frac{\|p_{l_1}\|_\nu}{l_1!} \Big)^{2s_1} \cdots \Big(\frac{\|p_{l_k}\|_\nu}{l_k!} \Big)^{2s_k} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} \widehat{f^{(n)} g^{(m)}} (\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \\ &\times \lambda^{l_1}(u_1) \cdots \lambda^{l_k}(u_{s_1 + \dots + s_k}) \, du_1 \cdots du_{s_1 + \dots + s_k}. \end{aligned} \tag{2.25}$$

Without loss of generality one can think that $m \geq n$, and representatives $\dot{f}^{(n)} \in F^{(n)}$, $\dot{g}^{(m)} \in G^{(m)}$ are symmetric functions. Taking into consideration this symmetry we can write (see [8] for a detailed explanation)

$$\begin{aligned} \widehat{f^{(n)} g^{(m)}} (u_1, \dots, u_{n+m}) &= \frac{n! m!}{(n+m)!} \\ \times \sum_{\substack{1 \leq p_1, \dots, p_n \leq n, n+1 \leq q_1, \dots, q_m \leq n+m \\ 0 \leq r \leq n, p_1 < \dots < p_r, p_{r+1} < \dots < p_n, q_1 < \dots < q_{n-r}, q_{n-r+1} < \dots < q_m}} & \widetilde{f^{(n)} g^{(m)}} (u_{p_1}, \dots, u_{p_r}, u_{q_1}, \dots, u_{q_{n-r}}; \\ & u_{p_{r+1}}, \dots, u_{p_n}, u_{q_{n-r+1}}, \dots, u_{q_m}) \end{aligned} \tag{2.26}$$

(for $r = n$ the argument in the right hand side of (2.26) is $(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m})$; for $r = 0$ this argument is $(u_{q_1}, \dots, u_{q_n}; u_1, \dots, u_n, u_{q_{n+1}}, \dots, u_{q_m})$). To put it in another way, the arguments of $\widetilde{f^{(n)} g^{(m)}}$ in the right hand side of (2.26) are u_j, $j \in \{1, \dots, n+m\}$, where the indexes of n first and m last arguments (before and after ';') are (independently) ordered in ascending. (Note that we selected arrangement in ascending when we used the symmetric property of $\dot{f}^{(n)}$ and $\dot{g}^{(m)}$ because this is convenient for a consequent calculation.) Substituting (2.26) in (2.25), we obtain

$$
\begin{aligned}
&(F^{(n)}\widehat{\diamond}G^{(m)},\lambda^{\otimes n+m})_{\mathrm{ext}}\\
&= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+m}} \frac{n!m!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k}\\
&\times\Big[\int_{\mathbb{R}_+^{s_1+\dots+s_k}}\widetilde{f^{(n)}g^{(m)}}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})\\
&\times\lambda^{l_1}(u_1)\cdots\lambda^{l_k}(u_{s_1+\dots+s_k})\,du_1\cdots du_{s_1+\dots+s_k}+\cdots\Big].
\end{aligned}
\tag{2.27}
$$

We say that a collection of equal to each other arguments (e.g., $(\underbrace{u_1,\dots,u_1}_{l_1})$) is called a *procession*. It follows from the ordering in ascending of indexes in (2.26) and in the statement for $(\cdot,\cdot)_{\mathrm{ext}}$ (see (1.8)) that in summands in interior sums $[\cdots]$ from (2.27) processions can "tear" only so that different parts of a "torn" procession will be for different parties from ';'; processions being for one side from ';' do not switch places; and elements in processions do not switch places. Further, it follows from a construction of $\widetilde{f^{(n)}g^{(m)}}$ that summands in interior sums $[\cdots]$ from (2.27), in which a procession is divided by ';', are equal to zero. Another summands (if there exist for a collection k,l_j,s_j) disintegrate on groups of equal to each other integrals. These groups arise by means of mutual transpositions of processions with equal quantity of members, which are placed before ';' and after ';'. It is clear that if there are s' processions of length l before ';' and s'' processions of length l after ';' then by means of mutual transpositions of these processions one can obtain

$$\frac{(s'+s'')!}{s'!s''!}$$

equal to each other summands.

So, nonzero summands in the right hand side of (2.27) are related to the expressions

$$l_1s_1+\dots+l_ks_k=n+m \tag{2.28}$$

that can be presented in the form

$$
\begin{gathered}
l'_1s'_1+\dots+l'_{k'}s'_{k'}=n,\quad l''_1s''_1+\dots+l''_{k''}s''_{k''}=m,\\
k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''}\in\mathbb{N},\\
l'_1>\dots>l'_{k'},\quad l''_1>\dots>l''_{k''}
\end{gathered}
\tag{2.29}
$$

(the first sum in (2.29) corresponds to first n arguments of $\widetilde{f^{(n)}g^{(m)}}$, the second one corresponds to last m arguments). Now for each s_j from (2.28) either there exists $s'_i=s_j$ ($l'_i=l_j$) or there exists $s''_i=s_j$ ($l''_i=l_j$) or there exist s'_i and s''_w such that

$$s'_i+s''_w=s_j$$

($l'_i=l''_w=l_j$). Inequalities for $l'_\cdot$, $l''_\cdot$ in (2.29) follow from inequalities $l_1>\dots>l_k$ and ordering of indexes in (2.26) (more long processions have smaller indexes of arguments).

We will replace each group of equal to each other summands in the right hand side of (2.27) by a representative multiplied by a quantity of summands in the group. Now, taking into account that $w^{s'+s''} = w^{s'}w^{s''}$, one can rewrite (2.27) in the form

$$
\begin{aligned}
&(F^{(n)}\widehat{\diamond}G^{(m)}, \lambda^{\otimes n+m})_{\mathrm{ext}} \\
&= \sum_{\substack{k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''}\in\mathbb{N},\\ l'_1>\dots>l'_{k'},\ l''_1>\dots>l''_{k''},\\ l'_1s'_1+\dots+l'_{k'}s'_{k'}=n,\ l''_1s''_1+\dots+l''_{k''}s''_{k''}=m}} \frac{n!m!}{s'_1!\dots s'_{k'}!s''_1!\dots s''_{k''}!} \\
&\quad\times \left(\frac{\|p_{l'_1}\|_\nu}{l'_1!}\right)^{2s'_1} \dots \left(\frac{\|p_{l'_{k'}}\|_\nu}{l'_{k'}!}\right)^{2s'_{k'}} \left(\frac{\|p_{l''_1}\|_\nu}{l''_1!}\right)^{2s''_1} \dots \left(\frac{\|p_{l''_{k''}}\|_\nu}{l''_{k''}!}\right)^{2s''_{k''}} \\
&\quad\times \int_{\mathbb{R}_+^{s'_1+\dots+s'_{k'}}} \dot{f}^{(n)}(\underbrace{u_1,\dots,u_1}_{l'_1},\dots,\underbrace{u_{s'_1+\dots+s'_{k'}},\dots,u_{s'_1+\dots+s'_{k'}}}_{l'_{k'}}) \\
&\quad\times \lambda^{l'_1}(u_1)\cdots\lambda^{l'_{k'}}(u_{s'_1+\dots+s'_{k'}})\,du_1\cdots du_{s'_1+\dots+s'_{k'}} \\
&\quad\times \int_{\mathbb{R}_+^{s''_1+\dots+s''_{k''}}} \dot{g}^{(m)}(\underbrace{u_{n+1},\dots,u_{n+1}}_{l''_1},\dots,\underbrace{u_{n+s''_1+\dots+s''_{k''}},\dots,u_{n+s''_1+\dots+s''_{k''}}}_{l''_{k''}}) \\
&\quad\times \lambda^{l''_1}(u_{n+1})\cdots\lambda^{l''_{k''}}(u_{n+s''_1+\dots+s''_{k''}})\,du_{n+1}\cdots du_{n+s''_1+\dots+s''_{k''}} \\
&= (\dot{f}^{(n)},\lambda^{\otimes n})_{\mathrm{ext}}(\dot{g}^{(m)},\lambda^{\otimes m})_{\mathrm{ext}} = (F^{(n)},\lambda^{\otimes n})_{\mathrm{ext}}(G^{(m)},\lambda^{\otimes m})_{\mathrm{ext}} \\
&= \langle F^{(n)},\lambda^{\otimes n}\rangle_{\mathrm{ext}}\langle G^{(m)},\lambda^{\otimes m}\rangle_{\mathrm{ext}}
\end{aligned}
$$

(here we used a nonatomicity of the Lebesgue measure). Comparing this result with (2.24), we obtain (2.23).

Further, $F^{(n)}\widehat{\diamond}G^{(m)} \in \mathcal{H}_{\mathrm{ext}}^{(n+m)} \subset \mathcal{D}'^{(n+m)}$ generates a linear continuous functional on $\mathcal{D}^{\widehat{\otimes}n+m}$ by a formula $\widehat{l}(\cdot) := (F^{(n)}\widehat{\diamond}G^{(m)}, \cdot)_{\mathrm{ext}}$. On the other hand, at the same time $F^{(n)} \diamond G^{(m)} \in \mathcal{H}_{-\tau}^{(n+m)} \subset \mathcal{D}'^{(n+m)}$ generates a linear continuous functional on $\mathcal{D}^{\widehat{\otimes}n+m}$ by a formula $l(\cdot) := \langle F^{(n)} \diamond G^{(m)}, \cdot\rangle_{\mathrm{ext}}$. By (2.23) $\widehat{l} = l$ on a total in $\mathcal{D}^{\widehat{\otimes}n+m}$ set $\{\lambda^{\otimes n+m} : \lambda \in \mathcal{D}\}$, therefore these *linear continuous* functionals coincide on $\mathcal{D}^{\widehat{\otimes}n+m}$, whence (2.22) follows. □

As a corollary of this Proposition we obtain the next statement.

Theorem 2.4. *For arbitrary $n \in \mathbb{N}$ and $F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)} \subset \mathcal{H}_{-\tau}^{(n)}$ the restriction of the operator $(\widehat{D}^n\circ)(F^{(n)})$ to the space $(\mathcal{H}_\tau)_q$ coincides with the operator $(D^n\circ)(F^{(n)})$.*

Proof. By (2.6) and (2.21) it is sufficient to show that for arbitrary $m > n$ and $f^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes}m} \subset \mathcal{H}_{\mathrm{ext}}^{(m)}$ $\langle F^{(n)}, f^{(m)}\rangle_{\mathrm{ext}} = (F^{(n)}, f^{(m)})_{\mathrm{ext}}$ in $\mathcal{H}_{\mathrm{ext}}^{(m-n)}$. In fact, by (2.3), (2.22) and (2.19) for arbitrary $G^{(m-n)} \in \mathcal{H}_{\mathrm{ext}}^{(m-n)} \subset \mathcal{H}_{-\tau}^{(m-n)}$ we obtain

$$
\begin{aligned}
(G^{(m-n)}, \langle F^{(n)}, f^{(m)}\rangle_{\mathrm{ext}})_{\mathrm{ext}} &= \langle G^{(m-n)}, \langle F^{(n)}, f^{(m)}\rangle_{\mathrm{ext}}\rangle_{\mathrm{ext}} = \langle F^{(n)} \diamond G^{(m-n)}, f^{(m)}\rangle_{\mathrm{ext}} \\
= \langle F^{(n)}\widehat{\diamond}G^{(m-n)}, f^{(m)}\rangle_{\mathrm{ext}} &= (F^{(n)}\widehat{\diamond}G^{(m-n)}, f^{(m)})_{\mathrm{ext}} = (G^{(m-n)}, (F^{(n)}, f^{(m)})_{\mathrm{ext}})_{\mathrm{ext}},
\end{aligned}
$$

whence the result follows. □

Acknowledgments. I am very grateful to the Referee for many useful remarks and advices.

References

1. F. E. Benth, *The Gross derivative of generalized random variables*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **2** (1999), no. 3, 381–396.

2. F. E. Benth, G. Di Nunno, A. Lokka, B. Oksendal, F. Proske, *Explicit representation of the minimal variance portfolio in markets driven by Lévy processes*, Math. Finance **13** (2003), no. 1, 55–72.
3. Yu. M. Berezansky, Yu. G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis,* Vols. 1, 2, Kluwer Academic Publishers, Dordrecht–Boston–London, 1995. (Russian edition: Naukova Dumka, Kiev, 1988)
4. Yu. M. Berezansky, Z. G. Sheftel, G. F. Us, *Functional Analysis,* Vols. 1, 2, Birkhäuser Verlag, Basel–Boston–Berlin, 1996; 3rd ed., Institute of Mathematics NAS of Ukraine, Kyiv, 2010. (Russian edition: Vyshcha Shkola, Kiev, 1990)
5. J. Bertoin, *Lévy Processes*, Cambridge University Press, Cambridge, 1996.
6. G. Di Nunno, B. Oksendal, F. Proske, *Malliavin Calculus for Lévy Processes with Applications to Finance*, Universitext, Springer-Verlag, Berlin, 2009.
7. G. Di Nunno, B. Oksendal, F. Proske, *White noise analysis for Lévy processes*, J. Funct. Anal. **206** (2004), no. 1, 109–148.
8. M. M. Dyriv, N. A. Kachanovsky, *On operators of stochastic differentiation on spaces of regular test and generalized functions of Lévy white noise analysis*, Carpathian Math. Publ. **6** (2014), no. 2, 212–229.
9. M. M. Dyriv, N. A. Kachanovsky, *Operators of stochastic differentiation on spaces of regular test and generalized functions in the Lévy white noise analysis*, Research Bulletin of National Technical University of Ukraine "Kyiv Polytechnic Institute" 2014, (4), 36–40.
10. M. M. Dyriv, N. A. Kachanovsky, *Stochastic integrals with respect to a Levy process and stochastic derivatives on spaces of regular test and generalized functions*, Research Bulletin of National Technical University of Ukraine "Kyiv Polytechnic Institute" 2013, (4), 27–30.
11. I. M. Gelfand, N. Ya. Vilenkin, *Generalized Functions,* Vol. IV*: Applications of Harmonic Analysis*, Academic Press, New York–London, 1964. (Russian edition: Fizmatgiz, Moscow, 1961)
12. I. I. Gihman, A. V. Skorohod, *The Theory of Stochastic Processes,* Vol. 2, Nauka, Moscow, 1973. (Russian); English transl. Springer-Verlag, Berlin—Heidelberg—New York, 1975.
13. H. Holden, B. Oksendal, J. Uboe, T.-S. Zhang, *Stochastic Partial Differential Equations – a Modeling, White Noise Functional Approach*, Birkhäuser, Boston, 1996.
14. K. Itô, *Spectral type of the shift transformation of differential processes with stationary increments*, Trans. Amer. Math. Soc. **81** (1956), 253–263.
15. Yu. M. Kabanov, *On extended stochastic integrals*, Teorija Verojatnostej i ee Pril. **20** (1975), no. 4, 725–737. (Russian); English transl. Theory Probab. Appl. **20** (1975), no. 4, 710–722.
16. Yu. M. Kabanov, A. V. Skorohod, *Extended stochastic integrals*, Proc. School-Symposium "Theory Stoch. Proc." (Druskininkai, Lietuvos Respublika, November 25–30, 1974). Inst. Phys. Math., Vilnus, 1975, pp. 123–167.
17. N. A. Kachanovsky, *A generalized Malliavin derivative connected with the Poisson- and Gamma-measures*, Methods Funct. Anal. Topology **9** (2003), no. 3, 213–240.
18. N. A. Kachanovsky, *A generalized stochastic derivative on the Kondratiev-type space of regular generalized functions of Gamma white noise*, Methods Funct. Anal. Topology **12** (2006), no. 4, 363–383.
19. N. A. Kachanovsky, *Generalized stochastic derivatives on a space of regular generalized functions of Meixner white noise*, Methods Funct. Anal. Topology **14** (2008), no. 1, 32–53.
20. N. A. Kachanovsky, *Generalized stochastic derivatives on parametrized spaces of regular generalized functions of Meixner white noise*, Methods Funct. Anal. Topology **14** (2008), no. 4, 334–350.
21. N. A. Kachanovsky, *Extended stochastic integrals with respect to a Lévy process on spaces of generalized functions*, Mathematical Bulletin of Taras Shevchenko Scientific Society **10** (2013), 169–188.
22. N. A. Kachanovsky, *On an extended stochastic integral and the Wick calculus on the connected with the generalized Meixner measure Kondratiev-type spaces*, Methods Funct. Anal. Topology **13** (2007), no. 4, 338–379.
23. N. A. Kachanovsky, *On extended stochastic integrals with respect to Lévy processes*, Carpathian Math. Publ. **5** (2013), no. 2, 256–278.
24. N. A. Kachanovsky, V. A. Tesko, *Stochastic integral of Hitsuda–Skorokhod type on the extended Fock space*, Ukrainian Math. J. **61** (2009), no. 6, 873–907.
25. E. W. Lytvynov, *Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), no. 1, 73–102.

26. P. A. Meyer, *Quantum Probability for Probabilists*, Lect. Notes in Math., vol. 1538, Springer–Verlag, Berlin, 1993.
27. D. Nualart, W. Schoutens, *Chaotic and predictable representations for Lévy processes*, Stochastic Process. Appl. **90** (2000), no. 1, 109–122.
28. P. Protter, *Stochastic Integration and Differential Equations*, Springer-Verlag, Berlin, 1990.
29. K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999.
30. W. Schoutens, *Stochastic Processes and Orthogonal Polynomials*, Lect. Notes in Statist., vol. 146, Springer–Verlag, New York, 2000.
31. A. V. Skorohod, *Integration in Hilbert Space*, Springer–Verlag, Berlin—New York—Heidelberg, 1974. (Russian edition: Nauka, Moscow, 1974).
32. A. V. Skorohod, *On a generalization of a stochastic integral*, Teorija Verojatnostej i ee Pril. **20** (1975), no. 2, 223–238. (Russian); English transl. Theory Probab. Appl. **20** (1975), no. 2, 219–233.
33. J. L. Solé, F. Utzet, J. Vives, *Chaos expansions and Malliavin calculus for Lévy processes*, Stoch. Anal. and Appl., Abel Symposium, vol. 2, Springer, Berlin, 2007, pp. 595–612.
34. D. Surgailis, *On L^2 and non-L^2 multiple stochastic integration*, Lect. Notes in Control and Information Sciences, vol. 36, Springer-Verlag, New York, 1981, 212–226.
35. A. S. Ustunel, *An Introduction to Analysis on Wiener Space*, Lect. Notes in Math., vol. 1610, Springer-Verlag, Berlin, 1995.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: nkachano@gmail.com

Received 06/02/2015; Revised 16/02/2015