WEAK DEPENDENCE FOR A CLASS OF LOCAL FUNCTIONALS OF MARKOV CHAINS ON $\mathbb{Z}^d$

C. BOLDRIGHINI, A. MARCHESIELLO, AND C. SAFFIRIO

Dedicated to the 90th anniversary of Academician Yu. M. Berezansky

ABSTRACT. In many models of Mathematical Physics, based on the study of a Markov chain $\widehat{\eta} = \{\eta_t\}_{t=0}^{\infty}$ on $\mathbb{Z}^d$, one can prove by perturbative arguments a contraction property of the stochastic operator restricted to a subspace of local functions $\mathcal{H}_M$ endowed with a suitable norm. We show, on the example of a model of random walk in random environment with mutual interaction, that the condition is enough to prove a Central Limit Theorem for sequences $\{f(S^k\widehat{\eta})\}_{k=0}^{\infty}$, where S is the time shift and f is strictly local in space and belongs to a class of functionals related to the Hölder continuous functions on the torus T^1.

1. INTRODUCTION

Many problems in Physics and other sciences lead to consider Markov chains on the d-dimensional lattice $\mathbb{Z}^d$ with local interaction (see [15]). The states of the chain are random fields $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}^d\}$, $t \in \mathbb{Z}_+ = \{0, 1, \ldots\}$, with $\eta_t(x) \in S$, where S is usually a finite or countable set. In many models, notably in the work of R. A. Minlos and collaborators (see, e.g. [1, 3, 4, 6, 13, 14, 15] and references therein) one can prove, usually by perturbative arguments, the existence of an invariant measure Π on the state space $\Omega = S^{\mathbb{Z}^d}$, and of a subspace of local functions $\mathcal{H}_M \subset L_2(\Omega, \Pi)$, invariant with respect to the stochastic operator $\mathcal{T}$ and such that for all $F \in \mathcal{H}_M$ with zero average $\langle F \rangle_\Pi = 0$, we have, for some constant $\bar{\mu} \in (0, 1)$,

$$(1) \qquad\qquad |(\mathcal{T}F)(\xi)| \leq \bar{\mu}\|F\|_M, \quad \xi \in \Omega.$$

Here $\|\cdot\|_M$ is a suitable norm on $\mathcal{H}_M$, which is as a rule identified with the help of an expansion in a natural basis.

If one considers sums of functionals depending on the space-time field $\widehat{\eta} = \{\eta_t\}_{t=0}^{\infty} \in \widehat{\Omega} = S^{\mathbb{Z}^d \times \mathbb{Z}_+}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, of the type $\sum_{t=0}^{T} f(S^t\widehat{\eta})$, where S is the time shift, $S\widehat{\eta} = \{\eta_t\}_{t=1}^{\infty}$ and f is a functional which is local in space, one cannot in general obtain a Central Limit Theorem (CLT) by relying on properties such as strong mixing and the like [11], which need requirements that may not apply or may be difficult to prove [7]. The aim of the present paper is to establish properties which hold in the framework described above and are sufficient for the CLT to hold.

The models to which our description above applies are of different nature, and the space $\mathcal{H}_M$ is based on explicit constructions, so that it is convenient to work on the example of a particular model. The model that we consider here is a random walk in dynamical random environment with mutual interaction introduced in the papers [2, 3]: the Markov chain $\eta_t, t \in \mathbb{Z}_+$, describes the "environment from the point of view of the

random walk", an object which plays an important role in the analysis of random walks in random environment [12].

Our results are inspired by a classical result on the CLT for functionals of independent variables by Ibragimov and Linnik [11] (Th. 19.3.1).

In the next section we describe the model, which is a perturbation of an independent model, and present the main features which are relevant to our analysis. In § 3 we prove some preliminary results and in the final section § 4 we prove our main results.

2. DESCRIPTION OF THE MODEL

We consider a version of the model studied in [2, 3], which describes a discrete-time random walk $X_t \in \mathbb{Z}^d$, $d \geq 1$, $t \in \mathbb{Z}_+$, evolving in mutual interaction with a random field $\xi_t = \{\xi_t(x) : x \in \mathbb{Z}^d\}$, with $\xi_t(x) \in S = \{\pm 1\}$. The state space is $\Omega = S^{\mathbb{Z}^d}$, and the space of the "trajectories" (or "histories") of the environment $\hat{\xi} = \{\xi_t : t \in \mathbb{Z}_+\}$ is $\widehat{\Omega} = S^{\mathbb{Z}^d \times \mathbb{Z}_+}$. Measurability is understood with respect to the σ-algebra generated by the cylinder sets.

The pair $(X_t, \xi_t), t \in \mathbb{Z}_+$ is a conditionally independent Markov chain [15], i.e., if $A \subset \Omega$ is a measurable set, we have

$$
\begin{aligned}
(2) \quad & P(X_{t+1} = x + u, \xi_{t+1} \in A \mid X_t = x, \xi_t = \bar{\xi}) \\
& \qquad = P(X_{t+1} = x + u \mid X_t = x, \xi_t = \bar{\xi}) \, P(\xi_{t+1} \in A \mid X_t = x, \xi_t = \bar{\xi}) \, .
\end{aligned}
$$

If $\hat{\xi} \in \widehat{\Omega}$ is fixed, the first factor on the right of (2) defines the "quenched" random walk, for which we assume the simple form

$$
(3) \quad P(X_{t+1} = x + u \mid X_t = x, \xi_t = \bar{\xi}) = P_0(u) + \varepsilon c(u)\bar{\xi}(x), \quad u \in \mathbb{Z}^d, \quad \xi \in \Omega.
$$

Here $\varepsilon > 0$ is a small parameter, P_0 is a probability distribution on $\mathbb{Z}^d$ and c is a real function on $\mathbb{Z}^d$, such that $P_0(u) \pm \varepsilon c(u) \in [0, 1)$, $u \in \mathbb{Z}^d$. We also assume that P_0 is even and c odd in u, and that both are finite range. By homogeneity in space it is not restrictive to assume $X_0 = 0$.

For the random walk transition probability P_0, with characteristic function $\tilde{p}_0(\lambda) = \sum_{u \in \mathbb{Z}^d} P_0(u)e^{i(\lambda, u)}$ we assume that it is non-degenerate, i.e., $|\tilde{p}_0(\lambda)| = 1$ if and only if $\lambda = 0$, and, in order to meet a technical assumption in [3], we also need that the Fourier coefficients of the function $\frac{1}{\tilde{p}_0(\lambda)}$ are absolutely summable.

The evolution of the environment is independent at each site, so that $P(\xi_{t+1} \in A \mid X_t = x, \xi_t = \bar{\xi})$ is a sum of products of the factors

$$
(4) \quad P(\xi_{t+1}(y) = s \mid X_t = x, \xi_t = \bar{\xi}) = (1 - \delta_{x,y})Q_0(\bar{\xi}(y), s) + \delta_{x,y}Q_1(\bar{\xi}(y), s),
$$

where $s = \pm 1$, Q_0, Q_1 are symmetric 2×2 matrices, Q_0 has eigenvalues $1, \mu$, $|\mu| \in (0, 1)$, and Q_1 is such that $Q_1 - Q_0 = \mathcal{O}(\varepsilon)$. In words, at each site $x \in \mathbb{Z}^d$ the evolution is given by the transition matrix Q_0, except at the site where the random walk is located, where the transition matrix is Q_1.

A natural probability measure on the state space Ω is the product $\Pi_0 = \pi_0^{\mathbb{Z}^d}$, with $\pi_0 = (1/2, 1/2)$. If $Q_0 = Q_1$ (no reaction on the environment) Π_0 is invariant.

The model just described was first considered in [3] both for the annealed and quenched case. If there is no reaction on environment (i.e., $Q_0 = Q_1$) the CLT for the annealed and quenched asymptotics of the random walk was obtained in a general setting [8]. A non-perturbative result was obtained in [9].

The field $\eta_t(x) = \xi_t(X_t + x)$, $t \in \mathbb{Z}_+$ is the "environment from the point of view of the particle". $\{\eta_t : t \in \mathbb{Z}_+\}$ is also a Markov chain with state space Ω, and it can be shown [6, 9] that it is equivalent to the full process (X_t, ξ_t), i.e, for all $T \in \mathbb{Z}_+$, $T \geq 1$, given the sequence $\eta_0, \ldots, \eta_T$ one can reconstruct $(X_0, \xi_0), \ldots, (X_T, \xi_T)$, almost-surely.

The stochastic operator $\mathcal{T}$ on the Hilbert space $\mathcal{H} = L_2(\Omega; \Pi_0)$, is defined as

$$(5) \qquad (\mathcal{T}f)(\bar{\eta}) = \langle f(\eta_{t+1}) | \eta_t = \bar{\eta} \rangle, \quad f \in \mathcal{H},$$

where the average $\langle \cdot \rangle$ is w.r.t. the transition probability (3). By our assumptions $\mathcal{T}$ preserves parity under the exchange $\eta \to -\eta$.

In $\mathcal{H}$ we introduce a convenient basis. As Q_0 is symmetric, its eigenvectors are $e_0 = (1,1)$ and $e_1 = (1,-1)$ with corresponding eigenvalues 1 and μ. We denote their components as $e_j(s)$, so that $e_1(s) = s$, $e_0(s) = 1$, $s = \pm 1$, and set

$$(6) \qquad \Phi_\Gamma(\bar{\eta}) = \prod_{x \in \Gamma} e_1(\bar{\eta}(x)) = \prod_{x \in \Gamma} \bar{\eta}(x), \quad \Gamma \in \mathfrak{G},$$

where $\mathfrak{G}$ is the collection of the finite subsets of $\mathbb{Z}^d$, with $\Phi_\emptyset = 1$. $\{\Phi_\Gamma : \Gamma \in \mathfrak{G}\}$ is a discrete orthonormal complete basis in $\mathcal{H}$, and for $f \in \mathcal{H}$ we write $f(\eta) = \sum_{\Gamma \in \mathfrak{G}} f_\Gamma \Phi_\Gamma$.

For $M > 1$ the dense subspace $\mathcal{H}_M \subset \mathcal{H}$ is defined as

$$(7) \qquad \mathcal{H}_M = \{f = \sum_\Gamma f_\Gamma \Phi_\Gamma \ : \ \|f\|_M = \sum_\Gamma |f_\Gamma| M^{|\Gamma|} < \infty\}.$$

$\mathcal{H}_M$ equipped with the norm $\|\cdot\|_M$ is a Banach space. As $|\Phi_\Gamma(\eta)| = 1$, we have

$$(8) \qquad \|f\|_{\mathcal{H}} \leq \|f\|_\infty \leq \|f\|_M, \quad f \in \mathcal{H}_M.$$

Moreover $\mathcal{H}_M$ is closed under multiplication. In fact, as it is to see,

$$\Phi_\Gamma \Phi_{\Gamma'} = \Phi_{\Gamma \triangle \Gamma'}, \quad \Gamma \triangle \Gamma' = \Gamma \setminus \Gamma' \ \cup \ \Gamma' \setminus \Gamma,$$

so that if $f, g \in \mathcal{H}_M$ and $f = \sum_\Gamma f_\Gamma \Phi_\Gamma$, $g = \sum_\Gamma g_\Gamma \Phi_\Gamma$, we have

$$(9) \qquad \|fg\|_M = \sum_{\Gamma \Gamma'} |f_\Gamma g_{\Gamma'}| M^{|\Gamma \triangle \Gamma'|} \leq \|f\|_M \|g\|_M.$$

In the paper [3] an analysis of the expression of the matrix elements of $\mathcal{T}$ and its adjoint $\mathcal{T}^*$, relying on their spectral properties for $\varepsilon = 0$, leads to the following results.

Theorem 2.1. *If ε and $|\mu|$ are small enough, the space $\mathcal{H}_M$ is invariant under $\mathcal{T}$, and there is an invariant probability measure Π for the chain $\{\eta_t\}$ which is absolutely continuous with respect to Π_0 with uniformly bounded density $v(\eta)$. Moreover $\mathcal{H}_M$ can be decomposed as*

$$\mathcal{H}_M = \mathcal{H}_M^{(0)} + \widehat{\mathcal{H}}_M,$$

where $\mathcal{H}_M^{(0)}$ is the space of the constants, and on $\widehat{\mathcal{H}}_M$ the restriction of $\mathcal{T}$ acts as a contraction

$$(10) \qquad \|\mathcal{T}f\|_M \leq \bar{\mu} \|f\|_M, \quad f \in \widehat{\mathcal{H}}_M,$$

$\bar{\mu} \in (0,1)$, $\bar{\mu} = |\mu| + \mathcal{O}(\varepsilon)$. Furthermore if $f = f_0 + \widehat{f}$, $f_0 \in \mathcal{H}_M^{(0)}$, $\widehat{f} \in \widehat{\mathcal{H}}_M$, then

$$f_0 = \int f(\eta) \, d\Pi(\eta) = \int f(\eta) v(\eta) \, d\Pi_0(\eta).$$

3. Preliminary estimates

We denote by $\mathcal{P}_\Pi$ the probability measure on $\widehat{\Omega} = \{\pm 1\}^{\mathbb{Z}^d \times \mathbb{Z}_+}$ generated by the initial distribution Π, and by $\mathfrak{M}_{t_0}^{t_1}$, $0 \leq t_0 \leq t_1$ the σ-algebra of subsets of $\widehat{\Omega}$ generated by $\{\eta_t\}_{t=t_0}^{t_1}$. As Π is invariant, $\mathcal{P}_\Pi$ is invariant under the time shift.

We consider functionals f which depend only on the values of the field at the origin, i.e., on the sequence of random variables $\{\eta_t(0)\}_{t=0}^\infty$. We set for brevity $\zeta_t = \eta_t(0)$ and $\widehat{\zeta} = \{\zeta_t : t \in \mathbb{Z}_+\} \in \Omega_+ = \{\pm 1\}^{\mathbb{Z}_+}$. $\mathcal{M}_{t_0}^{t_1}$, $0 \leq t_0 < t_1$ will denote the σ-algebra generated by the variables $\{\eta_t(0)\}_{t=t_0}^{t_1}$, which is a subalgebra of $\mathfrak{M}_{t_0}^{t_1}$.

By abuse of notation, $f(\widehat{\zeta})$ may denote a function on $\widehat{\Omega}$ or on Ω_+, according to the circumstances, and similarly for the σ-algebras $\mathcal{M}_{t_0}^{t_1}$, $0 \le t_0 < t_1$. We also write $\mathcal{M}_t$ and $\mathfrak{M}_t$ for $\mathcal{M}_t^t$ and $\mathfrak{M}_t^t$, respectively.

In what follows if f is a function on $\widehat{\Omega}$ we introduce the notation $\langle f(\cdot)|\mathfrak{M}_0\rangle(\eta) = G^{(f)}(\eta)$, $\eta \in \Omega$. The following lemma is a simple consequence of Theorem 2.1.

Lemma 3.1. *Let $f(\widehat{\zeta})$ be a cylinder function on Ω_+, depending only on the variables $\zeta_0, \ldots, \zeta_{m-1}$, $m \ge 1$. Then $G^{(f)}(\eta) \in \mathcal{H}_M$ and*

$$\left\| G^{(f)} \right\|_M \le C \max_{\gamma \in \{0,\ldots,m-1\}} |f_\gamma|(1 + \mu_*)^m, \tag{11}$$

where $\mu_* = M\sqrt{\bar{\mu}(1 + 2\bar{\mu})}$ and $C > 0$ is a constant.

Proof. As f depends only on $\zeta_0, \ldots, \zeta_{m-1}$ it can be written in the form

$$f(\widehat{\zeta}) = \sum_{\gamma \subset \{0,\ldots,m-1\}} f_\gamma \Psi_\gamma(\widehat{\zeta}), \tag{12}$$

where the sum runs over the subsets of $\{0, \ldots, m-1\}$, and the functions

$$\Psi_\gamma(\widehat{\zeta}) = \prod_{t \in \gamma} \zeta_t, \quad \gamma \ne \emptyset, \quad \Psi_\emptyset(\widehat{\zeta}) = 1 \tag{13}$$

are called "Walsh functions". The first assertion follows from the fact that for any subset $\gamma = \{t_0, t_1, \ldots, t_k\} \subset \mathbb{Z}_+$, $t_0 < t_1 < \cdots < t_k$, we have

$$G_\gamma(\bar{\eta}) := \langle \Psi_\gamma|\mathfrak{M}_{t_0}\rangle \in \mathcal{H}_M, \quad \bar{\eta} \in \Omega. \tag{14}$$

In fact, if $r_j = t_{k-1-j} - t_{k-j}$, $j = 1, \ldots, k$, G_γ can be written as

$$G_\gamma(\bar{\eta}) = \Phi_{\{0\}}(\bar{\eta}) \left[\mathcal{T}^{r_k}\Phi_{\{0\}} \ldots \mathcal{T}^{r_1}\Phi_{\{0\}} \right](\bar{\eta}), \quad \bar{\eta} \in \Omega, \tag{15}$$

i.e., G_γ is obtained by successive applications of $\mathcal{T}$ and of the multiplication operator by $\Phi_{\{0\}}$. As both operations leave $\mathcal{H}_M$ invariant, $G_\gamma \in \mathcal{H}_M$.

Moreover the following inequality is proved in the Appendix:

$$\|G_\gamma\|_M \le M^{|\gamma|} \bar{\mu}^{[\frac{|\gamma|}{2}]}(1 + 2\bar{\mu})^{[\frac{|\gamma|-1}{2}]} \le C \mu_*^{|\gamma|}, \tag{16}$$

where $[\cdot]$ denotes the integer part, and $C > 0$ is a constant which is easily worked out.

The proof of the lemma follows by observing that the inequality (16) implies

$$\|\langle f(\cdot)|\mathfrak{M}_0\rangle\|_M \le C \max_{\gamma \in \{0,\ldots,m-1\}} |f_\gamma| \sum_{\gamma \subset \{0,\ldots,m-1\}} \mu_*^{|\gamma|}. \qquad \square \tag{17}$$

We denote by $\wp$ the probability measure induced by $\mathcal{P}_\Pi$ on Ω_+. $\wp$ is stationary with respect to the time shift on Ω_+: $S\widehat{\zeta} = \{\zeta_1, \zeta_2, \ldots\}$.

The following assertion is a simple consequence of the previous lemma.

Lemma 3.2. *Under the assumptions of the previous lemma, if $\bar{\mu}$ is so small that $\mu_* < 1$, then the probability measure $\wp$ on Ω_+ is continuous.*

Proof. We need to prove that any point $\widehat{\zeta}^{(0)} = \{\bar{\zeta}_k\}_{k=0}^\infty \in \Omega_+$ has zero $\wp$-measure. Consider the cylinders $Z_n(\widehat{\zeta}^{(0)}) = \{\zeta_j = \bar{\zeta}_j : j = 0, 1, \ldots, n-1\}$, which are decreasing $Z_{n+1}(\widehat{\zeta}^{(0)}) \subset Z_n(\widehat{\zeta}^{(0)})$ and such that $\cap_n Z_n(\widehat{\zeta}^{(0)}) = \{\widehat{\zeta}^{(0)}\}$. The probabilities

$$\wp\left(Z_n(\widehat{\zeta}^{(0)})\right) = \frac{1}{2^n} \left\langle \prod_{j=0}^{n-1} (1 + \bar{\zeta}_j\zeta_j) \right\rangle_\wp \tag{18}$$

are computed by expanding the internal product in terms of the functions Ψ_γ

$$\prod_{j=0}^{n-1} \left(1 + \bar{\zeta}_j \zeta_j\right) = \sum_{\gamma \subset \{0,\ldots,n-1\}} \Psi_\gamma(\widehat{\bar{\zeta}})\Psi_\gamma(\widehat{\zeta}), \quad \widehat{\bar{\zeta}} = \{\bar{\zeta}_j\}_{j=0}^{n-1}.$$

Recalling that $|\Psi_\gamma(\widehat{\zeta})| = 1$, we have

$$\left| \left\langle \sum_{\gamma \subset \{0,\ldots,n-1\}} \Psi_\gamma(\widehat{\bar{\zeta}})\Psi_\gamma(\widehat{\zeta}) \right\rangle_\wp \right| \leq \sum_{\gamma \subset \{0,\ldots,n-1\}} \left| \left\langle \Psi_\gamma(\widehat{\zeta}) \right\rangle_\wp \right|$$

$$= \sum_{\gamma \subset \{0,\ldots,n-1\}} \left| \langle \langle \Psi_\gamma | \mathfrak{M}_0 \rangle \, (\cdot) \rangle_\Pi \right| = \sum_{\gamma \subset \{0,\ldots,n-1\}} \left| \langle G_\gamma(\cdot) \rangle_\Pi \right|.$$

Therefore by the inequality (16) the right side is bounded by

$$\frac{C}{2^n} \sum_{\gamma \subset \{0,\ldots,n-1\}} \mu_*^{|\gamma|} = C \left(\frac{1 + \mu_*}{2} \right)^n.$$

Hence if $\mu_* < 1$, the right side tends to 0 as $n \to \infty$, which proves the lemma. $\square$

From now on we assume that $\mu_* < 1$.

We pass to consider functions for which the expansion (12) is infinite, i.e., γ runs over the collection $\mathfrak{g}$ of the finite subsets $\mathbb{Z}_+$. The functions $\{\Psi_\gamma : \gamma \in \mathfrak{g}\}$, are an orthonormal basis in $L_2(\Omega_+, \wp_0)$, where $\wp_0 = \pi^{\mathbb{Z}_+}$ is the probability measure on Ω_+ corresponding to the random variables $\{\zeta_k\}_{k=0}^\infty$ being i.i.d. with distribution $\pi(\pm 1) = \frac{1}{2}$. The corresponding series is called "Fourier-Walsh expansion" [10].

A map $\mathcal{F} : \Omega_+ \to T^1$, where $T^1 = [0, 1) \mod 1$ is the one-dimensional torus, is defined by associating to a point $\widehat{\zeta} \in \Omega_+$ the binary expansion $x = 0, a_0 a_1 \ldots \in [0, 1]$, with $a_t = \frac{1 - \zeta_t}{2}$, $t \in \mathbb{Z}_+$. $\mathcal{F}$ is not invertible because the dyadic points of T^1 have two binary expansions, but it becomes invertible if we exclude the sequences such that $\zeta_t = -1$ for all t large enough. Such sequences are a countable set, which has zero $\wp_0$-measure, and also, by Lemma 3.2, zero $\wp$-measure.

Under the map $\mathcal{F}$ the basis functions Ψ_γ go into the functions

$$\psi_\gamma(x) = \prod_{t \in \gamma} \phi_t(x), \quad \gamma \in \mathfrak{g},$$

where $\phi_t(x)$ is the image of ζ_t, $t \in \mathbb{Z}_+$, i.e.,

$$\phi_0(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1 \end{cases}$$

and for $t \geq 0$, $\phi_t(x) = \phi_0(2^t x)$, where $2^t x$ is understood $\mod 1$.

If $f \in L_2(\Omega_+, \wp_0)$ then $\tilde{f}(x) = f(\mathcal{F}^{-1} x) \in L^2(T^1, dx)$ and can be expanded in the orthonormal basis $\{\psi_\gamma : \gamma \in \mathfrak{g}\}$, with coefficients

$$(19) \qquad f_\gamma = \int_{\Omega_+} f(\widehat{\zeta}) \, d\wp_0(\widehat{\zeta}) = \int_0^1 \tilde{f}(x) \psi_\gamma(x) \, dx.$$

A natural way of ordering the collection $\mathfrak{g}$ of the finite subsets of $\mathbb{Z}_+$, which plays an important role in the theory, is obtained by setting $\gamma_0 = \emptyset$ and $\gamma_n = \{t_1, t_2, \ldots, t_r\}$, where r and $0 \leq t_1 < t_2 < \cdots < t_r$ are uniquely defined by the relation $n = 2^{t_1} + \cdots + 2^{t_r}$. We call Walsh series both the expansion

$$(20) \qquad f(\widehat{\zeta}) = \sum_{\gamma \in \mathfrak{g}} f_\gamma \Psi_\gamma(\widehat{\zeta}) = \sum_{n=0}^\infty f_{\gamma_n} \Psi_{\gamma_n}(\widehat{\zeta})$$

and the corresponding expansions of $\tilde{f}(x)$. For the latter, an important role is played by a particular set of partial sums

$$(21) \qquad \Sigma_{2^k}(\tilde{f};x) = \sum_{\gamma \subset \{0,1,\ldots,k-1\}} f_\gamma \psi_\gamma(x) = \sum_{n=0}^{2^k-1} f_{\gamma_n} \psi_{\gamma_n}(x)$$

for which it can be seen [10] that

$$(22) \qquad \Sigma_{2^k}(\tilde{f};x) = 2^k \int_{\alpha_k}^{\beta_k} \tilde{f}(y)\,dy, \quad \alpha_k = m\,2^{-k}, \quad \beta_k = (m+1)2^{-k},$$

where the integer m is such that $\alpha_k \leq x < \beta_k$.

The following result is proved in [10]. We repeat it here, with a shorter proof based on conditional probabilities.

Lemma 3.3. *Let $\tilde{f}(x)$ be a bounded function. Then its Walsh-Fourier coefficients f_γ, given by (19), satisfy the following inequality:*

$$(23) \qquad |f_\gamma| \leq \frac{\omega(\tilde{f}; 2^{-n-1})}{2^{n+2}}, \quad n = \max\{t : t \in \gamma\},$$

where $\omega(f;\delta)$ is the modulus of continuity of $\tilde{f}$

$$(24) \qquad \omega(f;\delta) = \sup_{\substack{x,x' \in T^1 \\ |x-x'|=\delta}} \frac{|f(x) - f(x')|}{\delta}.$$

Proof. We have

$$f_\gamma = \left\langle f(\widehat{\zeta}) \prod_{t \in \gamma} \zeta_t \right\rangle_{\wp_0} = \left\langle \prod_{t \in \gamma \setminus \{n\}} \zeta_t \left\langle f(\widehat{\zeta})\zeta_n | \mathcal{M}_0^{n-1} \right\rangle \right\rangle_{\wp_0}.$$

Going back to T^1, and setting $x_n = \frac{a_0}{2} + \cdots + \frac{a_{n-1}}{2^n}$, $a_j = \frac{1-\zeta_j}{2}$, we have

$$(25) \qquad \begin{aligned} \left| \left\langle f(\widehat{\zeta})\zeta_n | \mathcal{M}_0^n \right\rangle \right| &= 2^n \int_{x_n}^{x_n+2^{-n}} \tilde{f}(x)\,(1 - 2\phi_n(x))\,dx \\ &= 2^n \int_{x_n}^{x_n+2^{-n-1}} \left[\tilde{f}(x) - \tilde{f}(x + 2^{-n-1}) \right] dx, \end{aligned}$$

from which, taking into account (24), the inequality (23) follows immediately. $\qquad \square$

The results above allow us to prove the analogue of Lemma 3.1 for functions f such that $\tilde{f}(x) = f(\mathcal{F}^{-1}x)$ is Hölder continuous: $\tilde{f} \in \mathcal{C}^\alpha(T^1)$, $\alpha \in (0,1)$. In what follows if $g \in \mathcal{C}^\alpha(T^1)$ we denote by $\|g\|_{\mathcal{C}^\alpha}$ the norm and by $\|g\|_\alpha$ the semi–norm

$$\|g\|_\alpha = \sup_{x,y \in T^1} \frac{|g(x) - g(y)|}{|x - y|^\alpha}.$$

Lemma 3.4. *Let f be a function on Ω_+, such that $\tilde{f} \in \mathcal{C}^\alpha(T^1)$, $\alpha \in (0,1)$. If $\bar{\mu}$ is so small that $\kappa := 2^{-\alpha}(1 + \mu_*) < 1$, then $G^{(f)} \in \mathcal{H}_M$ and the following inequality holds:*

$$(26) \qquad \left\| G^{(f)} \right\|_M \leq \frac{C_\alpha}{1 - \kappa} \|\tilde{f}\|_{\mathcal{C}^\alpha},$$

where $C_\alpha > 0$ is a positive constant.

Proof. If $2^k \leq n < 2^{k+1}$ the Fourier coefficient γ_n in the Walsh series (20) is such that $\max\{t \in \gamma_n\} = k$. Hence, as $\delta \, \omega(\tilde{f}; \delta) \leq \delta^\alpha \|\tilde{f}\|_\alpha$, the inequality (23) gives

$$\tag{27} |f_{\gamma_n}| \leq \frac{\|\tilde{f}\|_\alpha}{2^{1+\alpha}} 2^{-k\alpha}, \quad 2^k \leq n < 2^{k+1}.$$

Therefore we have

$$\left\| \sum_{n=2^k}^{2^{k+1}-1} f_{\gamma_n} \langle \Psi_{\gamma_n} | \mathfrak{M}_0 \rangle \right\|_M \leq \frac{\|\tilde{f}\|_\alpha}{2^{1+\alpha}} 2^{-k\alpha} \sum_{n=2^k}^{2^{k+1}-1} \|\langle \Psi_{\gamma_n} | \mathfrak{M}_0 \rangle\|_M.$$

Observe moreover that the number of elements of γ_n is $r_n = |\gamma_n| = u_n - 1$ where u_n is the number of "1" in the binary expansion of n. Hence, by the inequality (16) we find

$$\sum_{n=2^k}^{2^{k+1}-1} \|\langle \Psi_{\gamma_n} | \mathfrak{M}_0 \rangle\|_M \leq C \sum_{s=0}^{k-1} \binom{k-1}{s} \mu_*^s = C(1 + \mu_*)^{k-1},$$

which, as $|f_\emptyset| \leq \|f\|_\infty$, together with (27), implies (26). $\square$

4. WEAK DEPENDENCE AND THE CENTRAL LIMIT THEOREM

In the present paragraph we prove our main results for sums of sequences of the type $f(S^t \widehat{\zeta})$, $t = 0, 1, \ldots$ As $\mathcal{P}_\Pi$ and the measure $\wp$ induced by it on Ω_+ are invariant under time shift, the sequence is stationary in distribution.

In what follows $\langle \cdot \rangle$ denotes an average with respect to $\wp$, $\mathcal{P}_\Pi$ or Π, according to the context. Moreover we denote by $c_i, i = 1, 2, \ldots$, and sometimes by const, different constants which depend on the parameters of the model. Let f be a bounded measurable function on Ω_+ with $\langle f \rangle_\wp = 0$, and

$$\tag{28} S_n(\widehat{\zeta}|f) = \sum_{t=0}^{n-1} f(S^t \widehat{\zeta}), \quad n = 1, 2, \ldots$$

If f admits a Walsh expansion (20) then $\sum_{\gamma \in \mathfrak{g}} f_\gamma \langle \Psi_\gamma \rangle_\wp = \langle f \rangle_\wp = 0$, so that

$$\tag{29} f(\widehat{\zeta}) = \sum_{\gamma \in \mathfrak{g}, \, \gamma \neq \emptyset} f_\gamma \widehat{\Psi}_\gamma(\widehat{\zeta}), \quad \widehat{\Psi}_\gamma(\widehat{\zeta}) = \Psi_\gamma(\widehat{\zeta}) - \langle \Psi_\gamma(\cdot) \rangle_\wp.$$

In what follows we make repeated use of the fact that if f is a function on $\widehat{\Omega}$ and $G^{(f)} := \langle f(\cdot) | \mathfrak{M}_0 \rangle \in \mathcal{H}_M$, then, by Theorem 2.1, $\langle f(S^{t+h} \cdot) | \mathfrak{M}_h \rangle = \mathcal{T}^t G^{(f)} \in \mathcal{H}_M$.

Theorem 4.1. *Let f be a function on Ω_+, depending only on $\zeta_0, \ldots, \zeta_{m-1}$, $m \geq 1$, and such that $\langle f \rangle_\wp = 0$. Then the dispersion of normalized sums $\frac{S_n(\widehat{\zeta}|f)}{\sqrt{n}}$ tends, as $n \to \infty$, to a finite non-negative limit*

$$\tag{30} \sigma_f^2 = \langle f^2(\cdot) \rangle_\wp + 2 \sum_{t=1}^\infty \langle f(\cdot) f(S^t \cdot) \rangle_\wp$$

and the series is absolutely convergent. Moreover, if $\sigma_f^2 > 0$, the sequence $\frac{S_n(\widehat{\zeta}|f)}{\sqrt{n}}$ tends weakly to the centered Gaussian distribution with dispersion σ_f^2.

Proof. The proof of the theorem is based on two basic inequalities.

$$\tag{31} \left\| \langle f(\cdot) f(S^t \cdot) | \mathfrak{M}_0 \rangle \right\|_M \leq c_1 \|f\|_\infty^2 \bar{\mu}^{\max\{0, t-m+1\}} (1 + \mu_*)^{2m},$$

$$\tag{32} \left\| G^{(S_n)} \right\|_M \leq c_2 \|f\|_\infty (1 + \mu_*)^m, \quad \left\| G^{(\widehat{S}_n^2)} \right\|_M \leq c_3 \|f\|_\infty^2 m (1 + \mu_*)^m,$$

where $\widehat{S}_n^2(\widehat{\zeta}|f) = S_n^2(\widehat{\zeta}|f) - \langle S_n^2(\cdot|f)\rangle$ and c_1, c_2, c_3 are constants independent of m. For the proof of (31), observe that if $t \leq m - 1$, then $f_t^{(2)}(\widehat{\zeta}) := f(\widehat{\zeta})f(S^t(\widehat{\zeta}))$ is a cylinder function $\mathcal{M}_0^{t+m-1}$- measurable and bounded by $\|f\|_\infty^2$. Hence, by Lemma 3.1 and (19), $\|f_t^{(2)}\|_M \leq \mathrm{const}\|f\|_\infty^2(1 + \mu_*)^{m+t}$, and (31) holds for $t \leq m - 1$.

If $t \geq m$ taking the expectation with respect to $\mathfrak{M}_0^{m-1}$ we have

$$\langle f(\cdot)f(S^t\cdot)|\mathfrak{M}_0\rangle = \left\langle f(\widehat{\zeta})[\mathcal{T}^{t-m+1}G^{(f)}](\eta_{m-1})|\mathfrak{M}_0\right\rangle.$$

As $G^{(f)} \in \widehat{\mathcal{H}}_M$, expanding f in Walsh series and using Proposition 5.1 in the Appendix and Lemma 3.1 we see that Inequality (31) also holds for $t \geq m - 1$.

The first inequality in (32) is a simple consequence of the inequality $\|\langle f(S^t\cdot)|\mathfrak{M}_0\|_M = \|\mathcal{T}^t G^{(f)}\|_M \leq \bar{\mu}^t\|G^{(f)}\|_M$, of Lemma 3.1 and of the inequality $|f_\gamma| \leq \|f\|_\infty$ (see (19)). Moreover, setting $\widehat{f^2}(\widehat{\zeta}) = f^2(\widehat{\zeta}) - \langle f^2(\cdot)\rangle$ and $\widehat{f_t^{(2)}}(\widehat{\zeta}) = f_t^{(2)}(\widehat{\zeta}) - \langle f_t^{(2)}(\cdot)\rangle$, we have

$$(33) \qquad \left\|G^{(\widehat{S}_n^2)}\right\|_M \leq \sum_{j=0}^{n-1}\left\|\mathcal{T}^j G^{(\widehat{f^2})}\right\|_M + 2\sum_{j=0}^{n-2}\sum_{t=1}^{n-j-1}\left\|\mathcal{T}^j G^{(\widehat{f_t^{(2)}})}\right\|_M,$$

and the second inequality in (32) follows by observing that $\widehat{f^2}$ is a cylinder function with zero average and $\|\widehat{f^2}\|_\infty \leq \|f\|_\infty^2$, and using the estimate (31) for $\|G^{(\widehat{f_t^{(2)}})}\|_M$.

Passing to the assertions of the theorem, observe first that, by the property (8) of $\mathcal{H}_M$, the absolute convergence of the series in (30) follows from Inequality (31).

Assuming that $\sigma_f^2 > 0$, for the proof of the CLT we adopt a Bernstein scheme. Let $p_n = [n^\beta]$, $q_n = [n^\delta]$, $k_n = [\frac{n}{p(n)+q(n)}]$, with $0 < \delta < \beta < 1/4$. The interval of integers $[0, n-1]$ is divided into subintervals of length p_n and q_n

$$I_\ell = [(\ell-1)(p_n + q_n), \ \ell p_n + (\ell-1)q_n - 1], \quad J_\ell = [\ell p_n + (\ell-1)q_n, \ \ell(p_n + q_n) - 1],$$

$\ell = 1, \ldots, k_n$, and the rest $J_* = [0, n-1] \setminus \cup_{j=1}^{k_n}[I_j \cup J_j]$.

The sum (28) is then written as $S_n(\widehat{\zeta}|f) = S_n^{(M)}(\widehat{\zeta}|f) + S_n^{(R)}(\widehat{\zeta}|f)$ where

$$(34) \qquad S_n^{(M)}(\widehat{\zeta}|f) = \sum_{\ell=1}^{k_n} S_{I_\ell}(\widehat{\zeta}|f), \quad S_n^{(R)}(\widehat{\zeta}|f) = \sum_{\ell=1}^{k_n} S_{J_\ell}(\widehat{\zeta}|f) + S_{J_*}(\widehat{\zeta}|f)$$

and $S_{I_\ell}, S_{J_\ell}, S_{J_*}$ denote the sums over the corresponding subinterval.

We first prove that the L_2-norm of $S_n^{(R)}/\sqrt{n}$ vanishes as $n \to \infty$, i.e.,

$$(35) \quad \left\langle\left(\sum_{\ell=1}^{k_n} S_{J_\ell}(\cdot|f)\right)^2\right\rangle = k_n\left\langle S_{J_1}^2(\cdot|f)\right\rangle + 2\sum_{1\leq s<t\leq k_n}\langle S_{J_s}(\cdot|f)S_{J_t}(\cdot|f)\rangle = \mathcal{O}(n^{1-\beta+\delta}).$$

For the proof, observe that Inequality (31) implies that $\langle S_{J_1}^2(\cdot|f)\rangle = \mathcal{O}(q_n)$, so that the first term on the right of (35) is of the order $k_n q_n \sim n^{1-\beta+\delta}$.

For the second term, observe that, by translation invariance, recalling that $S_{q_n}(\widehat{\zeta}|f)$ is $\mathfrak{M}_0^{q_n+m-2}$-measurable and taking the corresponding conditional probability,

$$(36) \quad \langle S_{J_s}(\cdot|f)S_{J_t}(\cdot|f)|\mathfrak{M}_0\rangle = \left\langle S_{q_n}(\cdot|f)\left[\mathcal{T}^{(t-s)\ell_n+p_n-m+2}G^{(S_{q_n})}\right](\eta_{q_n-m+2})|\mathfrak{M}_0\right\rangle,$$

where $\ell_n = p_n + q_n$. Therefore, recalling the inequalities (8), we get the estimate

$$(37) \qquad |\langle S_{J_s}(\cdot|f)S_{J_t}(\cdot|f)\rangle| \leq \mathrm{const}\, (1+\mu_*)^m\, \|f\|_\infty^2\, q_n\bar{\mu}^{(t-s)\ell_n+p_n-m}.$$

As $k_n q_n \bar{\mu}^{p_n} \leq \mathrm{const}\, \bar{\mu}^{\frac{p_n}{2}}$, the double sum on the right of (35) is of the order $\mathcal{O}(\bar{\mu}^{\frac{p_n}{2}})$, so that (35) is proved.

As for S_{J_*}, (31) implies $\langle S^2_{J_*}(\cdot|f)\rangle \leq \langle S^2_{p_n+q_n}(\cdot|f)\rangle = \mathcal{O}(n^{-1+\beta})$. This fact, together with (35), proves that $\langle (S_n^{(R)}(\widehat{\zeta}|f))^2\rangle/n = \mathcal{O}(n^{-\beta+\delta})$, and, as $\beta > \delta$, $S_n^{(R)}$ does not contribute to the limiting distribution.

We now show that the random variables $\{S_{I_\ell}\}_{\ell=1}^{k_n}$ are almost independent for large n, i.e., for the characteristic functions $\phi_n^{(\ell)}(\lambda|\widehat{\zeta}) = \exp\{i\frac{\lambda}{\sqrt{n}}S_{I_\ell}(\widehat{\zeta}|f)\}$ we have

$$(38) \qquad \left\langle \prod_{\ell=1}^{k_n} \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) \right\rangle - \prod_{\ell=1}^{k_n} \left\langle \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) \right\rangle \to 0, \quad n \to \infty.$$

We proceed by iteration. As a first step we consider the difference

$$(39) \qquad \left\langle \prod_{\ell=1}^{k_n} \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) \right\rangle - \left\langle \prod_{\ell=1}^{k_n-1} \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) \right\rangle \left\langle \phi_n^{(k_n)}(\lambda|\widehat{\zeta}) \right\rangle = \left\langle \prod_{\ell=1}^{k_n-1} \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) \, \widehat{\phi}_n^{(k_n)}(\lambda|\widehat{\zeta}) \right\rangle,$$

$$\widehat{\phi}_n^{(\ell)}(\lambda|\widehat{\zeta}) = \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) - \left\langle \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) \right\rangle.$$

We expand $\widehat{\phi}_n^{(k_n)}(\lambda|\widehat{\zeta})$ in Taylor series at $\lambda = 0$, we have, for some λ_*, $|\lambda_*| \leq |\lambda|$,

$$(40) \quad \widehat{\phi}_n^{(k_n)}(\lambda|\widehat{\zeta}) = i\frac{\lambda}{\sqrt{n}}S_{I_{k_n}}(\widehat{\zeta}|f) - \frac{\lambda^2}{2n}\left(S^2_{I_{k_n}}(\widehat{\zeta}|f) - \left\langle (S^2_{I_{k_n}}(\cdot|f))\right\rangle\right) + i^3\frac{\lambda^3}{n^{\frac{3}{2}}3!}R_n(\lambda_*,\widehat{\zeta}),$$

$$(41) \qquad R_n(\lambda_*,\widehat{\zeta}) = S^3_{I_\ell}(\widehat{\zeta}|f)\exp\{i\frac{\lambda_*}{\sqrt{n}}S_{I_\ell}(\widehat{\zeta}|f)\} - \left\langle S^3_{I_\ell}(\widehat{\zeta}|f)\exp\{i\frac{\lambda_*}{\sqrt{n}}S_{I_\ell}(\widehat{\zeta}|f)\}\right\rangle.$$

Clearly $|R_n(\lambda_*,\widehat{\zeta})| \leq 2p_n^3|\lambda|^3\|f\|_\infty^3 = \mathcal{O}(n^{3\beta})$, so that, as $\beta < 1/4$, we need only consider the first two terms of the expansion (40).

The product of the first $k_n - 1$ factors in the expectation in (39) is measurable with respect to $\mathfrak{M}_0^{t_n}$, where $t_n = (k_n - 1)p_n + (k_n - 2)q_n + m - 2$. Taking the corresponding conditional expectation, by Inequality (32) we get for the first order term the estimate

$$(42) \qquad \left|\left\langle \prod_{\ell=1}^{k_n-1} \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) \left[\mathcal{T}^{q_n-m+2}G^{(S_{p_n})}\right](\eta_{t_n}) \right\rangle\right| \leq c_4\,\bar{\mu}^{q_n-m}\,(1+\mu_*)^m\|f\|_\infty.$$

For the second order term, proceeding in the same way, and taking into account the second inequality (33) we come to the estimate

$$(43) \qquad \left|\left\langle \prod_{\ell=1}^{k_n-1} \phi_n^{(\ell)}(\lambda|\widehat{\zeta}) \left[\mathcal{T}^{q_n-m+2}G^{(\widehat{S}^2_{p_n})}\right](\eta_{t_n}) \right\rangle\right| \leq c_5\,\bar{\mu}^{q_n-m}\,m(1+\mu_*)^m\|f\|_\infty^2.$$

Iterating the procedure for the remaining product $\langle \prod_{\ell=1}^{k_n-1} \phi_n^{(\ell)}(\lambda|\widehat{\zeta})\rangle$, we see that the quantity on the left of (38) is of the order $\mathcal{O}(k_n n^{-3(\frac{1}{2}-\beta)}) = \mathcal{O}(n^{-\frac{1}{2}+2\beta})$, so that , as $\beta < 1/4$, it vanishes as $n \to \infty$.

We are left with a sum $\tilde{S}_n^{(M)}$ of $k(n)$ independent variables distributed as $S_{p_n}(\widehat{\zeta}|f)$. The log of the characteristic function of the corresponding normalized sum is

$$(44) \qquad k_n\,\psi_n(\frac{\lambda}{\sqrt{n}}|f), \quad \psi_n(\lambda|f) = \log\left\langle e^{i\lambda S_{q_n}^{(f)}(\cdot)}\right\rangle.$$

Expanding ψ_n in Taylor series at $\lambda = 0$, we see, in analogy to the proof above, that the third order remainder is of order $\mathcal{O}(n^{-3(\frac{1}{2}-\beta)})$, so that it does not contribute to the limit. The first order term vanishes, and we see that

$$\lim_{n\to\infty} k_n\psi_n(\frac{\lambda}{\sqrt{n}}|f) = -\frac{\lambda^2}{2}\lim_{n\to\infty}\frac{k_n}{n}\left[p_n\left\langle f^2(\cdot)\right\rangle + 2\sum_{j=0}^{p_n-1}\sum_{k=1}^{p_n-j-1}\left\langle f(\cdot)f(S^k)\cdot\right\rangle\right].$$

As $\frac{k_n p_n}{n} \to 1$, the expression on the right tends to $-\frac{\lambda^2}{2}\sigma_f^2$. The theorem is proved. $\qquad\square$

Theorem 4.2. *Let f be a function on Ω_+, satisfying the assumptions of Lemma 3.4 with $\alpha > 1/2$ and such that $\langle f \rangle_\wp = 0$. Then, if $\bar{\mu}$ is small enough, the dispersion of the normalized sums $\frac{S_n(\widehat{\zeta}|f)}{\sqrt{n}}$ tends, as $n \to \infty$, to a finite non-negative limit*

$$(45) \qquad \sigma_f^2 = \left\langle f^2(\cdot)\right\rangle_\wp + 2\sum_{t=1}^\infty \left\langle f(\cdot)f(S^t\cdot)\right\rangle_\wp,$$

where the series on the right is absolutely convergent. Moreover, if $\sigma_f^2 > 0$, the sequence $S_n^{(f)}(\widehat{\zeta})$ tends weakly to the centered Gaussian distribution with dispersion σ_f^2.

Proof. The proof repeats the pattern of the previous proof, to which we refer. Inequalities (31) and (32) are replaced by

$$(46) \qquad \left\|\left\langle f(\cdot)f(S^t\cdot)|\mathfrak{M}_0\right\rangle\right|_M \le c_6\|\tilde{f}\|_{\mathcal{C}^\alpha}^2\kappa^t,$$

$$(47) \qquad \left\|G^{(S_n)}\right\|_M \le c_7\,\|\tilde{f}\|_{\mathcal{C}^\alpha}, \quad \left\|G^{(\widehat{S}_n^2)}\right\|_M \le c_8\,\|\tilde{f}\|_{\mathcal{C}_\alpha}^2.$$

The proof of the estimate (46) is deferred to the Appendix. The first inequality (47) is proved as in the previous theorem, recalling Lemma 3.4.

The second inequality (47) follows from Inequality (33), observing that $\widehat{f}^2(\mathcal{F}^{-1}\cdot) \in \mathcal{C}^\alpha$ and using Inequality (46).

For the estimate (35) observe that (46) again implies that $\langle S_{J_1}^2(\cdot|f)\rangle = \mathcal{O}(q_n)$. For the second term on the right of (35) we need, as in [11], that the functions are well approximated by their conditional probabilities on finite σ-algebras. This property is provided by the representation (22) for the partial sums, which gives

$$(48) \qquad \left|f(\widehat{\zeta}) - \Sigma_{2^n}(f;\widehat{\zeta})\right| \le \|\tilde{f}\|_\alpha 2^{-\alpha n}.$$

Let $m_n = [\frac{4}{\alpha}\log_2 n]$, where $[\cdot]$ denotes the integer part. In the expression (36), in the sum $S_{J_s}(\widehat{\zeta}|f)$, we replace the function f by its partial sum $\Sigma_{2^{m_n}}$. The corresponding sum is denoted $\tilde{S}_{J_s}$. By Inequality (48) we have

$$\langle S_{J_s}(\cdot|f)S_{J_t}(\cdot|f)|\mathfrak{M}_0\rangle = \left\langle\tilde{S}_{J_s}(\cdot|f)S_{J_t}(\cdot|f)|\mathfrak{M}_0\right\rangle + \mathcal{O}\left(q_n^2/n^4\right).$$

$\tilde{S}_{J_s}(\cdot|f)$ can be treated as $S_{J_s}(\cdot|f)$ in the previous proof, so that the corresponding conditional expectation is written, if n is so large that $p_n > m_n$, as

$$(49) \qquad \left\langle\tilde{S}_{q_n}(\cdot|f)\left[\mathcal{T}^{(t-s)\ell_n+p_n-m_n+2}\,G^{(\tilde{S}_{q_n})}\right](\eta_{q_n-m_n+2})|\mathfrak{M}_0\right\rangle$$

(the tilde in $\tilde{S}_{q_n}$ again denotes that f is replaced by $\Sigma_{2^{m_n}}$). By the first inequality (47)

$$\left|\left\langle\tilde{S}_{J_s}(\cdot|f)S_{J_t}(\cdot|f)|\mathfrak{M}_0\right\rangle\right| \le \text{const}\,\|\tilde{f}\|_{\mathcal{C}^\alpha}^2\bar{\mu}^{(t-s)\ell_n+p_n-m_n}$$

and, as $k_n q_n \bar{\mu}^{p_n-m_n} \le \text{const}\,\bar{\mu}^{\frac{p_n}{2}}$, we see that the same (35) holds in this case. The estimate for S_{I_*} is obvious, so that the negligibility of $S_n^{(R)}$ is proved.

Further, we pass to the variables $\tilde{S}_{I_\ell}$, $\ell = 1, \ldots, k_n$, obtained, as before, by replacing f with the partial sum $\Sigma_{2^{m_n}}$. The correction is of order $\mathcal{O}(n^{-3})$, so that it can be neglected. The rest of the proof repeats the previous steps, with the only changes that m is replaced by m_n and we use the estimates (47). We omit the obvious details. $\qquad\square$

5. Appendix

Proof of inequality (16)**.** Observe that, by symmetry with respect to the change of sign $\bar{\eta}(x) \to -\bar{\eta}(x)$, $x \in \mathbb{Z}^d$, the density $v(\bar{\eta})$ is even. Moreover any finite trajectory of the Markov chain has the same probability of the trajectory obtained by sign exchange.

The functions Φ_Γ defined by (6) are even (odd) for $|\Gamma|$ even (odd). Therefore for $|\Gamma|$ odd we have $\langle \Phi_\Gamma \rangle_\Pi = 0$, and also $\langle \mathcal{T}^r \Phi_\Gamma \rangle_\Pi = 0$, $r > 0$. The functions Ψ_γ are also even (odd) for $|\gamma|$ even (odd), and for $|\gamma|$ odd $\langle \Psi_\gamma \rangle_\wp = \langle G_\gamma \rangle_\Pi = 0$.

For $|\gamma|$ even we set

$$(50) \qquad\qquad G_\gamma = \langle G_\gamma \rangle_\Pi + \widehat{G}_\gamma, \quad \widehat{G}_\gamma \in \widehat{\mathcal{H}}_M.$$

If $\gamma = \{t_0, \ldots, t_k\}$, $k \geq 1$, we have, by (15), $G_\gamma(\bar{\eta}) = \Phi_{\{0\}}(\bar{\eta})[\mathcal{T}^{r_k} G_{\gamma \setminus \{t_0\}}](\bar{\eta})$. Therefore, if $|\gamma| \geq 2$ is even we have

$$(51) \qquad\qquad \|G_\gamma\|_M \leq M \bar{\mu}^{r_k} \|G_{\gamma \setminus \{t_0\}}\|_M$$

and if $|\gamma| > 1$ is odd

$$(52) \qquad \|G_\gamma\|_M \leq M \left(|\langle G_{\gamma \setminus \{t_0\}} \rangle| + \bar{\mu}^{r_k} \|\widehat{G}_{\gamma \setminus \{t_0\}}\|_M \right) \leq M(1 + 2\bar{\mu}^{r_k}) \|G_{\gamma \setminus \{t_0\}}\|_M,$$

where in the second inequality we take into account that $|\langle G_\gamma \rangle| \leq \|G_\gamma\|_\infty \leq \|G_\gamma\|_M$.

For $|\gamma| = 1$, $G_{\{t_0\}}(\bar{\eta}) = \Phi_{\{0\}}(\bar{\eta})$ so that $\|G_{\{t_0\}}\|_M = M$, and for $|\gamma| = 2$ we have $\|G_\gamma\|_M \leq M \|\mathcal{T}^{r_1} \Phi_{\{0\}}\|_M \leq M^2 \bar{\mu}^{r_1}$. Inequalities (51) and (52) imply that

$$(53) \qquad\qquad \|G_\gamma\|_M \leq M^{|\gamma|} \prod_{j \text{ odd}} \bar{\mu}^{r_j} \prod_{j \text{ even}} (2\bar{\mu}^{r_j} + 1),$$

which implies (16). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The following proposition is a simple consequence of the previous proof.

Proposition 5.1. *Under the assumptions of Lemma 3.1, if* $\gamma = \{t_0, \ldots, t_k\}$, $G \in \widehat{\mathcal{H}}_M$, *and* $t \geq t_k$, *the following inequality holds, for some positive constant* C_*:

$$(54) \qquad\qquad \|\langle \Psi_\gamma(\zeta) \, G(\eta_t) | \mathfrak{M}_0 \rangle\|_M \leq C_* \, \|G\|_M \, \bar{\mu}^{t-t_k} \, \mu_*^{|\gamma|}.$$

Proof. Proceeding as in the previous proof, we see that if G is odd and $|\gamma| > 1$, we get, in analogy with (53),

$$(55) \qquad \|\langle \Psi_\gamma(\zeta) \, G(\eta_t) | \mathfrak{M}_0 \rangle\|_M \leq M^{|\gamma|} \|G\|_M \, \bar{\mu}^{t-t_k} \prod_{j \text{ even}} \bar{\mu}^{r_j} \prod_{j \text{ odd}} (1 + 2\bar{\mu}^{r_j}),$$

and if G is even, by an obvious modification of the proof,

$$(56) \qquad \|\langle \Psi_\gamma(\zeta) \, G(\eta_t) | \mathfrak{M}_0 \rangle\|_M \leq M^{|\gamma|} \|G\|_M \, \bar{\mu}^{t-t_k} \prod_{j \text{ odd}} \bar{\mu}^{r_j} \prod_{j \text{ even}} (1 + 2\bar{\mu}^{r_j}).$$

Writing $G(\eta) = G^{(+)}(\eta) + G^{(-)}(\eta)$, where $G^{(\pm)}(\eta) = \frac{G(\eta) \pm G(-\eta)}{2} \in \widehat{\mathcal{H}}_M$, and observing that $\|G\|_M = \|G^{(+)}\|_M + \|G^{(-)}\|_M$, we get the result (54). $\qquad\qquad \square$

Proof of Inequality 46. We denote by m_γ, M_γ the minimum and maximum of the set $\gamma \in \mathfrak{g}$, $\gamma \neq \emptyset$, and if $\gamma = \{t_0, \ldots, t_k\}$, then $\gamma + t = \{t_0 + t, \ldots, t_k + t\}$, $t \geq -t_0$.

Using the Walsh expansion (29) we have $f(S^t \widehat{\zeta}) = \sum_\gamma f_\gamma \widehat{\Psi}_{\gamma+t}(\widehat{\zeta})$, and we write

$$f(\widehat{\zeta}) f(S^t \widehat{\zeta}) = f_\emptyset f(S^t \widehat{\zeta}) + C_t^{(1)}(\widehat{\zeta}) + C_t^{(2)}(\widehat{\zeta}) - R_t(\widehat{\zeta}),$$

$$(57) \quad C_t^{(1)}(\widehat{\zeta}) = \sum_{\substack{\gamma, \gamma' \neq \emptyset \\ M_\gamma < m_{\gamma'+t}}} f_\gamma f_{\gamma'} \Psi_\gamma(\widehat{\zeta}) \widehat{\Psi}_{\gamma'+t}(\widehat{\zeta}), \quad C_t^{(2)}(\widehat{\zeta}) = \sum_{\substack{\gamma, \gamma' \neq \emptyset \\ M_\gamma \geq m_{\gamma'+t}}} f_\gamma f_{\gamma'} \Psi_\gamma(\widehat{\zeta}) \Psi_{\gamma'+t}(\widehat{\zeta})$$

and $R_t(\widehat{\zeta}) = \sum_{\gamma, \gamma' \neq \emptyset} f_\gamma \Psi_\gamma(\widehat{\zeta}) f_{\gamma'} \langle \Psi_{\gamma'} \rangle \chi(M_\gamma \geq m_{\gamma'+t})$, where χ is the indicator function.

As $|\langle \Psi_{\gamma\prime} \rangle| \leq \|\langle \Psi_{\gamma\prime}|\mathfrak{M}_0 \rangle\|_M$ we see, by (16) and (27), that

$$(58) \qquad \|\langle R_t(\cdot)|\mathfrak{M}_0 \rangle\|_M \leq \sum_{\gamma\prime \neq \emptyset} |f_{\gamma\prime} \langle \Psi_{\gamma\prime} \rangle| \sum_{\gamma: M_\gamma \geq t} |f_\gamma| \, \|\langle \Psi_\gamma | \mathfrak{M}_0 \rangle\|_M \; \leq \; \text{const} \, \|\tilde{f}\|_{\mathcal{C}^\alpha}^2 \, \kappa^t.$$

Passing to $C_t^{(1)}$, let $\gamma, \gamma\prime \in \mathfrak{g}$ be such that $M_\gamma = r$, $m_{\gamma\prime} = m$ and $r < m + t$. Taking the conditional expectation with respect to $\mathfrak{M}_0^r$ we get by Proposition 5.1

$$\left| \left\langle \Psi_\gamma \widehat{\Psi}_{\gamma\prime+t} | \mathfrak{M}_0 \right\rangle \right| = \left| \left\langle \Psi_\gamma \left[\mathcal{T}^{t+m-r} \widehat{G}_{\gamma\prime} \right] (\eta_r) | \mathfrak{M}_0 \right\rangle \right| \leq C_* \, \mu_*^{|\gamma|} \bar{\mu}^{t+m-r} \|\widehat{G}_{\gamma\prime}\|_M,$$

where $\widehat{G}_{\gamma\prime}$ is defined in (50). Therefore, again by Inequalities (16) and (27), we see that

$$\left\| \left\langle C_t^{(1)}(\cdot) | \mathfrak{M}_0 \right\rangle \right\|_M \leq \text{const} \, \|\tilde{f}\|_\alpha^2 \sum_{m=0}^\infty \sum_{r=0}^{t+m-1} \sum_{k=0}^\infty \bar{\mu}^{t+m-r} 2^{-\alpha(r+m+k)} A_0^r A_m^{m+k},$$

where, for $0 \leq j \leq k \in \mathbb{Z}_+$ we set $A_j^k := \sum_{\gamma \in \mathfrak{g}} \mu_*^{|\gamma|} \chi(m_\gamma = j, M_\gamma = k) \; < (1 + \mu_*)^{k-j+1}$. As $\bar{\mu} < \kappa = 2^{-\alpha}(1 + \mu_*) < 1$, we get the estimate

$$(59) \qquad \left\| \left\langle C_t^{(1)}(\cdot) | \mathfrak{M}_0 \right\rangle \right\|_M \leq \text{const} \, \|f\|_\alpha^2 \sum_{m=0}^\infty 2^{-\alpha m} \sum_{r=0}^{t+m-1} \kappa^r \bar{\mu}^{t+m-r} \leq \text{const} \, \|f\|_\alpha^2 \, \kappa^t.$$

Turning to $C_t^{(2)}$, observe that $\Psi_\gamma \Psi_{\gamma\prime} = \Psi_{\gamma\Delta\gamma\prime}$, where $\gamma\Delta\gamma\prime = \gamma \setminus \gamma\prime \, \cup \, \gamma\prime \setminus \gamma$, so that

$$(60) \qquad \left\| \left\langle C_t^{(2)}(\cdot) | \mathfrak{M}_0 \right\rangle \right\|_M \leq \sum_{m,s,k=0}^\infty \sum_{\gamma: M_\gamma = t+m+s} |f_\gamma| \sum_{\substack{\gamma\prime: m_{\gamma\prime} = m \\ M_{\gamma\prime} = m+k}} |f_{\gamma\prime}| \, \left| \left\langle \Psi_{\gamma\Delta\{\gamma\prime+t\}} | \mathfrak{M}_0 \right\rangle \right|_M.$$

Let $n = \min\{s, k\}$, $N = \max\{s, k\}$, $\Omega_n = \{m, \ldots, m+n\}$ and $\gamma_1 = \gamma \cap \{0, \ldots, m-1\}$, $\gamma_{11} = \gamma \cap \Omega_n$, $\gamma_{12} = \gamma\prime \cap \Omega_n$, $\gamma_2 = \gamma \cup \{\gamma\prime + t\} \cap \{t+n+1, \ldots, t+N\}$. If $\bar{\gamma} = \gamma_{11}\Delta\gamma_{12}$ we have $\gamma\Delta\{\gamma\prime + t\} = \gamma_1 \cup \bar{\gamma} \cup \gamma_2$, and the sets $\gamma_1, \bar{\gamma}, \gamma_2$ have no common elements.

It is not hard to see by induction that

$$\sum_{\gamma_{11}, \gamma_{12} \subseteq \Omega_n} \mu_*^{|\gamma_{11}\Delta\gamma_{12}|} = (2(1 + \mu_*))^{n+1},$$

so that the sum on the right of (60), for fixed m, s, k, is bounded by

$$\text{const} \, \|\tilde{f}\|_\alpha^2 \, \kappa^{t+m} 2^{-\alpha m} \kappa^{N-n} (2^{-2\alpha+1}(1 + \mu_*))^n.$$

If $\alpha > 1/2$ and $\bar{\mu}$ is so small that $2^{-2\alpha+1}(1 + \mu_*) < 1$, all series converge and we get

$$(61) \qquad \left\| \left\langle C_t^{(2)}(\cdot) | \mathfrak{M}_0 \right\rangle \|_M \right\| \leq \text{const} \, \|\tilde{f}\|_\alpha^2 \, \kappa^t.$$

Finally, the inequality $|f_\emptyset| \|\langle f(S^t \cdot | \mathfrak{M}_0) \|_M \leq \text{const} \, \bar{\mu}^t \|\tilde{f}\|_{\mathcal{C}^\alpha}^2$ is an immediate consequence of Theorem 2.1 and Lemma 3.4. The proof of (46) follows from this estimate, together with the previous estimates (58), (59) and (61). □

References

1. N. Angelescu, R. A. Minlos, V. A. Zagrebnov, *The one-particle energy spectrum of weakly coupled quantum rotators*, J. Math. Phys. **41** (2000), no. 1, 1–23.

2. C. Boldrighini, R. A. Minlos, and A. Pellegrinotti, *Interacting random walk in a dynamical random environment.* I. *Decay of correlations*, Ann. Inst. Henri Poincaré, Sec. Probabilités **30** (1994), 519–558.

3. C. Boldrighini, R. A. Minlos, and A. Pellegrinotti, *Interacting random walk in a dynamical random environment.* II, Ann. Inst. Henri Poincaré, Sec. Probabilités **30** (1994), 559–605.

4. C. Boldrighini, R. A. Minlos, and A. Pellegrinotti, *Random walk in a fluctuating random environment with Markov evolution*, On Dobrushin's way. From probability theory to statistical physics. (Eds. R. A. Minlos, Senya Shlosman, Yu. M. Suhov), Amer. Math. Soc. Transl. Ser. 2, vol. 198, Amer. Math. Soc., Providence, RI, 2000, pp. 13–35.

5. C. Boldrighini, R. A. Minlos, and A. Pellegrinotti, *Random walks in a random (fluctuating) environment*, Russian Math. Surveys **62** (2007), no. 4, 663–712.

6. C. Boldrighini, R. A. Minlos, and A. Pellegrinotti, *Random walks in random environment with Markov dependence on time*, Condensed Matter Physics **11** (2008), no. 2, 209–221.

7. Yu. A. Davydov, *Mixing conditions for Markov chains*, Teor. Veroyatnost. i Primenen. **18** (1973), no. 2, 321-338. (Russian); English transl. Theory Probab. Appl. **18** (1973), no. 2, 312–328.

8. D. Dolgopyat, G. Keller, C. Liverani, *Random walk in Markovian environment*, Ann. Probab. **36** (2008), no. 5, 1676–1710.

9. D. Dolgopyat, C. Liverani, *Non perturbative approach for random walk in Markovian environment*, Electronic Communications in Probability **14** (2009), 245–251.

10. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372–414.

11. I. A. Ibragimov, Yu. V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff Publishing, Groningen, 1971.

12. C. Kipnis, S. R. S. Varadhan, *Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions*, Commun. Math. Phys. **104** (1986), 1–19.

13. Yu. G. Kondratiev, R. A. Minlos, *One-particle subspaces in the stochastic XY model*, J. Statist. Phys. **87** (1997), no. 3–4, 613–642.

14. Yu. G. Kondratiev, R. A. Minlos, E. Zhizhina, *One-particle subspace of the Glauber dynamics generator for continuous particle systems*, Rev. Math. Phys. **16** (2004), no. 9, 1073–1114.

15. V. A. Malyshev and R. A. Minlos, *Linear Infinite-Particle Operators*, Nauka, Moscow, 1994; English transl. Transl. Math. Monographs, vol. 143, Amer. Math. Soc., Providence, RI, 1995.

Dipartimento di Matematica G. Castelnuovo, Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Roma; GNFM, Istituto Nazionale di Alta Matematica, Piazzale Aldo Moro 5, 00185 Roma
E-mail address: `boldrigh@mat.uniroma1.it`

Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Děčín Branch, Pohranicní 1, 40501 Děčín
E-mail address: `anto.marchesiello@gmail.com`

Institut für Mathematik Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich
E-mail address: `chiara.saffirio@math.uzh.ch`