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PERCOLATIONS AND PHASE TRANSITIONS IN A CLASS OF RANDOM SPIN SYSTEMS

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Dedicated to Yuri Berezansky on occasion of his 90th Birthday

Abstract. The aim of this paper is to give a review of recent results of Yu. Kondratiev, Yu. Kozitsky, T. Pasurek and myself on the multiplicity of Gibbs states (phase transitions) in infinite spin systems on random configurations, and provide a 'pedestrian' route following Georgii–Häggström approach to (closely related to phase transitions) percolation problems for a class of random point processes.

1. Introduction

In recent papers [3, 4] we studied equilibrium states of the following random particle system. A countable number of point particles is chaotically distributed over a Euclidean space X and modeled by a random point process in X. Realisations γ of this process belong to the space $\Gamma(X)$ of all locally finite subsets (configurations) of X, that is,

$$\Gamma(X) = \{\gamma \subset X : \ N(\gamma_\Lambda) < \infty \text{ for any } \Lambda \in \mathcal{B}_c(X)\},$$

where $\mathcal{B}_c(X)$ is the collection of compact subsets of X and $N(\gamma_\Lambda)$ denotes cardinality of $\gamma_\Lambda := \gamma \cap \Lambda$.

Each particle $x \in \gamma$ possesses internal structure described by a mark (spin) $\sigma(x)$ taking values in a single-spin (Euclidean) space S, and is characterized by a single-spin probability measure χ on S. The system as a whole is described by the law μ of the underlying point process and a spin-spin pair interaction $W_{xy} : S \times S \to \mathbb{R}$, which depends on the location of the particles $x, y \in X$.

For a fixed configuration γ, the 'quenched' system is described by the formal energy

$$E^\gamma(\sigma) := \sum_{\{x,y\}\subset\gamma} W_{xy}(\sigma(x), \sigma(y)), \ \sigma = (\sigma(x))_{x\in\gamma}.$$

The equilibrium states of the system are given by the corresponding Gibbs measures η_γ on the product space $S^\gamma = \prod_{x\in\gamma} S_x, \ S_x = S$. Heuristically, these are probability measures defined by the formula

$$\eta_\gamma(d\sigma) := Z^{-1} \exp\left(-\beta E^\gamma(\sigma)\right) \bigotimes_{x\in\gamma} \chi(d\sigma(x)), \tag{1}$$

where β denotes the inverse temperature of the system and Z is the normalizing factor. Clearly, $E^\gamma(\sigma)$ and thus the right-hand side of (1) has only heuristic meaning for all but finite configurations γ (which form null-set for the majority of interesting point processes). The rigorous definition of η_γ is based on the Dobrushin-Lanford-Ruelle approach, see Section 3.1. We denote by $\mathcal{G}(S^\gamma, W, \chi)$ the space of all such Gibbs measures.

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In the spirit of [2], the elements of $\mathcal{G}(S^\gamma, W, \chi)$ can be called quenched Gibbs states of our system.

The study of the structure of the set $\mathcal{G}(S^\gamma, W, \chi)$ is of a great importance. In particular, there are three fundamental questions arising here:

(E) *Existence*: is $\mathcal{G}(S^\gamma, W, \chi)$ not empty?
(U) *Uniqueness: is* $\mathcal{G}(S^\gamma, W, \chi)$ a singleton?
(M) *Multiplicity: does* $\mathcal{G}(S^\gamma, W, \chi)$ contain at least two elements? Positive answer indicates the appearance of phase transition in the system.

Sufficient conditions of the existence of $\eta \in \mathcal{G}(S^\gamma, W, \chi)$ have been derived in [3] for a wide class of underlying point processes γ. In [4], we discussed question (M) for Poisson γ and ferromagnetic interaction

$$W_{xy}(u, v) = -\phi(|x - y|)uv, \tag{2}$$

where $S = \mathbb{R}^1$ and $\phi : \mathbb{R} \to \mathbb{R}_+$ is a bounded measurable function with supp $\phi \subset [0, R]$ for some fixed $R > 0$, by using the relationship with bond percolation problem (see [10]). Observe that any $\eta \in \mathcal{G}(S^\gamma, W, \chi)$ with W given by (2) can be understood as a Gibbs measure with nearest neighbour interaction on the geometric graph $(\gamma)_R := (\gamma, \varepsilon)$ with vertex set γ and edge set $\varepsilon = \{\{x, y\} \subset \gamma : |x - y| \le R\}$. It turns out that the methods developed in [4] can be applied to a wider class of underlying random point processes, once the occurrence of bond percolation for a typical realisation of $(\gamma)_R$ is known.

For a general infinite graph with vertex set G and edge set $\mathcal{E} \subset G \times G$ the classical bond percolation problem is posed in the following way. Declare each edge $e \in \mathcal{E}$ open with probability q and closed otherwise, independently of all other edges. Construct a random graph $(G, \mathcal{E})_q$ by removing all closed edges. If there exists $q_c < 1$ such that $(G, \mathcal{E})_q$, $q > q_c$, contains an infinite connected component with positive probability, we say that the bond percolation with critical value q_c occurs. In this case, the infinite connected component is unique and occurs with probability 1 (which follows from the Kolmogorov $0 - 1$ Law).

In a situation when the underlying graph is random itself, it is important to understand whether bond percolation can be found in its typical realisation. In particular, the question of occurrence of bond percolation in the geometric graph $(\gamma)_R$ for μ-a.a. γ is indirectly related to the continuum percolation problem for μ, see [14]. This relationship was used in the proof of the appearance of bond percolation for a.a. γ distributed according to a Poisson measure on $\Gamma(X)$ in [15]. For a class of Gibbs measures on $\Gamma(X)$, the proof is given in [8] as the particular case of a more general result. In Section 2 we show that the approach of [8] is applicable to a wider class of point processes.

In Section 3 we discuss the application of the above results to the study of the multiplicity of Gibbs measures $\eta_\gamma \in \mathcal{G}(S^\gamma, W, \chi)$, for a typical realisation γ of a random point process that satisfies conditions of Section 2.

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2. Percolations in a random geometric graph

In this section, we discuss the problem of occurrence of bond percolation for a typical configuration $\gamma \in \Gamma(X)$ endowed with a natural graph structure. We closely follow the scheme proposed in [8].

Recall that the configuration space $\Gamma(X)$ is endowed with the vague topology, which is the weakest topology that makes continuous all mappings

$$\Gamma(X) \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x), \quad f \in C_0(X),$$

where $C_0(X)$ is the set of all continuous functions on $f : X \to \mathbb{R}$ with compact support. It is known that this topology is completely metrizable, which makes $\Gamma(X)$ a Polish space (see, e.g., [11, Section 15.7.7] or [17, Proposition 3.17]). An explicit construction of the appropriate metric can be found in [13]. By $\mathcal{P}(\Gamma(X))$ we denote the space of all probability measures on the Borel σ-algebra $\mathcal{B}(\Gamma(X))$ of subsets of $\Gamma(X)$.

2.1. **Setting and the main result.** We start with the construction of the probability space associated with the random graph $(\gamma)_R$, cf. [4]. Let $X^{(2)}$ be the space of two-element subsets of X and $E := \Gamma(X^{(2)})$, so that $\varepsilon \in E$ for any $\gamma \in \Gamma(X)$. Any probability measure $\varpi \in \mathcal{P}(E)$ can be characterized by its Laplace transform

$$L_\varpi(f) := \int_E \exp\left[\sum_{\{x,y\}\in\varepsilon} \log\left(1 + f(x,y)\right)\right] \varpi(d\varepsilon),$$

where f runs the set of all measurable symmetric functions $X \times X \to (-1, 0]$.

For a given $\gamma \in \Gamma(X)$, let $\varpi_\gamma \in \mathcal{P}(E)$ be the Dirac measure concentrated at ε_γ. Its Laplace transform has the form

$$L_{\varpi_\gamma}(f) = \exp\left[\sum_{\{x,y\}\in\varepsilon} \log\left(1 + \mathbf{1}_R(x-y) f(x,y)\right)\right],$$

where $\mathbf{1}_R$ is the indicator of the ball of radius R centered at zero in X.

For a constant $q \in [0,1]$, let us consider the independent q-thinning of ϖ_γ, that is, a measure $\varpi_\gamma^q \in \mathcal{P}(E)$, cf. [6, Section 11.2], defined by the relation

$$L_{\varpi_\gamma^q}(f) = L_{\varpi_\gamma}(qf).$$

The q-thinning of ϖ_γ means that each configuration ε distributed according to ϖ_γ is 'thinned' in the sense that $\{x,y\} \in \varepsilon$ is removed from the edge configuration with probability $1-q$ and is kept with probability q. The probability distribution of such 'thinned' configurations is then ϖ_γ^q. In these notations, the occurrence of bond percolation on (γ, ε) with critical probability q_c means that

$$\varpi_\gamma^q\left(\{\varepsilon : (\gamma,\varepsilon) \text{ contains infinite connected component}\}\right) > 0.$$

Assume now that γ is distributed according to a probability measure ω on $\Gamma(X)$. Then the probability distribution of the graph (γ, ε) is defined by the measure

$$\zeta(d\gamma, d\varepsilon) = \varpi_\gamma(d\varepsilon)\omega(d\gamma).$$

Denote

$$\zeta^q(d\gamma, d\varepsilon) := \varpi_\gamma^q(d\varepsilon)\omega(d\gamma). \tag{3}$$

In what follows, the notation $x \leftrightarrow \infty$ will indicate that $x \in \gamma$ belongs to the infinite connected component of (γ, ε).

We assume from now on that the underlying point process depends on a parameter $z \in \mathbb{R}^+$, that is, $\omega = \omega_z$. We will omit the subscript z whenever possible.

Definition 1. We say that ω is l-dense, $l > 0$, if for any $M \in (0,1)$ and $n_* \in \mathbb{N}$ there exists $z_* = z_*(M, n_*)$ such that

$$\omega_z(N(\gamma_Q) \geq n_* \,|\gamma_{Q^c} = \eta) \geq M,$$

for all $z \geq z_*$, any cube $Q \subset X$ with side length l (cf. (5)) and a.a. boundary conditions $\eta \in \Gamma(X \setminus Q)$.

Let $l_R := 2^{-1} d^{-1/2} R$. Our aim is to prove the following result.

Theorem 2. *Assume that ω is l_R-dense. Then for any $q_c \in (0,1)$ there exist $z_c > 0$ and $\theta > 0$ such that*

$$\zeta_z^q\left(\{(\gamma,\varepsilon) : x \leftrightarrow \infty \text{ for all } x \in \gamma_Q\}\right) \geq \theta, \tag{4}$$

for all $z \geq z_c$, $q \geq q_c$ and any cube Q with side length l_R.

The proof of this result will be given in Section 2.3 after some technical preparations.

The next statement follows directly from Theorem 2 and establishes the existence of bond percolation for a class of configurations γ.

Corollary 3. *For any z, q and Q as above there exists a set $\mathcal{C} = \mathcal{C}(z,q,Q) \subset \Gamma(X)$ such that $\omega_z(\mathcal{C}) \geq \theta/2$ and for any $\gamma \in \mathcal{C}$ its restriction $\gamma_Q \neq \emptyset$, and*

$$\varpi_\gamma^q\left(\{\varepsilon : x \leftrightarrow \infty \text{ for all } x \in \gamma_Q\}\right) \geq \theta/2.$$

Proof. The statement follows directly from the disintegration formula $\zeta_z^q(d\gamma, d\varepsilon) := \varpi_\gamma^q(d\varepsilon)\omega_z(d\gamma)$ (cf. (3)) and bound (4). Indeed, for $A_\gamma := \{\varepsilon : x \leftrightarrow \infty \text{ for all } x \in \gamma_Q\}$ set

$$B_+ := \left\{\gamma :\ \varpi_\gamma^q(A_\gamma) \geq \theta/2\right\}, \quad B_- := \left\{\gamma :\ \varpi_\gamma^q(A_\gamma) < \theta/2\right\}$$

and assume that $\omega(B_+) \leq \theta/2$. Then

$$\begin{aligned}\zeta_z^q\left(\{(\gamma,\varepsilon) : x \leftrightarrow \infty \text{ for all } x \in \gamma_\Delta\}\right) &= \int_{B_+} \varpi_\gamma^q(A_\gamma)\omega(d\gamma) + \int_{B_-} \varpi_\gamma^q(A_\gamma)\omega(d\gamma) \\ &\leq \omega(B_+) + \frac{\theta}{2}(1-\omega(B_+)) \leq \theta - \left(\frac{\theta}{2}\right)^2 < \theta,\end{aligned}$$

which contradicts (4). □

Corollary 4. *Assume in addition that ω_z is ergodic with respect to translations by elements of X. Then*

$$\varpi_\gamma^q\left(\{\varepsilon : (\gamma,\varepsilon) \text{ contains infinite connected component}\}\right) = 1,$$

for ω_z-a.a. $\gamma \in \Gamma(X)$.

Proof. Observe that the set $\bar{\mathcal{C}} := \cup_\Delta \mathcal{C}(z,q,Q)$ is translation invariant, which implies that $\omega_z(\bar{\mathcal{C}}) = 1$. The statement follows now from the Kolmogorov $0-1$ Law applied to the bond percolation problem on γ. □

2.2. **Auxiliary facts.** In this section we give some general known facts, see [8].

2.2.1. *Combinatorics.*

Lemma 5. *Let G be a n-element set. Introduce a random graph structure on G by declaring any two elements $x, y \in \mathsf{G}$ connected by an edge with probability q, independently of other edges. Then*

$$\rho(n,q) = \text{Prob}\ (\mathsf{G} \textit{ is connected}) \to 1, \quad n \to \infty.$$

Proof. It is obvious that $\rho(1,q) = 1$ and $\rho(2,q) = q$. For $n \geq 3$, $\rho(n,q)$ is bounded from below by the probability that a fixed element of G is connected to any other element by a path of length 2, which is greater than $1-(n-1)(1-q^2)^{(n-2)}$. Thus we have

$$\rho(n,q) \geq 1-(n-1)(1-q^2)^{(n-2)} \to 1, \quad n \to \infty.$$

□

Corollary 6. *We have*

$$\hat{\rho}(n,q) := \inf_{k \geq n} \rho(k,q) \to 1, \quad n \to \infty.$$

2.2.2. *Bond-Site percolation.* The mixed bond-site percolation problem on $\mathcal{Z} := \mathbb{Z}^d$ is posed in the following way, see [8]. Declare each vertex $k \in \mathbb{Z}^d$ and each edge (k,j), $|k-j| = 1$, open with probability p and closed otherwise, independently of other vertices and edges. Construct a random graph $(\mathcal{Z})_p = (G, \mathcal{E}) \in \Gamma(\mathcal{Z}) \times \Gamma(\mathcal{Z}^{(2)})$ (with $\mathcal{Z}^{(2)}$:= the space of two-element subsets of $\mathcal{Z}$) by setting

$$G := \{\text{all open vertices}\}\,, \qquad \mathcal{E} := \{\text{all open edges}\}\,.$$

The corresponding probability space can be introduced in a similar way to Section 2.1.

Lemma 7. [8]. *There exists a critical value $p_c < 1$ such that, for any $p > p_c$, the origin $o \in \mathcal{Z}$ belongs to an infinite connected component of $(\mathcal{Z})_p$ with probability $\theta(p) > 0$, that is,*

$$\mathbb{P}\left(o \leftrightarrow \infty \ \textit{in}\ (\mathcal{Z})_p\right) = \theta(p).$$

Corollary 8. *Let us denote by $(\mathcal{Z})_{\geq p}$ the class of random graphs $(G, \mathcal{E}) \in \Gamma(\mathcal{Z}) \times \Gamma(\mathcal{Z}^{(2)})$ satisfying the relations*

$$\begin{aligned} \mathbb{P}\left(k \in G \ \left|G \setminus k = g\right.\right) &\geq p, \\ \mathbb{P}\left((k,j) \in \mathcal{E} \ \left|k, j \textit{ open}, \mathcal{E} \setminus (k,j) = e\right.\right) &\geq p, \end{aligned}$$

for any $k \in \mathcal{Z}$ and $(k,j) \in \mathcal{Z}^{(2)}$, $|k-j| = 1$, and all boundary conditions $g \in \Gamma(\mathcal{Z} \setminus k)$, $e \in \Gamma(\mathcal{Z}^{(2)} \setminus (k,j))$. It can be proved that the distribution of any $(G, \mathcal{E}) \in (\mathcal{Z})_{\geq p}$ stochastically dominates the distribution of random graph $(\mathcal{Z})_p$. Thus for any $(G, \mathcal{E}) \in (\mathcal{Z})_{\geq p}$ we have

$$\mathbb{P}\left(o \leftrightarrow \infty \ \textit{in}\ (G, \mathcal{E})\right) \geq \theta(p),$$

cf. [9].

2.2.3. *Cell model.* Our next goal is to construct an embedding $(\gamma, \varepsilon) \mapsto (G, \mathcal{E})$. We start with the following definitions.

Definition 9. Define the collection of sets

$$G(\gamma, \varepsilon; n) := \{A \subset X : N(\gamma_A) \geq n,\ (\gamma_A, \varepsilon_A) \text{ is connected}\}\,.$$

Definition 10. We say that two sets $A, B \subset X$ are (γ, ε)-connected and write $A \overset{(\gamma,\varepsilon)}{\sim} B$ if the exists $a \in \gamma_A$ and $b \in \gamma_B$ such that a and b are connected by a path in (γ, ε).

Let us introduce a partition of X by 'elementary' volumes. For a fixed $l > 0$, denote by Q_k the cube in X with side length l, centered at point lk, $k = (k^{(1)}, \ldots, k^{(d)}) \in \mathbb{Z}^d$, defined by the formula

$$Q_k := \left\{x = (x^{(1)}, \ldots, x^{(d)}) \in X :\ x^{(i)} \in \left[l(k^{(i)} - 1/2), l(k^{(i)} + 1/2)\right)\right\}. \tag{5}$$

We will use the notation $\gamma_k := \gamma_{Q_k}$.

In what follows, we choose $l = l_R := 2^{-1} d^{-1/2} R$, so that the distance between any two elements of adjacent cubes is no more than R.

Proposition 11. *Assume that ω is l_R-dense. Then for any $n_* \in \mathbb{N}$ and $M \in (0,1)$ there exists z_c such that for all $z \geq z_c$ we have*
(i)

$$\zeta_z^q\left(\{(\gamma, \varepsilon) : Q_k \in G(\gamma, \varepsilon)\}\,|\eta\right) \geq M \hat{\rho}(n_*, q), \tag{6}$$

for any boundary condition $\eta \in \Gamma(X \setminus Q_k)$ and all $k \in \mathbb{Z}^d$;
(ii)

$$\zeta_z^q\left(\left\{(\gamma, \varepsilon) : Q_k \overset{(\gamma,\varepsilon)}{\sim} Q_j\right\} |N(\gamma_k) \geq n, N(\gamma_j) \geq n\right) \geq \lambda(n, q), \tag{7}$$

where $\lambda(n,q) := 1 - (1-q)^{n^2}$, for all k, j such that $|k - j| = 1$.

Proof. We have

$$\begin{aligned}\zeta_z^q\left(\{(\gamma,\varepsilon): Q_k\in G(\gamma,\varepsilon)\}\,|\eta\right) &= \int \rho(N(\gamma_k),q)\omega_z(d\gamma\,|\eta)\\ &\geq \hat\rho(n_*,q)\ \omega_z(N(\gamma_k)\geq n_*\,|\eta).\end{aligned}$$

Estimate (6) follows now from l_R-density of ω_z.

Observe that $|a-b|\leq R$ for any $a\in\gamma_k$ and $b\in\gamma_j$. Then, for a fixed $\gamma\in\Gamma(X)$ such that $N(\gamma_k)\geq n,\ N(\gamma_j)\geq n$ we have

$$\varpi_\gamma^q\left(Q_k \overset{(\gamma,\varepsilon)}{\sim} Q_j\right)\geq 1-(1-q)^{n^2}$$

and the bound (7) follows. □

2.3. **Occurrence of the percolation.** Now we are in a position to prove Theorem 2**.**

Proof. For any configuration $(\gamma,\varepsilon)\in\Gamma(X)\times\Gamma(X^{(2)})$ define a graph $(\mathcal{V}_{(\gamma,\varepsilon)},\mathcal{E}_{(\gamma,\varepsilon)})$ with vertex set

$$\mathcal{V}_{(\gamma,\varepsilon)}:=\left\{Q_k: Q_k\in G(\gamma,\varepsilon),\ k\in\mathbb{Z}^d\right\}$$

and edge set

$$\mathcal{E}_{(\gamma,\varepsilon)}:=\left\{\{Q_k,Q_j\}: Q_k \overset{(\gamma,\varepsilon)}{\sim} Q_j,\quad k,j\in\mathbb{Z}^d,\ |k-j|=1\right\}.$$

This is a random graph with the probability distribution induced by the measure ζ_z^q on $\Gamma(X)\times\Gamma(X^{(2)})$.

Fix $q_c\in(0,1)$ and choose n_* such that $\lambda(n_*,q_c)>p_c$, where p_c is defined in Lemma 7. Then set $M=\frac{p_c}{\hat\rho(n_*,q_c)}$ and $z_c=z_c(M,n_*)$, cf. Definition 1. Observe $\hat\rho(n_*,q)$ and $\lambda(n_*,q)$ are non-decreasing functions of q. Then

$$p_*(q):=\min\ (M\hat\rho(n_*,q),\lambda(n_*,q))>p_c,$$

for any $q\geq q_c$. Then, according to estimate (6),

$$\zeta_z^q\left(\{(\gamma,\varepsilon): Q_k\in G(\gamma,\varepsilon)\}\,|\eta\right)\geq p_*,\quad \eta\in\Gamma(X\setminus Q_k),\quad z\geq z_c.$$

On the other hand, for any k,j such that $|k-j|=1$ the probability of the event that $Q_k \overset{(\gamma,\varepsilon)}{\sim} Q_j$ depends only on the number of elements in the cells Q_k,Q_j and not on the fact that the graphs $\gamma_{Q_k},\gamma_{Q_j}$ are connected, so that

$$\begin{aligned}&\zeta_z^q\left(\left\{(\gamma,\varepsilon): Q_k \overset{(\gamma,\varepsilon)}{\sim} Q_j\right\}\ |\ Q_k,Q_j\in G(\gamma,\varepsilon)\right)\\ &\qquad=\zeta_z^q\left(\left\{(\gamma,\varepsilon): Q_k \overset{(\gamma,\varepsilon)}{\sim} Q_j\right\}|\ N(\gamma_k)\geq n_*, N(\gamma_j)\geq n_*\ \right)\geq\lambda(n_*,q)\geq p_*.\end{aligned}$$

Thus $(\mathcal{V}_{(\gamma,\varepsilon)},\mathcal{E}_{(\gamma,\varepsilon)})\in(\mathcal{Z})_{\geq p}$. The statement follows now from Corollary 8. □

2.4. **Examples.** In this section, we give several examples of l-dense random point fields on X.

2.4.1. *Poisson measure.* A Poisson measure on the configuration space Γ_X is defined descriptively as follows (cf. [5, § 2.4]). Let τ be a σ-finite measure in $(X,\mathcal{B}(X))$. The Poisson measure π_τ with intensity τ is a probability measure on $\mathcal{B}(\Gamma_X)$ satisfying the following condition: for any pairwise disjoint sets $B_1,\dots,B_k\in\mathcal{B}(X)$ such that $\tau(B_i)<\infty$ $(i=1,\dots,k)$, and any $n_1,\dots,n_k\in\mathbb{Z}_+$, we have

$$\pi_\tau\left(\{\gamma: N(\gamma_{B_i}=n_i,\ i=1,\dots,k\}\right)=\prod_{i=1}^k\frac{\tau(B_i)^{n_i}\,\mathrm{e}^{-\tau(B_i)}}{n_i!}. \tag{8}$$

That is, for disjoint sets B_i the values $N(\gamma_{B_i})$ are mutually independent Poisson random variables with parameters $\tau(B_i)$, respectively.

Theorem 12. *Assume* $\sup_{k\in\mathbb{Z}^d}\ \tau(Q_k)<\infty$, *cf. (5) (which obviously holds in the case where τ is translation invariant). Then $\pi_{z\tau},\ z>0$, is l-dense.*

Proof. The result follows directly from (8). □

2.4.2. *Gibbs measures.* Fix a measurable function $\Phi: X\times X\to\mathbb{R}$. For any $\Delta\in\mathcal{B}_c(X)$ consider the relative local interaction energy

$$H_\Delta(\gamma_\Delta\,|\eta)=\sum_{\{x,y\}\in\gamma_\Delta}\Phi(x,y)+\sum_{x\in\gamma_\Delta}\sum_{y\in\eta_{\Delta^c}}\Phi(x,y).$$

Introduce a measure $\Pi_\Delta\,(\cdot\,|\eta)$ on $\Gamma(X)$ via the integral relation

$$\int F(\gamma)\Pi_\Delta\,(d\gamma\,|\eta)=Z_\Delta(\eta)^{-1}\int F(\gamma_\Delta\cup\eta_{\Delta^c})\exp\ (-H_\Delta(\gamma_\Delta\,|\eta))\ \lambda_z(d\gamma_\Delta),$$

where F is a positive measurable function on $\Gamma(X)$. Here λ_z is the Lebesgue-Poisson measure on the space of finite configurations $\Gamma_0(X)$ defined by the formula

$$\int_{\Gamma_0(X)}F(\gamma)\lambda_z(d\gamma)=F(\emptyset)+\sum_{k=1}^{\infty}\frac{z^k}{k!}\int_{X^k}F(x_1,\dots,x_k)dx_1\dots dx_k$$

and

$$Z_\Delta(\eta):=\int\exp\ (-H_\Delta(\gamma_\Delta\,|\eta))\ \lambda_z(d\gamma_\Delta)$$

is the normalizing factor (called the partition function) making $\Pi_\Delta\,(\cdot\,|\eta)$ a probability measure on $\Gamma(X)$.

The family $\Pi:=\{\Pi_\Delta\,(\cdot\,|\eta)\}_{\Delta\in\mathcal{B}_c(X),\eta\in\Gamma(X)}$ is a Gibbsian specification on $\Gamma(X)$ (see e.g. [7, 16]). We say that a probability measure ν on $\Gamma(X)$ is a Gibbs measure associated with the specification Π if it satisfies the Dobrushin-Lanford-Ruelle (DLR) equation

$$\nu(B)=\int_{\Gamma(X,S)}\Pi_\Delta\,(B\,|\eta)\,\nu(d\gamma),\tag{9}$$

for all $B\in\mathcal{B}(\Gamma(X))$ and $\Delta\in\mathcal{B}_c(X)$. We denote by $\mathcal{G}(\Gamma(X),\Phi)$ the set of all such measures.

Assumptions on the interaction potentials are as follows:

(i) $\Phi(x,y)=\varphi(x-y)$; $\varphi(x)\le 0$ if $|x|\ge r$, for some $r>0$, and

$$\int_{|x|\ge\delta}\varphi_+(x)dx<\infty,\ \text{for any }\delta>0,$$

where φ_+ is the positive part of φ;

(ii) $\varphi\ge 0$ or φ is lower regular and superstable in the sense of Ruelle, see [19] and [8].

It is known that $\mathcal{G}(\Gamma(X),\Phi)\neq\emptyset$ under conditions (i) and (ii). The following result has been proved in [8].

Theorem 13. *Any Gibbs measure $\mu\in\mathcal{G}(\Gamma(X),\Phi)$ is l-dense provided $l>r/2$.*

2.4.3. *Cluster measures.* Let us recall the notion of a cluster point process with independent clusters (see, e.g., [5]). Its realisations are constructed in two steps: (i) a background random configuration of (invisible) 'centres' is obtained as a realisation of some point process γ_{c} governed by a probability measure μ on $\Gamma(X)$, and (ii) relative to each centre $x\in\gamma_{\mathrm{c}}$, a set of observable secondary points (referred to as a cluster centred at x) is generated, independently of all other clusters, according to a point process γ'_x with distribution μ_x on the space of finite configurations $\Gamma_0(X)$. The resulting (countable) assembly of random points, called the cluster point process (CPP), can be expressed symbolically as

$$\gamma = \bigsqcup_{x\in\gamma_c} \gamma'_x, \tag{10}$$

where the disjoint union signifies that multiplicities of points should be taken into account. Note that CPP configurations (10) may in principle have accumulation and/or multiple points due to the overlapping contributions from different clusters. In what follows we assume that CPP configurations are proper, that is, $\gamma \in \Gamma(X)$ a.s. (see [1] for necessary and sufficient conditions of properness of Gibbs CPP), and denote by $\mu_{cl} \in \mathcal{P}(\Gamma(X))$ the corresponding distribution.

In addition, we assume that random clusters are independent and identically distributed, being governed by a common probability law translated to the cluster centres, so that $\mu_x(A) = \mu_0(A - x)$ ($x \in X$, $A \in \mathcal{B}(\Gamma_0(X))$).

For a fixed $K \subset X$ introduce the set

$$\mathfrak{X}_K := \{\xi \in \Gamma_0(X) : \xi_K \neq \emptyset\}.$$

Theorem 14. *Assume that μ is l'-dense for some $l' < l$. Then μ_{cl} is l-dense.*

Proof. Fix a cube Q'' with side length $l'' < l - l'$ such that $\mu_0(\mathfrak{X}_{Q''}) > 0$. Then, for any cube Q with side length l, there exists a cube Q' with site length l' such that $Q' + Q'' \subset Q$. It is clear that

$$\mu_{cl}\left(N(\gamma_Q) \geq n_*\right) \geq \mu\left(N(\gamma_{Q'}) \geq n_*\right)\mu_0(\mathfrak{X}_{Q''})^{n_*}$$

and the statement follows. □

3. Phase transitions in infinite spin systems on random geometric graphs

In this section, we consider a class of spin systems on a typical geometric graph $(\gamma)_R$, where $\gamma \in \Gamma(X)$ is governed by a random point process $\mu \in \mathcal{P}(\Gamma(X))$, and derive sufficient conditions of the multiplicity of their equilibrium (Gibbs) states, following [3], [4].

3.1. Construction of Gibbs measures on geometric graphs.

To each $x \in \gamma$, we assign a variable (spin) $\sigma_x \in \mathbb{R}$. Then the configuration of spins corresponding to γ is $\sigma = (\sigma_x)_{x\in\gamma} \in \mathbb{R}^\gamma$. The set of spin configurations $\mathbb{R}^\gamma$ is equipped with the product topology and with the corresponding Borel σ-field $\mathcal{B}(\mathbb{R}^\gamma)$. For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, by σ_Λ we denote the 'configuration in Λ', i.e., $\sigma_\Lambda := (\sigma_x)_{x\in\gamma_\Lambda}$. Given two configurations σ and $\bar\sigma$, set $\sigma_\Lambda \times \bar\sigma_{\Lambda^c} = (s_x)_{x\in\gamma}$ with $s_x = \begin{cases} \sigma_x, x \in \gamma_\Lambda \\ \bar\sigma_x, x \in \gamma_{\Lambda_c} \end{cases}$, $\Lambda^c := \mathbb{R}^d \setminus \Lambda$. We assume that the spin-spin interaction is given by pair potential (2).

Let χ be a finite symmetric measure on $\mathbb{R}$. Our aim is to construct Gibbs measures on $\mathbb{R}^\gamma$ corresponding to the single-spin measures $\chi_x = \chi$ and to the following relative energy functionals

$$E^\gamma_\Lambda(\sigma_\Lambda|\bar\sigma_{\Lambda^c}) = -\sum_{\substack{\{x,y\}\subset\gamma\\|x-y|\leq R}} \phi(|x-y|)\sigma_x\sigma_y - \sum_{x\in\gamma_\Lambda}\sum_{y\in\partial x\cap\gamma_{\Lambda^c}} \phi(|x-y|)\sigma_x\bar\sigma_y, \tag{11}$$

where $\phi : [0, R] \to \mathbb{R}_+$ is a bounded measurable function. For any compact $\Lambda \subset X$ and $\bar\sigma \in \mathbb{R}^\gamma$, we define

$$\Pi_\Lambda(A|\bar\sigma) = \frac{1}{Z_\Lambda(\bar\sigma)} \int_{\mathbb{R}^{\gamma_\Lambda}} \mathbb{I}_A(\sigma_\Lambda \times \bar\sigma_{\Lambda^c}) \exp(-E^\gamma_\Lambda(\sigma_\Lambda|\bar\sigma_{\Lambda^c}))\chi_\Lambda(d\sigma_\Lambda),$$

where $\mathbb{I}_A$ is the indicator of $A \in \mathcal{B}(\mathbb{R}^\gamma)$, E^γ_Λ is as in (11), and

$$\chi_\Lambda(d\sigma_\Lambda) := \bigotimes_{x\in\gamma_\Lambda} \chi(d\sigma_x),$$

$$Z_\Lambda(\bar\sigma) := \int_{\mathbb{R}^\gamma} \exp(-E^\gamma_\Lambda(\sigma_\Lambda|\bar\sigma_{\Lambda^c}))\chi_\Lambda(d\sigma_\Lambda).$$

Thus, for any $A \in \mathcal{B}(\mathbb{R}^\gamma)$, $\Pi_\Lambda(A|\cdot)$ is $\mathcal{B}(\mathbb{R}^\gamma)$-measurable, and, for each $\bar\sigma \in \mathbb{R}^\gamma$, $\Pi_\Lambda(\cdot|\bar\sigma)$ is a probability measure on $(\mathbb{R}^\gamma, \mathcal{B}(\mathbb{R}^\gamma))$. The family $\{\Pi_\Lambda : \Lambda \in \mathcal{B}_c(X)\}$ is the Gibbs specification of the model (see e.g. [7, 16] and Introduction in [12]).

Similar to the case of continuum systems (Section 2.4.2), a probability measure η on $(\mathbb{R}^\gamma, \mathcal{B}(\mathbb{R}^\gamma))$ is said to be a Gibbs measure associated with Π if it satisfies the Dobrushin-Lanford-Ruelle equation

$$\eta(A) = \int_{\mathbb{R}^\gamma} \Pi_\Lambda(A|\sigma)\eta(d\sigma), \quad \text{for all} \quad A \in \mathcal{B}(\mathbb{R}^\gamma).$$

The set of all such measures will from now on be denoted by $\mathcal{G}(\mathbb{R}^\gamma, \phi, \chi)$ (with certain abuse of previous notations, cf. Introduction).

We study Gibbs measures with *a priori* prescribed support properties. We call them tempered Gibbs states. For an $\alpha > 0$, we define

$$\Sigma(\alpha, q) := \left\{ \sigma \in \mathbb{R}^\gamma : \sum_{x\in\gamma} |\sigma_x|^q e^{-\alpha|x|} < \infty \right\}$$

and set $\mathcal{G}_{\alpha,q}(\mathbb{R}^\gamma, \phi, \chi) = \{\mu \in \mathcal{G}(\mathbb{R}^\gamma, \phi, \chi) : \mu(\Sigma(\alpha, q)) = 1\}$. The following result has been proved in [3].

Theorem 15. *Let the single-spin measure χ be such that*

$$\int_{\mathbb{R}} \exp\left(\varkappa|u|^q\right)\chi(du) < \infty,$$

for some $q > 2$ and $\varkappa > 0$. Then there exists a set $\mathcal{C} \subset \Gamma(X)$ such that:
(i) for all $\gamma \in \mathcal{C}$, we have $\mathcal{G}_{\alpha,q}(\mathbb{R}^\gamma, \phi, \chi) \neq \emptyset$ and any $\eta \in \mathcal{G}_t(\mathbb{R}^\gamma, \phi, \chi)$ satisfies the following estimate:

$$\int_{\mathbb{R}^\gamma} \exp\left(\vartheta \sum_{x\in\gamma} |\sigma_x|^p e^{-\alpha|x|}\right) \eta(d\sigma) < \infty, \tag{12}$$

for any $\alpha > 0$;
(ii) for any measure $\mu \in \mathcal{P}(\Gamma(X))$ with bounded correlation functions we have $\mu(\mathcal{C}) = 1$.

Remark 16. The existence of Gibbs measures $\mu \in \mathcal{G}_{\alpha,q}(R^\gamma, \phi, \chi)$ follows from the relative compactness of the family $\Pi = \{\Pi_\Lambda(\cdot|\bar\sigma) : \Lambda \subset X, \text{ compact}\}$ in the topology of local set convergence, for a certain class of boundary conditions $\bar\sigma \in R^\gamma$. It can be shown by standard arguments that the corresponding accumulation points obey the Dobrushin-Lanford-Ruelle equation (9) and estimate (12). A typical choice of $\bar\sigma$ is constant boundary condition $\bar\sigma_x = s \in R$ for all $x \in \gamma$.

Remark 17. The framework of the paper [3] is more general: the spin space is assumed to be a general Euclidean space, and the existence result is proved for a general pair interaction potential $W_{xy}(u, v)$ of polynomial growth in u, v and finite range in x, y. Moreover, the existence of measurable selections $\gamma \mapsto \eta_\gamma \in \mathcal{G}(\mathbb{R}^\gamma, \phi, \chi)$ is shown.

Definition 18. For $a > 0$, by $\eta^{\pm a} \in G(R^\gamma, \phi, \chi)$ we denote the accumulation point of the family Π with constant boundary condition $\pm a$ respectively.

Now we turn to the single-spin measure χ and give two (quite different by their nature) examples.

Example 19. Set

$$\chi = \chi^{\text{Ising}} := \frac{1}{2}\left(\delta_{-1} + \delta_{+1}\right),$$

where δ_s is the Dirac measure centered at $s \in \mathbb{R}$. This example corresponds to an *Ising magnet and is very well studied*. We denote by $\mathcal{G}^{\text{Ising}}(\mathbb{R}^\gamma, \phi)$ the set of all the corresponding Gibbs measures.

Example 20. An example of a natural single-spin measure χ with non-compact support is given by the formula

$$\chi(du) = \exp\left(-V(u)\right) du,$$

where $V : \mathbb{R} \to \mathbb{R}$ is a measurable symmetric function such that: (a) the set $\{u \in \mathbb{R} : V(u) < +\infty\}$ is of positive Lebesgue measure; (b) $V(u)$ increases at infinity faster than $|u|^q$. This includes the case where V is a polynomial.

3.2. **The Phase Transition.** By a phase transition in our model we mean that the set of Gibbs measures $\mathcal{G}_{\alpha,q}(\mathbb{R}^\gamma, \phi, \chi)$ contains at least two (and hence infinitely many) elements for μ-a.a. $\gamma \in \Gamma(X)$.

In the case where μ is the Poisson measure, cf. Example 2.4.1, a sufficient condition of occurrence of phase transition has been derived in [4]. This result is based on (i) the relationship between the occurrence of Bernoulli bond percolation on a graph and the existence of multiple Gibbs states in the corresponding Ising model on that graph, see [10], on the one hand, and (ii) the Wells inequality that allows to estimate first moments of a Gibbs measure with general interaction by first moments of its Ising counterpart, see [20] and [4], on the other hand. We follow this scheme and prove the occurrence of a phase transitions γ governed by a general l_R-dense measure μ.

Theorem 21. *Let the measure χ be as in Theorem 15 and such that $\chi(\{0\}) < \chi(\mathbb{R})$ and $\phi(x) > \beta_* > 0$ for all $x \in [0, R]$. Assume that the underlying point process μ is l_R-dense. Then there exists $z_c > 0$ (which may depend on β_*) such that for any $z > z_c$ and μ-a.a. γ the set $\mathcal{G}_{\alpha,q}(\mathbb{R}^\gamma, \phi, \chi)$ contains at least two elements.*

Proof. Observe that it is sufficient to prove the existence $z_c > 0$ and $a > 0$ such that for any $z > z_c$ the estimate

$$\int_{\mathbb{R}^\gamma} \sigma_o \eta^{+a}(d\sigma) > 0 \tag{13}$$

holds for some $o \in \gamma$ (for μ-a.a. $\gamma \in \Gamma$). Indeed, in this case, by the symmetry of χ and of the interaction in (11), we have

$$\int_{\mathbb{R}^\gamma} \sigma_o \eta^{-a}(d\sigma) = -\int_{\mathbb{R}^\gamma} \sigma_o \eta^{+a}(d\sigma) < 0,$$

so that $\eta^{+a} \neq \eta^{-a}$.

In order to prove bound (13), let us first consider the Ising model associated with the single-spin measure $\chi = \chi^{\text{Ising}}$, cf. Example 19.

We denote by ν_ϕ^{+1} the element of $\mathcal{G}^{\text{Ising}}(\mathbb{R}^\gamma, \phi)$ that corresponds to the boundary condition $+1$, cf. Remark 18.

First assume that the function ϕ is constant, $\phi(x) = \beta > 0,\ x \in \mathbb{R}_+$. It is known (see [10, Theorem 2.1]) that the following statements are equivalent (for an arbitrary graph, and thus in particular for $(\gamma)_R$):

- the graph $(\gamma)_R$ admits bond percolation with critical probability $q_c < 1$;
- for any $\beta > \beta_c := -\frac{1}{2}\log(1-q_c)$ there exists a vertex $o \in \gamma$ such that the estimate

$$\int \sigma_o \nu_\beta^{+1}(d\sigma) > 0 \tag{14}$$

  holds.

Let us now set $q_c = 1 - e^{-2\beta_*}$ so that $\beta_c = \beta_*$. Theorem 2 implies that there exist $z_c > 0$ such that for any $z > z_c$ a typical geometric graph $(\gamma)_R$ admits bond percolation with critical probability q_c, which in turn implies that (14) holds for μ-a.a. γ.

In the case of a non-constant pair potential ϕ satisfying the inequality $\phi \geq \beta_*$, an application of the GKS inequality shows that

$$\int \sigma_o \nu_\phi^{+1}(d\sigma) > \int \sigma_o \nu_{\beta_*}^{+1}(d\sigma), \tag{15}$$

so that the corresponding inequality holds for ν_ϕ^{+1}.

Next, we will estimate the integral in (13) by the corresponding integral for the Ising model. Fix $a > 0$ such that

$$\chi([a\sqrt{2}, +\infty)) \geq \chi([0, a])$$

and consider Gibbs measures $\eta^{+a} \in \mathcal{G}(\mathbb{R}^\gamma, \phi, \chi)$ and $\nu_{a^2\phi}^{+1} \in \mathcal{G}^{\mathrm{Ising}}(\mathbb{R}^\gamma, a^2\phi)$. Then, for each $x \in \gamma$, we have the estimate

$$\int_{\mathbb{R}^\gamma} \sigma_x \eta^{+a}(d\sigma) \geq a \int_{\mathbb{R}^\gamma} \sigma_x \nu_{a^2\phi}^{+1}(d\sigma)$$

(the Wells inequality, see [20] and [4]). This inequality combined with (14) and (15) completes the proof of (13). □

Remark 22. This phase transition is achieved by choosing the intensity z of the underlying distribution of γ sufficiently large, for a fixed inverse temperature β_*, in contrast to [4], where z is fixed and β_* required to be large.

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