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ON A CLASS OF GENERALIZED STIELTJES CONTINUED FRACTIONS

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To Yu. M. Berezanskii with great respect and admiration

Abstract. With each sequence of real numbers $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$ two kinds of continued fractions are associated, — the so-called P-fraction and a generalized Stieltjes fraction that, in the case when $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$ is a sequence of moments of a probability measure on $\mathbb{R}_+$, coincide with the J-fraction and the Stieltjes fraction, respectively. A subclass $\mathcal{H}^{reg}$ of regular sequences is specified for which explicit formulas connecting these two continued fractions are found. For $\mathbf{s} \in \mathcal{H}^{reg}$ the Darboux transformation of the corresponding generalized Jacobi matrix is calculated in terms of the generalized Stieltjes fraction.

1. Introduction

Let $\mathcal{H}_0$ be the class of sequences $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$ of real numbers $s_j = \overline{s}_j$, such that the Hankel matrices $S_n = (s_{i+j})_{i,j=0}^{n-1}$ are nonnegative for all $n \in \mathbb{N}$. Denote

$$D_n = \det S_n \quad (n \in \mathbb{N}). \tag{1.1}$$

As is known [1], for every sequence $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}_0$ there exists a nonnegative measure $d\sigma$ on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} t^j d\sigma(t) = s_j, \quad j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}. \tag{1.2}$$

By Hamburger-Nevanlinna theorem the Stieltjes transformation f of the measure $d\sigma$ admits the asymptotic expansion

$$(f(z) :=) \int_{\mathbb{R}} \frac{d\sigma(t)}{t - z} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \cdots \quad (z\widehat{\rightarrow}\infty), \tag{1.3}$$

where $z\widehat{\rightarrow}\infty$ means that z tends to ∞ nontangentially, that is inside the sector $\varepsilon < \arg z < \pi - \varepsilon$ for some $\varepsilon > 0$.

Assume that $\operatorname{supp} \sigma$ is contained in a finite interval I and the moment problem (1.2) is nondegenerate, i.e.,

$$D_n > 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.4}$$

Then the function $f(z)$ can be expanded into an infinite J-fraction,

$$f(z) = \mathop{\mathbf{K}}_{0}^{\infty} \left(\frac{-b_j}{z - c_j} \right) := -\cfrac{b_0}{z - c_0 - \cfrac{b_1}{z - c_1 - \cfrac{b_2}{z - c_2 - \dots}}} \tag{1.5}$$

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with $c_j = \overline{c}_j$, $b_j > 0$ for all $j \in \mathbb{Z}_+$, $b_0 = 1$ and by Markov theorem [2] the convergents $f_n(z)$ of this J-fraction converge to $f(z)$ locally uniformly in $\overline{\mathbb{C}} \setminus I$.

If, in addition, supp σ is contained in the positive half-line $\mathbb{R}_+$, then

$$D_n^{(1)} := \det(s_{i+j+1})_{i,j=0}^{n-1} > 0 \quad \text{for all} \quad n \in \mathbb{Z}_+ \tag{1.6}$$

and the function $f(z)$ can be also expanded into a Stieltjes fraction (S-fraction)

$$f(z) = -\cfrac{1}{zm_1 - \cfrac{1}{l_1 - \cfrac{1}{zm_2 - \dots}}} \tag{1.7}$$

with $m_j > 0$, $l_j > 0$ for all $j \in \mathbb{N}$. The fraction (1.7) is connected with a system of two difference equations of the first order which describe oscillations of a Stieltjes string with masses m_j concentrated on a massless thread with distances l_j between masses m_j and m_{j+1} ($j \in \mathbb{N}$), see [24] or [1, Appendix]. As is known, cf. [25, Section 28], the $2n$-th convergent of the S-fraction (1.7) coincides with the n-th convergent of the J-fraction (1.7) and can be represented as a ratio

$$f_n(z) = -\frac{Q_n(z)}{P_n(z)} \quad (n \in \mathbb{N}) \tag{1.8}$$

of two polynomials $Q_n(z)$ and $P_n(z)$ of degrees $n-1$ and n, respectively. The polynomials $P_n(z)$ and $Q_n(z)$ are called polynomials of the first kind and the second kind, respectively, and can be found as solutions of the three-term difference equation

$$b_n y_{n-1}(z) + (c_n - z) y_n(z) + y_{n+1}(z) = 0 \quad (n \in \mathbb{Z}_+), \tag{1.9}$$

subject to the initial conditions

$$P_{-1}(z) \equiv 0, \quad P_0(z) \equiv 1 \quad \text{and} \quad Q_{-1}(z) \equiv -\frac{1}{b_0}, \quad Q_0(z) \equiv 0. \tag{1.10}$$

Let J be a *monic Jacobi matrix* associated with the three-term recurrence relation (1.9) and let $J_{[m,n]}$ ($m, n \in \mathbb{N}$, $m \le n$) be a shortened monic Jacobi matrix

$$J = \begin{pmatrix} c_0 & 1 & & \\ b_1 & c_1 & 1 & \\ & b_2 & c_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad J_{[m,n]} = \begin{pmatrix} c_m & 1 & & \\ b_{m+1} & c_{m+1} & 1 & \\ & \ddots & \ddots & 1 \\ & & b_n & c_n \end{pmatrix}. \tag{1.11}$$

Then the polynomials $P_n(z)$ and $Q_n(z)$ can be calculated by (see [3, Section 7.1.2])

$$P_n(z) = \det\,(zI_n - J_{[0,n-1]}), \quad Q_n(z) = \det\,(zI_{n-1} - J_{[1,n-1]}), \quad n \in \mathbb{N}. \tag{1.12}$$

Another formula for $P_n(z)$ can be written in terms of the sequence $\mathbf{s}$

$$P_n(z) = \frac{1}{D_n} \begin{vmatrix} s_0 & \dots & s_{n-1} & s_n \\ \vdots & \ddots & & \vdots \\ s_{n-1} & \dots & s_{2n-2} & s_{2n-1} \\ 1 & \dots & z^{n-1} & z^n \end{vmatrix}, \quad n \in \mathbb{N}. \tag{1.13}$$

It follows from (1.13) that the condition (1.6) can be rewritten as

$$(-1)^n P_n(0) = \frac{D_n^{(1)}}{D_n} > 0 \quad \text{for all} \quad n \in \mathbb{Z}_+. \tag{1.14}$$

In the present paper we consider the general class $\mathcal{H}$ of real sequences $\mathbf{s} = \{s_j\}_{j=0}^\infty$, i.e., sequences $\mathbf{s}$ with real s_j. Denote by $\mathcal{H}_\kappa$ ($\kappa \in \mathbb{Z}_+$) a subclass of sequences $s \in \mathcal{H}$, such that the number $\nu_-(S_n)$ of negative eigenvalues of S_n counting multiplicities does not exceed κ and coincides with κ for n large enough.

Clearly, the determinants $D_n = \det S_n$ $(n \in \mathbb{N})$ may vanish. An index $n \in \mathbb{N}$ is called a *normal index* of the sequence $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$, if $D_n \neq 0$. The set of all normal indices of $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$ is denoted by $\mathcal{N}(\mathbf{s})$. Let $(0 <) n_1 < n_2 < \cdots$ be the set of all normal indices of the moment sequence $\mathbf{s}$, i.e.,

$$D_{n_j} \neq 0 \quad (j = 1, 2, \ldots) \quad \text{and} \quad D_i = 0 \quad \text{for all} \quad i \neq n_j. \tag{1.15}$$

With each sequence $\mathbf{s} \in \mathcal{H}$ we associate the so-called P-fraction

$$\mathop{\mathbf{K}}_{0}^{\infty}\left(\frac{-b_j}{a_j(z)}\right) := -\cfrac{b_0}{a_0(z) - \cfrac{b_1}{a_1(z) - \cfrac{b_2}{a_2(z) - \dots}}}, \tag{1.16}$$

where $b_j \neq 0$ are real numbers and a_j are monic polynomials of degree $k_j = n_{j+1} - n_j$ $(j \in \mathbb{Z}_+,\ n_0 = 0)$, see Theorem 2.1. Fractions of the form (1.16) were introduced by A. Magnus in [21] and are called **P**-fractions, see also [22]. In the case when $\mathbf{s} \in \mathcal{H}_\kappa$ for some $\kappa \in \mathbb{N}$ such construction was presented in [8].

The j-th convergent $f_j(z)$ of the P-fraction (1.16) is a rational function of degree n_j, which can be represented as

$$f_j(z) = -\frac{Q_{n_j}(z)}{P_{n_j}(z)} \quad (j \in \mathbb{N}), \tag{1.17}$$

where $P_{n_j}(z)$ is a monic polynomial of degree n_j. The polynomials $P_{n_j}(z)$ and $Q_{n_j}(z)$ are solutions of the three-term recurrence relation

$$u_{j+1}(z) - a_j(z)u_j(z) + b_j u_{j-1}(z) = 0, \quad j \in \mathbb{Z}_+ \tag{1.18}$$

subject to the initial conditions

$$u_{-1}(z) \equiv 0, \quad u_0(z) \equiv 1 \quad \text{and} \quad u_{-1}(z) \equiv -1, \quad u_0(z) \equiv 0 \quad \text{respectively.} \tag{1.19}$$

As was shown in [8] the polynomials $P_{n_j}(z)$ and $Q_{n_j}(z)$ can be found by formulas analogous to (1.12), where J is a generalized Jacobi matrix, associated with the sequences b_j, a_j $(j \in \mathbb{Z}_+)$ via (2.20)–(2.22).

With each sequence $\mathbf{s} \in \mathcal{H}$ we associate also a generalized S-fraction

$$-\cfrac{1}{zm_1(z) - \cfrac{1}{l_1(z) - \cfrac{1}{zm_2(z) - \dots}}}, \tag{1.20}$$

where $m_j(z)$ and $l_j(z)$ are real polynomials $(j \in \mathbb{N})$, see Theorem 3.1. In the case when $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}_\kappa$ and $\{s_{j+1}\}_{j=0}^{\infty} \in \mathcal{H}_0$ for some $\kappa \in \mathbb{N}$ such a continuous fraction with constant polynomials $l_j(z) \equiv l_j$ was constructed in [19] and the case when $\mathbf{s} \in \mathcal{H}_\kappa$ and $\{s_{j+1}\}_{j=0}^{\infty} \in \mathcal{H}_k$ for some $k, \kappa \in \mathbb{N}$ was considered in [10].

The objective of the present paper is to find a connection between the P-fraction (1.16) and the generalized S-fraction (1.20). This connection is established in Theorem 4.1 for a subclass $\mathcal{H}^{reg}$ of regular sequences $\mathbf{s} \in \mathcal{H}^{reg}$. Let us say that a sequence $\mathbf{s} \in \mathcal{H}$ belongs to the class $\mathcal{H}^{reg}$, if

$$P_{n_j}(0) \neq 0 \quad \text{for all} \quad n_j \in \mathcal{N}(\mathbf{s}). \tag{1.21}$$

We show that these assumptions are equivalent to the fact that all the polynomials $l_j(z)$ are of degree 0. In the case when $\mathbf{s} \in \mathcal{H}^{reg}$ an explicit formulas connecting polynomials a_j from (1.16) and polynomials m_j from (1.20) are found. Moreover, it is shown that as well as in the classical case the $2j$-th convergent of the S-fraction (1.20) coincides with the j-th convergent of the P-fraction (1.16).

The assumption (1.21) naturally appears in the factorization problem for generalized Jacobi matrices studied in [16]. Using the connections between the polynomials a_j and the polynomials m_j we find explicit formulas for calculation of a generalized Jacobi matrix J and its Darboux transformation $\mathfrak{J}$ in terms of the parameters l_j and m_j ($j \in \mathbb{N}$) of the corresponding generalized Stieltjes fraction (1.20), see Proposition 5.2.

In the present paper we study algebraic aspects of the theory of generalized Stieltjes fractions. The analytical theory of generalized Stieltjes fractions will be published elsewhere.

2. Class $\mathcal{H}$ and P-fractions

Now we extend the above considerations from the classical case ($\mathbf{s} \in \mathcal{H}_{\mathbf{0}}$) to the case of arbitrary real sequence $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}$. Consider a formal asymptotic series

$$-\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_n}{z^{n+1}} - \cdots \tag{2.1}$$

corresponding to the sequence $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}$. In the following theorem a continued fraction is constructed such that its asymptotic expansion at ∞ coincides in some sense with the asymptotic series (2.1).

Theorem 2.1. *Let $\mathbf{s} \in \mathcal{H}$ and let $n_1 < n_2 < \cdots$ be the set of all normal indices of the moment sequence $\mathbf{s}$, i.e.,*

$$D_{n_j} \neq 0 \quad (j = 1, 2, \dots) \quad \text{and} \quad D_i = 0 \quad \text{for all} \quad i \neq n_j. \tag{2.2}$$

Then there exists a unique continued fraction $\mathop{\mathbf{K}}\limits_{0}^{\infty}\left(\frac{-b_j}{a_j(z)}\right)$ (see (1.16)), such that

1. *$a_j(\lambda)$ are monic polynomials of degree $n_{j+1} - n_j$ ($n_0 = 0$);*
2. *b_j are real numbers $b_j \neq 0$ for all $j \in \mathbb{Z}_+$;*
3. *the convergents f_k of the continued fraction (1.16) have the following asymptotic expansion:*

$$f_k(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n_k-1}}{z^{2n_k}} + O\left(\frac{1}{z^{2n_k+1}}\right) \quad (z \widehat{\rightarrow} \infty). \tag{2.3}$$

The proof of Theorem 2.1 is based on the Schur algorithm developed in [5]. In [8] this algorithm was applied to sequences from $\mathcal{H}_\kappa$. For arbitrary $\mathbf{s} \in \mathcal{H}$ the $\mathbf{P}$-fraction in (1.16) was introduced by A. Magnus in [21]. For the convenience of the reader we will briefly describe the construction of the $\mathbf{P}$-fraction in (1.16).

Proof. Let n_1 be the first normal index of $\mathbf{s} \in H$. Then

$$s_0 = s_1 = \ldots = s_{n_1-2} = 0 \quad \text{and} \quad b_0 := s_{n_1-1} \neq 0. \tag{2.4}$$

Let us choose $k \in \mathbb{N}$ and let us consider the rational function

$$g_0(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n_k-1}}{z^{2n_k}}. \tag{2.5}$$

Then the function $-\frac{b_0}{g_0(z)}$ can be represented as a sum

$$-\frac{b_0}{g_0(z)} = a_0(z) + g_1(z) \tag{2.6}$$

of a real monic polynomial $a_0(z)$ of degree n_1 and a real rational function $g_1(z)$ such that

$$g_1(z) = O\left(\frac{1}{z}\right) \quad \text{as} \quad z \widehat{\rightarrow} \infty. \tag{2.7}$$

Moreover, the function $g_1(z)$ admits the following asymptotic expansion:

$$g_1(z) = -\frac{s_0^{(1)}}{z} - \frac{s_1^{(1)}}{z^2} - \cdots - \frac{s_{2(n_k-n_1)-1}^{(1)}}{z^{2(n_k-n_1)}} + O\left(\frac{1}{z^{2(n_k-n_1)+1}}\right), \tag{2.8}$$

where $s_j^{(1)}$ are real numbers, such that the first $k-1$ normal indices of the sequence $\mathbf{s}^{(1)} = \left\{s_j^{(1)}\right\}_{j=0}^{2(n_k-n_1)-1}$ coincide with

$$n_j^{(1)} = n_{j+1} - n_1 \quad (j = 1, 2, \ldots, k-1). \tag{2.9}$$

Explicit formulas for $s_j^{(1)}$ are given in [5]. In particular, $n_1^{(1)} = n_2 - n_1$ and

$$s_0^{(1)} = s_1^{(1)} = \ldots = s_{n_1^{(1)}-2}^{(1)} = 0 \quad \text{and} \quad b_1 := s_{n_1^{(1)}-1}^{(1)} \neq 0. \tag{2.10}$$

By (2.6)

$$g_0(z) = -\frac{b_0}{a_0(z) + g_1(z)}. \tag{2.11}$$

Applying subsequently this algorithm one obtains on the second step

$$-\frac{b_1}{g_1(z)} = a_1(z) + g_2(z), \tag{2.12}$$

where $a_1(z)$ is the monic polynomial of degree $n_2 - n_1$. Hence,

$$g_1(z) = -\frac{b_1}{a_1(z) + g_2(z)}. \tag{2.13}$$

The k-th step yields

$$-\frac{b_{k-1}}{g_{k-1}(z)} = a_{k-1}(z) + g_k(z), \tag{2.14}$$

where $b_{k-1} \neq 0$, $a_{k-1}(z)$ is a real monic polynomial of degree $n_k - n_{k-1}$ and $g_k(z)$ is a rational function, such that

$$g_k(z) = O\left(\frac{1}{z}\right) \quad \text{as} \quad z \widehat{\rightarrow} \infty. \tag{2.15}$$

Hence,

$$g_{k-1}(z) = -\frac{b_{k-1}}{a_{k-1}(z) + g_k(z)}. \tag{2.16}$$

Combining (2.11)–(2.16), one obtains

$$g_0(z) = -\cfrac{b_0}{a_0(z) - \cfrac{b_1}{a_1(z) - \cdots - \cfrac{b_{k-1}}{a_{k-1}(z) + g_k(z)}}}. \tag{2.17}$$

Setting $g_k(z) \equiv 0$ in (2.17) one obtains the $k-$th convergent $f_k(z)$ of $g_0(z)$ which admits the asymptotic expansion (2.3).

The uniqueness of the expansion (1.16) is implied by [13, Theorem 1.20].

The $k-$th convergent $f_k(z)$ is a rational function of degree n_k. Denote its denominator and numerator by P_{n_k} and Q_{n_k}, respectively, so that f_k takes the form

$$f_k(z) = -\frac{Q_{n_k}(z)}{P_{n_k}(z)} \quad (k \in \mathbb{N}). \tag{2.18}$$

The polynomials $P_{n_k}(z)$ and $Q_{n_k}(z)$ of the convergent $f_k(z)$ of the **P**-fraction (3.12) are solutions of the three-term recurrence relation (1.18) subject to the initial conditions (1.19) (see [25, Section 1]).

The asymptotic expansion (2.3) for the function $f_k(z)$ in (2.18) was proved in [8, Proposition 6.1]. □

Let J be the monic generalized Jacobi matrix associated with the three-term recurrence relation (1.18) (see [8]), defined by the equalities

$$J = \begin{pmatrix} C_{a_0} & D_0 & & \\ B_1 & C_{a_1} & D_1 & \\ & B_2 & C_{a_2} & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{2.19}$$

where C_{a_j} is the companion matrix for the polynomial $a_j(z) = z^{k_j} + a^{(j)}_{k_j-1} z^{k_j-1} + \dots + a^{(j)}_0$

$$C_{a_j} = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ -a^{(j)}_0 & -a^{(j)}_1 & \cdots & -a^{(j)}_{k_j-1} \end{pmatrix} \quad (j \in \mathbb{Z}_+) \tag{2.20}$$

(see [12]) and B_j, D_{j-1} are $k_j \times k_{j-1}$ and $k_j \times k_{j+1}$ matrices, respectively, given by

$$B_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ b_j & 0 & \cdots & 0 \end{pmatrix}, \quad D_{j-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad (j \in \mathbb{N}). \tag{2.21}$$

Let $J_{[0,j-1]}$ be the shortened monic generalized Jacobi matrix, defined by

$$J_{[0,j-1]} = \begin{pmatrix} C_{a_0} & D_0 & & \\ B_1 & C_{a_1} & \ddots & \\ & \ddots & \ddots & D_{j-2} \\ & & B_{j-1} & C_{a_{j-1}} \end{pmatrix} \quad (j \in \mathbb{N}), \tag{2.22}$$

As was shown in [8, Proposition 2.3]

$$P_{n_j}(z) = \det\,(zI_{n_j} - J_{[0,j-1]}) \quad \text{and} \quad Q_{n_j}(z) = \det\,(zI_{n_j - n_1} - J_{[1,j-1]}). \tag{2.23}$$

Polynomials $P_{n_j}(z)$ can be calculated also in terms of the coefficients s_i $(0 \le i \le 2n_j - 1)$ by the formula

$$P_{n_j}(z) = \frac{1}{D_{n_j}} \begin{vmatrix} s_0 & \dots & s_{n_j-1} & s_{n_j} \\ \vdots & \ddots & & \vdots \\ s_{n_j-1} & \dots & s_{2n_j-2} & s_{2n_j-1} \\ 1 & \dots & z^{n_j-1} & z^{n_j} \end{vmatrix}, \quad j \in \mathbb{N}. \tag{2.24}$$

3. Generalized S-fractions

3.1. **Unwrapping transformation.** Let a function $f(z)$ be meromorphic on $\mathbb{C} \setminus \mathbb{R}_-$. Then the function

$$\widetilde{f}(z) := z f(z^2) \tag{3.1}$$

is called the unwrapping transformation of a function $f(z)$. Alongside with this notion we will use the following notion in the class $\mathcal{H}$ of sequences.

Definition 3.1. *Let $\boldsymbol{s} \in \mathcal{H}$. Then the new sequence*

$$\widetilde{\boldsymbol{s}} = \{\widetilde{s}_j\}_{j=0}^{\infty}, \quad \text{with} \quad \widetilde{s}_{2j} = s_j \quad \text{and} \quad \widetilde{s}_{2j+1} = 0, \quad j \in \mathbb{Z}_+ \tag{3.2}$$

is called the unwrapping transformation of $\boldsymbol{s}$.

The unwrapping transformation establishes a one-to-one correspondence between the class $\mathcal{H}$ and the class

$$\mathcal{H}^{sym} := \{\widetilde{\mathbf{s}} \in \mathcal{H} : \ \widetilde{s}_{2j+1} = 0 \text{ for all } j \in \mathbb{Z}_+\}. \tag{3.3}$$

The following two examples justify the above Definition 3.1.

Example 3.1. Recall (see [14]), that a function $f \in \mathbf{R}$ is said to belong to the class $\mathbf{S}^+$, if

$$zf(z) \in \mathbf{R}. \tag{3.4}$$

If a function $f(z)$ belongs to the class $\mathbf{S}^+$, then by M.G. Kreĭn theorem it admits a holomorphic continuation to $\mathbb{C}\backslash\mathbb{R}_-$ and its unwrapping transformation is also an $\mathbf{R}-$function (see [20]). Assume that $f \in \mathbf{S}^+$ admits the following asymptotic expansion

$$f(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots \quad (z \widehat{\to} \infty) \tag{3.5}$$

with $s_j \in \mathbb{R}$ for all $j \in \mathbb{Z}_+$. Then the unwrapping transformation $\widetilde{f}(z)$ of $f(z)$ has the following asymptotic expansion:

$$\widetilde{f}(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^3} - \frac{s_2}{z^5} - \cdots \quad (z \widehat{\to} \infty), \tag{3.6}$$

so that the coefficients in (3.6) coincide with the coefficients $\widetilde{s}_j$ defined by (3.2). Hence

$$\widetilde{f}(z) \sim -\frac{\widetilde{s}_0}{z} - \frac{\widetilde{s}_1}{z^2} - \frac{\widetilde{s}_2}{z^3} - \cdots \quad (z \widehat{\to} \infty) \tag{3.7}$$

and the sequence $\widetilde{\mathbf{s}}$ belongs to the class

$$\mathcal{H}_0^{sym} := \{\widetilde{\mathbf{s}} \in \mathcal{H}_0 : \ \widetilde{s}_{2j+1} = 0 \text{ for all } j \in \mathbb{Z}_+\}. \tag{3.8}$$

Example 3.2. Recall (see [18]) that a function f meromorphic on $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ is said to belong to the class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{Z}_+$), if the kernel

$$\mathsf{N}_\omega(\lambda) = \frac{f(z) - \overline{f(\omega)}}{z - \overline{\omega}} \tag{3.9}$$

has κ negative squares on $\mathbb{C}_+$. A function $f \in \mathbf{N}_\kappa$ is said to belong to the class $\mathbf{N}_\kappa^k$, if

$$zf(z) \in \mathbf{N}_k, \tag{3.10}$$

(see [10]). In particular, the class $\mathbf{N} := \mathbf{N}_0$ coincides with the class $\mathbf{R}$, and the class $\mathbf{N}_0^0$ coincides with the class $\mathbf{S}^+$. The classes $\mathbf{N}_\kappa^0$ and $\mathbf{N}_0^k$ were introduced in [17] and [11], respectively.

As was shown in [15] the unwrapping transformation $\widetilde{f}$ of $f \in \mathbf{N}_\kappa^k$, defined by (3.1), belongs to the class $\mathbf{N}_{\kappa+k}^{sym}$. Conversely, if $\widetilde{f} \in \mathbf{N}_{\widetilde{\kappa}}^{sym}$, then there are $\kappa, k \in \mathbb{Z}_+$ and a function $f \in \mathbf{N}_\kappa^k$ such that (3.1) holds and $\widetilde{\kappa} = \kappa + k$.

If $f \in \mathbf{N}_\kappa^k$ and, in addition, f admits the asymptotic expansion (3.5) in an angle

$$\varepsilon < \arg z < 2\pi - \varepsilon \quad (0 < \varepsilon < \pi),$$

then $\mathbf{s} \in \mathcal{H}_{\kappa'}$, where $\kappa' \leq \kappa$. The unwrapping transformation $\widetilde{f}$ of f has the asymptotic expansion (3.7) in the angle $\varepsilon/2 < \arg z < \pi - \varepsilon/2$ where the sequence $\widetilde{\mathbf{s}} = \{\widetilde{s}_j\}_{j=0}^{\infty}$ is the unwrapping transformation of $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$.

Since $\widetilde{f} \in \mathbf{N}^{sym}_{\kappa+k}$ then by [18] the sequence $\widetilde{\mathbf{s}}$ of the coefficients $\widetilde{s}_j$ in (3.7) belongs to the class

$$\mathcal{H}^{sym}_{\kappa''} := \{\widetilde{\mathbf{s}} \in \mathcal{H}_{\kappa''} :\ \widetilde{s}_{2j+1} = 0 \text{ for all } j \in \mathbb{Z}_+\} \tag{3.11}$$

with $\kappa'' \le \kappa + k$.

Proposition 3.1. *Let $\widetilde{\mathbf{s}} \in \mathcal{H}^{sym}$ and let $\mathcal{N}(\widetilde{\mathbf{s}}) = \{\widetilde{n}_j\}_{j=1}^{\infty}$ be the sequence of normal indices of $\widetilde{\mathbf{s}}$. Then the corresponding continued fraction (2.11)*

$$-\cfrac{\widetilde{b}_0}{\widetilde{a}_0(z) - \cfrac{\widetilde{b}_1}{\widetilde{a}_1(z) - \cfrac{\widetilde{b}_2}{\widetilde{a}_2(z) - \dots}}} \tag{3.12}$$

has the following properties:

1. *the polynomials $\widetilde{a}_j(z)$ are odd for all $j \in \mathbb{Z}_+$;*
2. *the indices $\widetilde{n}_{2j}$ are even and the indices $\widetilde{n}_{2j+1}$ are odd for all $j \in \mathbb{Z}_+$.*

Proof. Let a natural k be fixed. Since $\widetilde{\mathbf{s}} \in \mathcal{H}^{sym}$, then the function $g_0(z)$ determined by (2.17) is odd. Therefore, all the polynomials $\widetilde{a}_0(z), \dots, \widetilde{a}_{k-1}(z)$ defined by (2.6), (2.12), (2.14) by the Schur algorithm are also odd.

Since $\deg \widetilde{a}_j = \widetilde{n}_{j+1} - \widetilde{n}_j$, then the numbers

$$\widetilde{n}_1 = \deg \widetilde{a}_0, \quad \dots, \quad \widetilde{n}_{2j+1} = \sum_{i=0}^{2j} \deg \widetilde{a}_i$$

are odd and the numbers

$$\widetilde{n}_2 = \deg \widetilde{a}_0 + \deg \widetilde{a}_1, \quad \dots, \quad \widetilde{n}_{2j} = \sum_{i=0}^{2j-1} \deg \widetilde{a}_i$$

are even. This completes the proof. □

3.2. **Construction of a generalized S-fraction.** The continued fraction (3.12) can be rewritten as

$$-\cfrac{1}{a_0(z) - \cfrac{1}{a_1(z) - \cfrac{1}{a_2(z) - \ddots}}}, \tag{3.13}$$

where $a_0(z) = \widetilde{a}_0(z)/\widetilde{b}_0$ and

$$a_{2i-1}(z) = \frac{\widetilde{b}_0 \dots \widetilde{b}_{2i-2}}{\widetilde{b}_1 \dots \widetilde{b}_{2i-1}} \widetilde{a}_{2i-1}(z), \quad a_{2i}(z) = \frac{\widetilde{b}_1 \dots \widetilde{b}_{2i-1}}{\widetilde{b}_0 \dots \widetilde{b}_{2i}} \widetilde{a}_{2i}(z) \quad (i \in \mathbb{N}). \tag{3.14}$$

Let us set

$$D_n^{(\pm m)} = \det(s_{i+j\pm m})_{i,j=0}^{n-1} \quad (s_{-1} = \dots = s_{-m} = 0, \quad m \in \mathbb{N}). \tag{3.15}$$

Theorem 3.1. *Let $\boldsymbol{s} \in \mathcal{H}$, let $\widetilde{\boldsymbol{s}}$ be the unwrapping transformation of $\boldsymbol{s}$, let $\mathcal{N}(\widetilde{\boldsymbol{s}}) = \{\widetilde{n}_j\}_{j=1}^{\infty}$ be the set of normal indices of $\widetilde{\boldsymbol{s}}$ and let ν_j and μ_j be defined by the equalities*

$$\widetilde{n}_{2j-1} = 2\nu_j - 1 \quad \text{and} \quad \widetilde{n}_{2j} = 2\mu_j, \quad j \in \mathbb{N}. \tag{3.16}$$

Then

1. *$D_{\nu_j} \ne 0$ and $D^{(1)}_{\nu_j - 1} \ne 0$ for all $j \in \mathbb{N}$;*
2. *$D_{\mu_j} \ne 0$ and $D^{(1)}_{\mu_j} \ne 0$ for all $j \in \mathbb{N}$;*

(3) *$\mathcal{N}(\boldsymbol{s})$ is a union of two sets $\{\nu_j\}_{j=1}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$ and*

$$0 < \nu_1 \le \mu_1 < \nu_2 \le \mu_2 < \cdots ; \tag{3.17}$$

(4) *If the functions $m_j(z)$ and $l_j(z)$ are defined by*

$$za_{2(j-1)}(z) = z^2 m_j(z^2) \quad \textit{and} \quad \frac{a_{2j-1}(z)}{z} = l_j(z^2), \quad j \in \mathbb{N}, \tag{3.18}$$

then $m_j(z)$ and $l_j(z)$ are polynomials of degree

$$\textit{deg } m_j = \nu_j - \mu_{j-1} - 1, \quad \deg l_j = \mu_j - \nu_j, \quad j \in \mathbb{N}, \tag{3.19}$$

and the $\boldsymbol{P}$-fraction (3.13) can be rewritten as

$$-\cfrac{1}{zm_1(z) - \cfrac{1}{l_1(z) - \cfrac{1}{zm_2(z) - \ddots}}}; \tag{3.20}$$

(5) *The convergents φ_{2j} ($j \in \mathbb{N}$) of the continued fraction (3.12) have the following asymptotic expansions:*

$$\varphi_{2j}(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2\mu_j-1}}{z^{2\mu_j}} + O\left(\frac{1}{z^{2\mu_j+1}}\right), \quad (z \widehat{\to} \infty). \tag{3.21}$$

Proof. (1) & (2). Let us set

$$\widetilde{D}_n = \det\, (\widetilde{s}_{i+j})_{i,j=0}^{n-1}. \tag{3.22}$$

As was shown in [13, Lemma 1.33]

$$\widetilde{D}_{2\nu_j-1} = D_{\nu_j} D_{\nu_j-1}^{(1)}, \quad \widetilde{D}_{2\mu_j} = D_{\mu_j} D_{\mu_j}^{(1)}. \tag{3.23}$$

This proves the statements (1) and (2).

(3) Clearly, $\nu_j, \mu_j \in \mathcal{N}(\mathbf{s})$ for all $j \in \mathbb{N}$. Conversely, if n is a normal index of $\mathbf{s}$, then at least one of the determinants $D_n^{(1)}$ or $D_{n-1}^{(1)}$ is not vanishing. Indeed, if $D_n^{(1)} = D_{n-1}^{(1)} = 0$ then it follows from the Sylvester identity

$$D_n^2 = D_n^{(1)} D_n^{(-1)} - D_{n+1}^{(-1)} D_{n-1}^{(1)} \tag{3.24}$$

that D_n should be equal to 0, what contradicts to the fact that n is a normal index of $\mathbf{s}$. Therefore, either $D_{n-1}^{(1)} \ne 0$ or $D_n^{(1)} \ne 0$, and hence either $D_{2n-1} \ne 0$ or $D_{2n} \ne 0$, respectively. In the former case $n = \nu_j$ for some $j \in \mathbb{N}$, and in the latter case $n = \mu_j$ for some $j \in \mathbb{N}$. This proves the first part of statement (3).

Let n be the minimal normal index of $\mathbf{s}$. If $n = 1$, then $s_0 \ne 0$ and hence both $D_1 = s_0$ and $D_0^{(1)} = 1$ do not equal to 0. Therefore n coincides with $\nu_1 = 1$. If $n > 1$, then

$$s_0 = \cdots = s_{n-2} = 0, \quad s_{n-1} \ne 0.$$

This implies that $D_{n-1}^{(1)} \ne 0$ and hence n coincides with ν_1 by statement (1).

Now it follows from the inequality

$$\widetilde{n}_{2j-1} < \widetilde{n}_{2j}$$

and the equality (3.16) that $\nu_j < \mu_j + 1/2$ and hence $\nu_j \le \mu_j$ for all $j \in \mathbb{N}$.

Similarly, the inequality

$$\widetilde{n}_{2j} < \widetilde{n}_{2j+1}$$

and the equalities (3.16) imply that $\mu_j < \nu_{j+1} - 1/2$ and hence $\mu_j < \nu_{j+1}$ for all $j \in \mathbb{N}$. This completes the proof of (3).

(4) Since the polynomials $a_{2(j-1)}(z)$ and $a_{2j-1}(z)$ are odd then the polynomials $za_{2(j-1)}(z)$ and $\frac{a_{2j-1}(z)}{z}$ are even and admit the representations (3.18). The formula (3.20) is immediate from (3.1), (3.13) and (3.18).

Next since deg $a_j = \widetilde{n}_{j+1} - \widetilde{n}_j$, then it follows from (3.18)

$$\begin{aligned}\deg m_j &= \frac{1}{2}(\deg a_{2j-2} - 1) = \frac{1}{2}(\widetilde{n}_{2j-1} - \widetilde{n}_{2j-2} - 1)\\ &= \frac{1}{2}(2\nu_j - 1 - 2\mu_{j-1} - 1) = \nu_j - \mu_{j-1} - 1,\end{aligned}$$

and

$$\begin{aligned}\deg l_j &= \frac{1}{2}(\deg a_{2j-1} - 1) = \frac{1}{2}(\widetilde{n}_{2j} - \widetilde{n}_{2j-1} - 1)\\ &= \frac{1}{2}(2\mu_j - 2\nu_{j-1}) = \mu_j - \nu_j.\end{aligned}$$

(5) By Theorem 2.1 the $2j$-th convergent $\widetilde{f}_{2j}(z)$ of the P-fraction (2.11) has the following asymptotic expansion:

$$\widetilde{f}_{2j}(z) \sim -\frac{\widetilde{s}_0}{z} - \frac{\widetilde{s}_1}{z^2} - \cdots - \frac{\widetilde{s}_{2\widetilde{n}_{2j}-1}}{z^{2\widetilde{n}_{2j}}} + O\left(\frac{1}{z^{2\widetilde{n}_{2j}+1}}\right) \quad (z\widehat{\rightarrow}\infty). \tag{3.25}$$

Since $\widetilde{n}_{2i} = 2\mu_i$, $\widetilde{s}_0 = s_0$, $\widetilde{s}_1 = 0$, $\widetilde{s}_2 = s_1$, ..., $\widetilde{s}_{2\widetilde{n}_{2j}-2} = s_{2\mu_j-1}$ and $\widetilde{s}_{2\widetilde{n}_{2j}-1} = 0$, then the asymptotic expansion (3.25) takes the form

$$\widetilde{f}_{2j}(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^3} - \cdots - \frac{s_{2\mu_j-1}}{z^{4\mu_j-1}} + O\left(\frac{1}{z^{4\mu_j+1}}\right) \quad (z\widehat{\rightarrow}\infty). \tag{3.26}$$

The $2j$-th convergent $\varphi_{2j}(z)$ of the S-fraction (3.20) is connected with the $2j$-th convergent $\widetilde{f}_{2j}(z)$ of the P-fraction (2.11) by the formula

$$\widetilde{f}_{2j}(z) = z\varphi_{2j}(z^2).$$

Hence, in view of (3.2) the expansion (3.26) can be rewritten in the form

$$\varphi_{2j}(z^2) \sim -\frac{s_0}{z^2} - \frac{s_1}{z^4} - \cdots - \frac{s_{2\mu_j-1}}{z^{4\mu_j}} + O\left(\frac{1}{z^{4\mu_j+2}}\right) \quad (z\widehat{\rightarrow}\infty), \tag{3.27}$$

which is equivalent to (3.21). □

Remark 3.1. The statement (3) of Theorem 3.1 was proved in [10, Section 5.2] by using the following observation from [10, Lemma 5.1].

If $D_n \neq 0$, $D_{n+1} = \cdots = D_{m-1} = 0$, $D_m \neq 0$ then the following alternative holds:

(1) either $D_n^{(1)} = 0$ and hence $D_{n-1}^{(1)} \neq 0$, $D_n^{(1)} = D_{n+1}^{(1)} \cdots = D_{m-1}^{(1)} = 0$, $D_m^{(1)} \neq 0$;

(2) or $D_n^{(1)} \neq 0$ and hence $D_n^{(1)} \neq 0$, $D_{n+1}^{(1)} = \cdots = D_{m-2}^{(1)} = 0$, $D_{m-1}^{(1)} \neq 0$.

The present proof is based on the identities (3.23) which makes it essentially simpler.

3.3. The class $\mathcal{H}^{reg}$.

Lemma 3.1. *Let* $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}$, *let* $\{n_j\}_{j=1}^{\infty}$ *be the set of normal indices of* $\mathbf{s}$ *and let* $\{P_{n_j}\}_{j=1}^{\infty}$ *be polynomials of the first kind associated with the sequence* $\mathbf{s}$. *Then the following statements are equivalent:*

(1) $P_{n_j}(0) \neq 0$ *for every* $j \in \mathbb{N}$;

(2) $D_{n_j-1}^{(1)} \neq 0$ *for every* $j \in \mathbb{N}$;

(3) $D_{n_j}^{(1)} \neq 0$ *for every* $j \in \mathbb{N}$;

(4) n_j *is a normal index of the sequence* $\{s_{j+1}\}_{j=0}^{\infty}$ *for every* $j \in \mathbb{N}$;

(5) $n_j - 1$ *is a normal index of the sequence* $\{s_{j+1}\}_{j=0}^{\infty}$ *for every* $j \in \mathbb{N}$;

(6) *the set of normal indices of the sequence* $\{s_{j+1}\}_{j=0}^{\infty}$ *coincides with*

$$\bigcup_{j=1}^{\infty}\{n_j-1,n_j\}. \tag{3.28}$$

Proof. (1) ⇔ (2). The equivalence (1) ⇔ (2) follows from (2.24)

$$P_{n_j}(0)=(-1)^{n_j+1}\frac{D_{n_j}^{(1)}}{D_{n_j}}=\frac{(-1)^{n_j+1}}{D_{n_j}}\begin{vmatrix} s_1 & \cdots & s_{n_j}\\ \cdots & \cdots & \cdots\\ s_{n_j} & \cdots & s_{2n_j-1}\end{vmatrix}\quad (j=1,2,\dots). \tag{3.29}$$

(2) ⇔ (3). If the condition (2) holds, then all normal indices n_j of $\mathbf{s}$ coincide with the indices μ_j defined by (3.16), $(j=1,2,\dots)$. Thus, the indices ν_j defined by (3.16) satisfy the following

$$\nu_j=\mu_j=n_j\quad (j=1,2,\dots). \tag{3.30}$$

Therefore,

$$D_{n_j}^{(1)}\neq 0\quad (j=1,2,\dots). \tag{3.31}$$

Conversely, if the condition (3) hold, then $\{n_j\}_{j=0}^{\infty}=\{\nu_j\}_{j=0}^{\infty}$, hence $\mu_j=\nu_j=n_j$ $(j=1,2,\dots)$. This leads to the condition (2).

The equivalences (3) ⇔ (4), (2) ⇔ (5) follow from the definition of normal indices.

The implications (6) ⇒ (4) is obvious. Let us prove the implications (4) ⇒ (6). If the condition (4) hold, then

$$D_{n_{j-1}}\neq 0,\quad D_{n_{j-1}+1}=0,\quad \dots,\quad D_{n_j-1}=0,\quad D_{n_j}\neq 0\quad (j=2,1,\dots). \tag{3.32}$$

Due to (2), (3) $D_{n_{j-1}}^{(1)}\neq 0$, $D_{n_j-1}^{(1)}\neq 0$ and by Remark 3.1 we obtain

$$D_{n_{j-1}}^{(1)}\neq 0,\quad D_{n_{j-1}+1}^{(1)}=0,\quad \dots,\quad D_{n_j-2}^{(1)}=0,\quad D_{n_j-1}^{(1)}\neq 0,\quad D_{n_j}^{(1)}\neq 0. \tag{3.33}$$

Thus, there are only two normal indices n_j-1 and n_j in the interval $(n_{j-1},n_j]$, i.e., $\mathcal{N}(\widetilde{s})=\bigcup_{j=1}^{\infty}\{n_j-1,n_j\}$. □

Definition 3.2. *Let us say* $\mathbf{s}\in\mathcal{H}^{reg}$, *if* $\mathbf{s}\in\mathcal{H}$ *and one of the equivalent conditions of Lemma 3.1 holds.*

Proposition 3.2. *Let* $\mathbf{s}\in\mathcal{H}^{reg}$ *and let the polynomials* m_j, l_j *be determined by the S-fraction (3.20). Then* $m_1=n_1-1$ *and*

$$\deg m_j=n_j-n_{j-1}-1,\quad \deg l_j=0\quad (j\in\mathbb{N}). \tag{3.34}$$

4. Class $\mathcal{H}^{reg}$. Relations between generalized S-fractions and P-fractions

To any sequence $\mathbf{s}\in\mathcal{H}$ correspond two continued fractions: the **P**-fraction (3.12) constructed in Theorem 2.1 and the generalized **S**-fraction (3.20) constructed in Theorem 3.1. In the case when $\mathbf{s}\in\mathcal{H}^{reg}$ we will present explicit formulas which connect these two types of continued fractions.

Let $\mathbf{s}\in\mathcal{H}$. One can rewrite the continued fraction (3.20) as follows

$$\cfrac{1}{-zm_1(z)+\cfrac{1}{l_1(z)+\cfrac{1}{-zm_2(z)+\dots}}}. \tag{4.1}$$

If the j–th convergent of this continued fraction is denoted by $\frac{u_j}{v_j}$, then u_j, v_j can be found as solutions of the system (see [25, Section 1])

$$\begin{cases} y_{2j}-y_{2j-2}=l_j(z)y_{2j-1},\\ y_{2j+1}-y_{2j-1}=-zm_{j+1}(z)y_{2j}\end{cases} \tag{4.2}$$

subject to the following initial conditions:

$$u_{-1} \equiv 1, \quad u_0 \equiv 0; \qquad v_{-1} \equiv 0, \quad u_0 \equiv 1. \tag{4.3}$$

The first two convergents of the continued fraction (4.1) take the form

$$\frac{u_1}{v_1} = \frac{1}{-zm_1(z)}, \quad \frac{u_2}{v_2} = \frac{l_1(z)}{-zl_1(z)m_1(z)+1}. \tag{4.4}$$

Let t_j be a linear fractional transformation defined by

$$t_{2j-1}(w) = \frac{1}{-zm_j(z)+w}, \quad t_{2j} = \frac{1}{l_j(z)+w} \quad (j \in \mathbb{N}). \tag{4.5}$$

Then the $2j-$th convergent of the S-fraction (4.1) can be represented in the form

$$\frac{u_{2j}}{v_{2j}} = t_1 \circ t_2 \circ \dots \circ t_{2j}(0). \tag{4.6}$$

The following theorem establishes a connection between the continued fractions (4.1) and (3.12) in the case when $\mathbf{s} \in \mathcal{H}^{reg}$.

Theorem 4.1. *Let $\mathbf{s} \in \mathcal{H}^{reg}$. Then the $2j$-th convergent $\frac{u_{2j}}{v_{2j}}$ of the generalized S-fraction (4.1) coincides with the j-th convergent of the P-fraction* (3.12)

$$\frac{u_{2j}}{v_{2j}} = -\cfrac{b_0}{a_0(z) - \cfrac{b_1}{a_1(z) - \dots - \cfrac{b_{j-1}}{a_{j-1}(z)}}}, \tag{4.7}$$

corresponding to the sequence $\mathbf{s}$. The parameters l_j and $m_j(z)$ ($j \in \mathbb{Z}_+$) of the generalized S-fraction (4.1) are connected with the parameters b_j and $a_j(z)$ ($j \in \mathbb{N}$) of the P-fraction (3.12) *by the equalities*

$$b_0 = \frac{1}{d_1}, \quad a_0(z) = \frac{1}{d_1}\left(zm_1(z) - \frac{1}{l_1}\right), \tag{4.8}$$

$$b_{j-1} = \frac{1}{l_{j-1}^2 d_{j-1} d_j}, \quad a_{j-1} = \frac{1}{d_j}\left(zm_j(z) - \left(\frac{1}{l_{j-1}} + \frac{1}{l_j}\right)\right) \quad (j = 2, 3, \dots), \tag{4.9}$$

where d_j is the leading coefficient of $m_j(z)$ ($j \in \mathbb{N}$).

Proof. Let us set

$$s_1(w) = t_1(w), \quad s_2(w) = t_2 \circ t_3(w), \quad \dots, \quad s_j(w) = t_{2j-2} \circ t_{2j-1}(w). \tag{4.10}$$

Let us set $\widetilde{m}_i(z) := \frac{1}{d_i} m_i(z)$. Then by (4.6) and (4.10)

$$\begin{aligned} s_i(w) &= \cfrac{1}{l_{i-1} + \cfrac{1}{-zm_i(z)+w}} = \frac{1}{l_{i-1}} - \frac{1}{l_{i-1}(1 - zl_{i-1}m_i(z) + l_{i-1}w)} \\ &= \frac{1}{l_{i-1}} - \cfrac{\frac{1}{l_{i-1}^2 d_i}}{-z\widetilde{m}_i(z) + \frac{1}{l_{i-1}d_i} + \frac{1}{d_i}w} \\ &= \frac{1}{l_{i-1}} + \cfrac{\frac{1}{l_{i-1}^2 d_i}}{\frac{1}{d_i}\left(z\widetilde{m}_i(z) - \frac{1}{l_{i-1}} - w\right)}. \end{aligned} \tag{4.11}$$

It follows from (4.5), (4.6) and (4.10) that

$$\frac{u_{2j}}{v_{2j}} = s_1 \circ s_2 \circ \cdots \circ s_j \left(\frac{1}{l_j}\right). \tag{4.12}$$

In particular,

$$\frac{u_2}{v_2} = s_1\left(\frac{1}{l_1}\right) = \frac{-1/d_1}{z\widetilde{m}_1(z) - \cfrac{1}{l_1 d_1}}, \tag{4.13}$$

$$\frac{u_4}{v_4} = s_1 \circ s_2 \left(\frac{1}{l_j}\right) = \frac{-1/d_1}{z\widetilde{m}_1(z) - \cfrac{1}{l_1 d_1} - \cfrac{\frac{1}{l_1^2 d_1 d_2}}{z\widetilde{m}_2(z) - \frac{1}{d_2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)}}. \tag{4.14}$$

Substituting (4.11) into (4.12) one obtains

$$\frac{u_{2j}}{v_{2j}} = -\cfrac{\beta_0}{\alpha_0(z) - \cfrac{\beta_1}{\alpha_1(z) - \cdots - \cfrac{\beta_{j-1}}{\alpha_{j-1}(z)}}}, \tag{4.15}$$

where

$$\beta_0 = \frac{1}{d_1}, \quad \alpha_0(z) = \frac{1}{d_1}\left(zm_1(z) - \frac{1}{l_1}\right), \tag{4.16}$$

$$\beta_{j-1} = \frac{1}{l_{j-1}^2 d_{j-1} d_j}, \quad \alpha_{j-1}(z) = \frac{1}{d_j}\left(zm_j(z) - \left(\frac{1}{l_{j-1}} + \frac{1}{l_j}\right)\right) \quad (j = 2, 3, \dots). \tag{4.17}$$

Let $f_j(z)$ be the j-th convergent of the P-fraction (2.11) and let $\varphi_{2j}(z)$ be the $2j$-th convergent of the S-fraction (3.20). The functions $f_j(z) = -\frac{Q_{n_j}(z)}{P_{n_j}(z)}$ and $\varphi_{2j}(z) = \frac{u_{2j}}{v_{2j}}$ are rational functions of degree n_j which have the asymptotic expansions (2.3) and (3.21), respectively.

Since by Lemma 3.1 $n_j = \mu_j$, then these asymptotic expansions coincide and thus the functions $f_j(z)$ and $\varphi_{2j}(z)$ coincide, since they are uniquely determined by the expansions (2.3) and (3.21). Since the expansion into the P-fraction is unique, then

$$b_j = \beta_j \quad \text{and} \quad a_j(z) = \alpha_j(z) \quad (j \in \mathbb{Z}_+).$$

This proves (4.7). □

Corollary 4.1. *Let the assumptions of Theorem 4.1 hold. Then the polynomials $m_j(z)$ can be expressed in terms of $a_{j-1}(z)$ by the formulas*

$$\frac{m_j(z)}{d_j} = \frac{a_{j-1}(z) - a_{j-1}(0)}{z} \quad (j \in \mathbb{N}). \tag{4.18}$$

and

$$\prod_{i=1}^{j}(l_i d_i)^{-1} = (-1)^j P_{n_j}(0) \quad (j \in \mathbb{N}), \tag{4.19}$$

$$l_j = \frac{(-1)^j}{P_{n_{j-1}}(0) P_{n_j}(0)} \prod_{i=0}^{j-1} b_i, \quad d_j = (P_{n_{j-1}}(0))^2 \prod_{i=0}^{j-1} b_i^{-1} \quad (j \in \mathbb{N}). \tag{4.20}$$

Proof. The formula (4.18) is immediate from (4.9).

It follows from the three-term recurrence relation (1.18), (4.8) and (4.9) that

$$P_{n_1}(0) = a_0(0) = -\frac{1}{l_1 d_1}.$$

Next since $a_1(0) = -\frac{1}{d_2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)$ and $b_1 = \frac{1}{l_1^2 d_1 d_2}$ then

$$P_{n_2}(0) = -\frac{1}{d_2}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\left(-\frac{1}{l_1 d_1}\right) - \frac{1}{l_1^2 d_1 d_2} = \frac{1}{l_1 d_1}\frac{1}{l_2 d_2}.$$

The equality (4.19) is obtained by induction.

Now the equalities in (4.20) are implied by (4.19) and (4.9). □

Corollary 4.2. *Let the assumptions of Theorem 4.1 hold. Then*

(1) *the solution $\{u_j\}_{j=0}^{\infty}$ and $\{v_j\}_{j=0}^{\infty}$ of the system (4.2), (4.3) takes the form*

$$u_{2j} = -\frac{Q_{n_j}(z)}{P_{n_j}(0)} \quad u_{2j-1} = -\gamma_j \begin{vmatrix} Q_{n_j}(z) & Q_{n_{j-1}}(z) \\ P_{n_j}(0) & P_{n_{j-1}}(0) \end{vmatrix} \quad (j \in \mathbb{N}), \tag{4.21}$$

$$v_{2j} = \frac{P_{n_j}(z)}{P_{n_j}(0)} \quad v_{2j-1} = \gamma_j \begin{vmatrix} P_{n_j}(z) & P_{n_{j-1}}(z) \\ P_{n_j}(0) & P_{n_{j-1}}(0) \end{vmatrix} \quad (j \in \mathbb{N}), \tag{4.22}$$

where $\gamma_j = (-1)^{n_j + n_{j-1}} \frac{D_{n_{j-1}}}{D_{n_j}}$ *($j \in \mathbb{N}$).*

Proof. The j-th convergent of the P-fraction (3.12) is equal to $-\frac{Q_{n_j}(z)}{P_{n_j}(z)}$ (by Theorem 2.1) and is equal to $\frac{u_{2j}}{v_{2j}}$ (by Theorem 4.1). Since

$$\deg v_{2j}(z) = \deg P_{n_j}(z) = n_j,$$

then $v_{2j}(z)$ is proportional to $P_{n_j}(z)$ and $u_{2j}(z)$ is proportional to $-Q_{n_j}(z)$. The first formulas in (4.21) and (4.22) are implied now by the normalization condition $v_{2j}(0) = 1$ ($j \in \mathbb{N}$). The second formula in (4.22) follows from the first equality in (4.2), which takes the form

$$\frac{P_{n_j}(z)}{P_{n_j}(0)} - \frac{P_{n_{j-1}}(z)}{P_{n_{j-1}}(0)} = l_j v_{2j-1}(z) \quad (j \in \mathbb{N}).$$

Hence

$$v_{2j-1}(z) = \frac{1}{l_j P_{n_j}(0) P_{n_{j-1}}(0)} \begin{vmatrix} P_{n_j}(z) & P_{n_{j-1}}(z) \\ P_{n_j}(0) & P_{n_{j-1}}(0) \end{vmatrix} \quad (j \in \mathbb{N}).$$

Now the second formula in (4.22) is implied by (3.29) and (4.24) (see Remark 4.1 below). Similarly, one proves the second formula in (4.21). □

Remark 4.1. Explicit formulas for calculation of coefficients of the polynomials $l_j(z)$ and $m_j(z)$ in (4.2) ($j \in \mathbb{N}$) in terms of the sequence $\mathbf{s} = (s_i)_{i=0}^{\infty}$ are given in [10]. Let us write the formulas for l_j and d_j ($j \in \mathbb{N}$) in the case when $\mathbf{s} = (s_i)_{i=0}^{\infty} \in \mathcal{H}^{reg}$, and hence $\nu_j = \mu_j = n_j$ ($j \in \mathbb{N}$). Then $d_1 = 1/s_{n_1 - 1}$ and

$$l_j = \frac{D_{n_j}^{(-1)}}{D_{n_j - 1}^{(1)}} - \frac{D_{n_j + 1}^{(-1)}}{D_{n_j}^{(1)}}, \quad d_{j+1} = \frac{D_{n_j}^{(k_j + 1)}}{D_{n_j + 1}^{(k_j - 1)}} - \frac{D_{n_j - 1}^{(k_j + 1)}}{D_{n_j}^{(k_j - 1)}} \quad (j \in \mathbb{N}), \tag{4.23}$$

where $k_j = \deg m_{j+1} - 1 = \mu_{j+1} - \mu_j$ ($j \in \mathbb{N}$) and $D_n^{(\pm 1)}$ are defined by (3.15). By Sylvester identity (3.24) the above formulas take the form

$$l_j = \frac{D_{n_j}^2}{D_{n_j}^{(1)} D_{n_j - 1}^{(1)}}, \quad d_{j+1} = \frac{\left(D_{n_j}^{(k_j)}\right)^2}{D_{n_j}^{(k_j - 1)} D_{n_j + 1}^{(k_j - 1)}} \quad (j \in \mathbb{N}). \tag{4.24}$$

In the classical case $\mathbf{s} = (s_i)_{i=0}^{\infty} \in \mathcal{H}_0^0 \subset \mathcal{H}^{reg}$ one has

$$\nu_j = \mu_j = n_j, \quad k_j = 1 \quad (j \in \mathbb{N})$$

and the formulas (4.24) coincide with known formulas for lengths l_j and masses m_j of a Stieltjes string, see [18, (3.15)]

$$l_j = \frac{D_j^2}{D_j^{(1)} D_{j-1}^{(1)}}, \quad d_j = m_j = \frac{\left(D_{j-1}^{(1)}\right)^2}{D_{j-1} D_j} \quad (j \in \mathbb{N}). \tag{4.25}$$

In the case when $k_j = 2$ the latter formula in (4.24) coincides with the formula for d_j from [19, Section 5.3]

$$d_{j+1} = \frac{\left(D_{n_j}^{(2)}\right)^2}{D_{n_j}^{(1)} D_{n_j+1}^{(1)}} = -\frac{D_{n_j}^{(1)} D_{n_j+1}^{(1)}}{D_{n_j} D_{n_j+2}} \quad (j \in \mathbb{N}). \tag{4.26}$$

Let us summarize these results in the case when $\mathbf{s} \in \mathcal{H}^{reg}$ and $k_j = 2$ for all $j \in \mathbb{N}$, i.e., the corresponding Stieltjes string consists of dipoles only (see [19, Section 5.4]).

Proposition 4.1. *If $\mathbf{s} = (s_i)_{i=0}^{\infty} \in \mathcal{H}^{reg}$ and $k_j = 2$ for all $j \in \mathbb{N}$ then*

(1) *the normal indices of $\mathbf{s}$ are $\nu_j = \mu_j = n_j = 2j$ and the polynomials $m_j(z)$ are of degree 1 and*

$$m_j(z) = d_j z + m_j \quad \textit{for some} \quad d_j, m_j \in \mathbb{R} \quad (j \in \mathbb{N}); \tag{4.27}$$

(2) *the corresponding generalized Stieltjes fraction takes the form*

$$-\cfrac{1}{d_1 z^2 + m_1 z - \cfrac{1}{l_1 - \cfrac{1}{d_2 z^2 + m_2 z - \cfrac{1}{l_2 \dots}}}}, \tag{4.28}$$

where the coefficients d_j, m_j, and l_j can be found by $d_1 = 1/s_1$ and

$$d_j = \frac{\left(D_{2j-2}^{(2)}\right)^2}{D_{2j-2}^{(1)} D_{2j-1}^{(1)}}, \quad m_j = \frac{D_{2j-1}^{(2)}}{D_{2j}} - \frac{D_{2j-3}^{(2)}}{D_{2j-2}}, \quad l_j = \frac{D_{2j}^2}{D_{2j}^{(1)} D_{2j-1}^{(1)}} \quad (j \in \mathbb{N}). \tag{4.29}$$

Proof. Since $\mathbf{s} \in \mathcal{H}^{reg}$ then $\nu_j = \mu_j = n_j$. The equality $n_j = 2j$ is implied by the formula $k_j = n_j - n_{j-1}$ and the assumption $k_j = 2$.

This assumption implies also that $\deg m_{j+1} = k_j - 1 = 1$ for all $j \in \mathbb{N}$. Substituting (4.27) into (4.1) one obtains (4.28).

Formulas (4.29) are implied by (4.24) and [10, (5.19)]. □

5. Darboux transformation

In this section the Darboux transformation of monic generalized Jacobi matrices associated with P-fractions is calculated explicitly in terms of the corresponding generalized S-fractions.

In the classical case the Darboux transformation of a monic Jacobi matrix was introduced in [4] (see also [9], [7]). A monic Jacobi matrix of the form (1.11) is said to admit an $LU-$ factorization $J = LU$, if it can be represented as a product of a lower-triangular two-diagonal matrix L and an upper-triangular two-diagonal matrix U

$$J = LU = \begin{pmatrix} 1 & & & \\ \mathfrak{l}_1 & 1 & & \\ & \mathfrak{l}_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \mathfrak{u}_0 & 1 & & \\ & \mathfrak{u}_1 & 1 & \\ & & \mathfrak{u}_2 & \ddots \\ & & & \ddots \end{pmatrix}, \tag{5.1}$$

where all the non-specified elements are equal 0. As was shown in [4], a monic Jacobi matrix J admits an $LU-$factorization if and only if

$$P_n(0) \neq 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{5.2}$$

In this case $\mathfrak{l}_{j+1}$ and $\mathfrak{u}_j$ $(j \in \mathbb{Z}_+)$ can be found uniquely by

$$\mathfrak{u}_j = -\frac{P_{\mathfrak{n}_{j+1}}(0)}{P_{\mathfrak{n}_j}(0)}, \quad \mathfrak{l}_{j+1} = \frac{b_{j+1}}{\mathfrak{u}_j} \quad \text{for all} \quad j \in \mathbb{Z}_+ . \tag{5.3}$$

The monic Jacobi matrix $\mathfrak{J} := UL$ is called *the Darboux transformation of* J.

Let us list some formulas connected with the Darboux transformation of the monic Jacobi matrix J in the classical case which are implied by (4.25) and [16].

Proposition 5.1. *Let* $s = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}_0^{reg}$, *let* J *be the associated monic Jacobi matrix (1.11) and let* l_j *and* m_j *be determined by the S-fraction* (1.7). *Then*

(1) *The monic Jacobi matrix* J *admits the* $LU-$*factorization* (5.1) *with the entries*

$$\mathfrak{u}_{j-1} = \frac{1}{l_j d_j}, \quad \mathfrak{l}_j = \frac{1}{l_j d_{j+1}} \quad (j \in \mathbb{N}), \tag{5.4}$$

$$c_0 = \frac{1}{d_1 l_1}, \quad c_{j-1} = \frac{1}{d_j}\left(\frac{1}{l_j} + \frac{1}{l_{j-1}}\right), \quad b_j = \frac{1}{l_j^2 d_j d_{j+1}} \quad (j \in \mathbb{N}); \tag{5.5}$$

(2) *Darboux transformation of* J *is the monic Jacobi matrix*

$$\mathfrak{J} = UL = \begin{pmatrix} \frac{1}{l_1}\left(\frac{1}{d_1} + \frac{1}{d_2}\right) & 1 & & \\ \frac{1}{l_1 l_2 d_2^2} & \frac{1}{l_2}\left(\frac{1}{d_2} + \frac{1}{d_3}\right) & 1 & \\ & \frac{1}{l_2 l_3 d_3^2} & \frac{1}{l_3}\left(\frac{1}{d_3} + \frac{1}{d_4}\right) & \ddots \\ & & \ddots & \ddots \end{pmatrix}; \tag{5.6}$$

(3) *The parameters* $\mathfrak{b}_i$ *and* $\mathfrak{c}_i$ *of the J-fraction* $\mathop{\mathbf{K}}_{0}^{\infty}\left(\frac{-\mathfrak{b}_i}{\mathfrak{c}_i}\right)$, *corresponding to the Darboux transformation* $\mathfrak{J}$ *of* J *take the form*

$$\mathfrak{b}_0 = b_0, \quad \mathfrak{b}_i = \frac{1}{l_i l_{i+1} d_{i+1}^2} \quad \mathfrak{c}_i = \frac{1}{l_i}\left(\frac{1}{d_i} + \frac{1}{d_{i+1}}\right) \quad (i \in \mathbb{N}); \tag{5.7}$$

(4) *The monic Jacobi matrix* $\mathfrak{J}$ *corresponds to the sequence* $\mathfrak{s} = \{s_{j+1}\}_{j=0}^{\infty}$;

(5) *The k-th convergents of this J-fraction take the form*

$$\mathfrak{f}_k(z) = -\frac{z\left(Q_k(z)P_{k-1}(0) - P_k(0)Q_{k-1}(z)\right)}{P_k(z)P_{k-1}(0) - P_k(0)P_{k-1}(z)}, \quad k \in \mathbb{N}. \tag{5.8}$$

Consider now a monic generalized Jacobi matrix J associated with a sequence $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}^{reg}$ via the equalities (see [8])

$$J = \begin{pmatrix} C_{a_0} & D_0 & & \\ B_1 & C_{a_1} & D_1 & \\ & B_2 & C_{a_2} & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{5.9}$$

where C_{a_j} is the companion matrix for the polynomial $a_j(z) = z^{k_j} + a^{(j)}_{k_j-1} z^{k_j-1} + \cdots + a^{(j)}_0$

$$C_{a_j} = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ -a^{(j)}_0 & -a^{(j)}_1 & \cdots & -a^{(j)}_{k_j-1} \end{pmatrix}, \tag{5.10}$$

and B_j, D_{j-1} are $k_j \times k_{j-1}$ and $k_j \times k_{j+1}$ matrices, respectively, given by

$$B_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ b_j & 0 & \cdots & 0 \end{pmatrix}, \quad D_{j-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{5.11}$$

The following factorization result for generalized Jacobi matrices which meet the assumption (1.21) was proved in [16].

Theorem 5.1. *Let $\boldsymbol{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}^{req}$ and let J be a generalized Jacobi matrix associated with the moment sequence $\boldsymbol{s} = \{s_j\}_{j=0}^{\infty}$. Then J admits the following $LU-$factorization*

$$J = LU, \tag{5.12}$$

where where L and U are lower and upper two-diagonal block matrices, respectively

$$L = \begin{pmatrix} A_0 & & \\ L_1 & A_1 & \\ & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} U_0 & D_0 & \\ & U_1 & \ddots \\ & & \ddots \end{pmatrix}, \tag{5.13}$$

the blocks A_j and U_j are $k_j \times k_j$ matrices

$$A_j = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ -a^{(j)}_1 & -a^{(j)}_2 & \cdots & -a^{(j)}_{k_j-1} & 1 \end{pmatrix} \quad \text{and} \quad U_j = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 \\ \mathfrak{u}_j & 0 & \cdots & 0 \end{pmatrix}, \tag{5.14}$$

D_j are $k_j \times k_{j+1}$ matrices defined by (2.20), L_{j+1} are $k_{j+1} \times k_j$-matrices

$$L_{j+1} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \mathfrak{l}_{j+1} \end{pmatrix} \quad (j = 0, 1, \dots), \tag{5.15}$$

and $\mathfrak{u}_j$ $\mathfrak{l}_j$ can be found from the system of equations

$$\mathfrak{u}_0 = a_0(0), \quad \mathfrak{u}_j + \mathfrak{l}_j = -a_j(0), \quad \mathfrak{l}_{j+1} = \frac{b_{j+1}}{\mathfrak{u}_j} \quad \text{for all} \quad j \in \mathbb{Z}_+. \tag{5.16}$$

The transformation

$$J = LU \to \mathfrak{J} := UL \tag{5.17}$$

is called the Darboux transformation of the monic generalized Jacobi matrix J. As was shown in [16, Theorem 3.10] $\mathfrak{J}$ is also a monic generalized Jacobi matrix. In the following theorem we will reformulate the statement of [16, Theorem 3.10] in terms of the generalized S-fraction in a special case.

Proposition 5.2. ([16]). *Let $\boldsymbol{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}^{reg}$, such that $k_j = n_{j+1} - n_j \geq 2$ ($j \in \mathbb{Z}_+$, $n_0 = 0$), let J be a monic generalized Jacobi matrix associated with the sequence $\boldsymbol{s}$ and let l_j, d_j and $m_j(z)$ be determined by the generalized S-fraction* (4.1). *Then*

(a) *The monic generalized Jacobi matrix J admits the $LU-$factorization* (5.1) *with the entries $\mathfrak{u}_{j-1}$ and $\mathfrak{l}_j$ defined by* (5.4).

(b) *The Darboux transformation of J takes the form*

$$\mathfrak{J} = UL = \begin{pmatrix} C_{\widetilde{m}_1} & \mathfrak{D}_0 & & & \\ \mathfrak{B}_1 & 0 & \mathfrak{D}_1 & & \\ & \mathfrak{B}_2 & C_{\widetilde{m}_2} & \mathfrak{D}_2 & \\ & & \mathfrak{B}_3 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \tag{5.18}$$

where $C_{\widetilde{m}_i}$ are companion matrices for polynomials

$$\widetilde{m}_i(z) = \frac{1}{d_i} m_i(z), \qquad i = 1, 2, \dots .$$

The blocks $\mathfrak{B}_{2i-1}$, $\mathfrak{B}_{2i}$ are $1 \times (k_{i-1} - 1)$ and $(k_i - 1) \times 1$, respectively,

$$\mathfrak{B}_{2i-1} = \begin{pmatrix} \frac{1}{d_i l_i} & 0 & \dots & 0 \end{pmatrix}, \quad \mathfrak{B}_{2i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{d_{i+1} l_i} \end{pmatrix} \quad (i = 1, 2, \dots), \tag{5.19}$$

$\mathfrak{D}_{2j}$ and $\mathfrak{D}_{2j+1}$ are $(k_j - 1) \times 1$ and $1 \times (k_{j+1} - 1)$-matrices, respectively, of the form (5.11).

(c) *The matrix $\mathfrak{J}$ is associated with the following P-fraction*

$$\mathop{\mathbf{K}}_{0}^{\infty} \left(\frac{-\mathfrak{b}_j}{\mathfrak{a}_j(z)} \right) = -\cfrac{\mathfrak{b}_0}{\mathfrak{a}_0(z) - \cfrac{\mathfrak{b}_1}{\mathfrak{a}_1(z) - \ddots}}, \tag{5.20}$$

where

$$\mathfrak{b}_0 = b_0, \quad \mathfrak{b}_{2j-1} = \frac{1}{l_j d_j}, \quad \mathfrak{b}_{2j} = \frac{1}{l_j d_{j+1}} \quad (j \in \mathbb{N}). \tag{5.21}$$

$$\mathfrak{a}_{2j}(z) = \widetilde{m}_{j+1}(z), \quad \mathfrak{a}_{2j+1}(z) = z \quad (j \in \mathbb{Z}_+). \tag{5.22}$$

(d) *The monic Jacobi matrix $\mathfrak{J}$ corresponds to the sequence $\mathfrak{s} = \{s_{j+1}\}_{j=0}^{\infty}$.*

(e) *The k–th convergent of the P-fraction (5.20) is equal to*

$$\mathfrak{f}_k(z) = \begin{cases} -\dfrac{z\left(Q_{n_i}(z) P_{n_{i-1}}(0) - P_{n_i}(0) Q_{n_{i-1}}(z)\right)}{P_{n_i}(z) P_{n_{i-1}}(0) - P_{n_i}(0) P_{n_{i-1}}(z)}, & k = 2i - 1, \quad i \in \mathbb{N}, \\ -\dfrac{z Q_{n_i}(z)}{P_{n_i}(z)}, & k = 2i, \quad i \in \mathbb{N}. \end{cases} \tag{5.23}$$

Proof. (*a*) It follows from (5.4) and Corollary 4.1 that

$$\mathfrak{u}_{j-1} = -\frac{P_{n_j}(0)}{P_{n_{j-1}}(0)} = \frac{\prod_{i=1}^{j-1} l_i d_i}{\prod_{i=1}^{j} l_i d_i} = \frac{1}{d_j l_j} \quad (j \in \mathbb{N}).$$

Hence by (5.16) and the second equality in (5.4)

$$\mathfrak{l}_j = \frac{b_j}{\mathfrak{u}_{j-1}} = \frac{1}{l_j d_{j+1}} \quad (j \in \mathbb{N}).$$

This proves (5.4).

(*b*) It remains only to prove the statement for the entries $\mathfrak{b}_j$ of the subdiagonal blocks $\mathfrak{B}_j$ in (5.18). The rest is implied by Theorem 5.1. Indeed, it follows from the equality $\mathfrak{J} = UL$ that

$$\mathfrak{b}_{2j-1} = \mathfrak{u}_{j-1} = \frac{1}{l_j d_j}, \quad \mathfrak{b}_{2j} = \mathfrak{l}_j = \frac{1}{l_j d_{j+1}} \quad (j \in \mathbb{N}). \tag{5.24}$$

(*c*) The proof of (c) follows from (5.24) and the formula (5.18) for $\mathfrak{J}$.

(d) The statement is contained in [16, Theorem 3.13].

(e) Let $\mathfrak{P}_{\mathfrak{n}_k}(z)$ and $\mathfrak{Q}_{\mathfrak{n}_k}(z)$ be polynomials of the first and the second kind associated with the Jacobi matrix $\mathfrak{J}$. The k–th convergent of the P-fraction (5.20) is

$$\mathfrak{f}_k(z) = -\frac{\mathfrak{Q}_{\mathfrak{n}_k}(z)}{\mathfrak{P}_{\mathfrak{n}_k}(z)}, \tag{5.25}$$

where $\{\mathfrak{n}_k\}_{k=1}^{\infty}$ is the sequence of normal indices of the sequence $\{s_{j+1}\}_{j=0}^{\infty}$. Using the relations between $P_{n_i}(z)$, $Q_{n_i}(z)$ and $\mathfrak{P}_{n_i}(z)$, $\mathfrak{Q}_{n_i}(z)$ (see [16], Theorem 3.19 and Theorem 3.22), we get (5.23). This completes the proof. □

Corollary 5.1. *Let $\boldsymbol{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}^{reg}$, such that $k_j = 2$ for all $j = 0, 1, 2\ldots$, let J be a monic generalized Jacobi matrix associated with the sequence $\boldsymbol{s}$ and let l_j, d_j and $m_j(z)$ be determined by* (4.26). *Then*

(a) *The monic generalized Jacobi matrix J admits the $LU-$factorization* (5.13)

$$J = LU = \begin{pmatrix} 1 & 0 & & & & \\ -\frac{m_1}{d_1} & 1 & & & & \\ 0 & 0 & 1 & 0 & & \\ 0 & \mathfrak{l}_1 & -\frac{m_2}{d_2} & 1 & & \\ & & 0 & 0 & 1 & \\ & & 0 & \mathfrak{l}_2 & -\frac{m_2}{d_2} & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & & & & \\ \mathfrak{u}_0 & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & \mathfrak{u}_1 & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & \mathfrak{u}_2 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

with the entries $\mathfrak{u}_{j-1}$ and $\mathfrak{l}_j$ defined by (5.4).

(b) *The Darboux transformation of J is the monic Jacobi matrix*

$$\mathfrak{J} = UL = \begin{pmatrix} -\dfrac{m_1}{d_1} & 1 & & & \\ \dfrac{1}{l_1 d_1} & 0 & 1 & & \\ & \dfrac{1}{l_1 d_2} & -\dfrac{m_2}{d_2} & 1 & \\ & & \dfrac{1}{l_2 d_2} & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{5.26}$$

Remark 5.1. Let a function $f \in \mathbf{S}$ has an asymptotic behavior

$$f(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \cdots \quad (z\widehat{\rightarrow}\infty) \tag{5.27}$$

with some real s_j ($j \in \mathbb{Z}_+$). Then the function $f(z^2)$ has the asymptotic expansion

$$f(z^2) \sim -\frac{s_0}{z^2} - \frac{s_1}{z^4} - \frac{s_2}{z^6} - \cdots \quad (z\widehat{\rightarrow}\infty). \tag{5.28}$$

Let J be a monic generalized Jacobi matrix corresponding to the sequence $\{\widetilde{s}_j\}_{j=0}^{\infty}$ with $\widetilde{s}_{2j} = s_j$, $\widetilde{s}_{2j+1} = 0$ ($j \in \mathbb{Z}_+$). In [9, 6] J is also called a monic generalized Jacobi matrix corresponding to the function $f(z^2)$. In this partial case the Darboux transformation of the matrix J is a monic Jacobi matrix corresponding to the function $zf(z^2)$ which belongs to the class $\mathbf{R}$. Such interpretation of the unwrapping transformation $f \mapsto zf(z^2)$ was presented in [6].

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