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CONSERVATIVE L-SYSTEMS AND THE LIVŠIC FUNCTION

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Dedicated to Yury Berezansky, a remarkable Mathematician and Human Being, on the occasion of his 90th birthday

Abstract. We study the connection between the classes of (i) Livšic functions $s(z)$, i.e., the characteristic functions of densely defined symmetric operators $\dot{A}$ with deficiency indices $(1,1)$; (ii) the characteristic functions $S(z)$ of a maximal dissipative extension T of $\dot{A}$, i.e., the Möbius transform of $s(z)$ determined by the von Neumann parameter κ of the extension relative to an appropriate basis in the deficiency subspaces; and (iii) the transfer functions $W_\Theta(z)$ of a conservative L-system Θ with the main operator T. It is shown that under a natural hypothesis the functions $S(z)$ and $W_\Theta(z)$ are reciprocal to each other. In particular, $W_\Theta(z)=\frac{1}{S(z)}=-\frac{1}{s(z)}$ whenever $\kappa=0$. It is established that the impedance function of a conservative L-system with the main operator T belongs to the Donoghue class if and only if the von Neumann parameter vanishes ($\kappa=0$). Moreover, we introduce the generalized Donoghue class and obtain the criteria for an impedance function to belong to this class. We also obtain the representation of a function from this class via the Weyl-Titchmarsh function. All results are illustrated by a number of examples.

1. Introduction

Suppose that T is a densely defined closed operator in a Hilbert space $\mathcal{H}$ such that its resolvent set $\rho(T)$ is not empty and assume, in addition, that $\operatorname{Dom}(T)\cap\operatorname{Dom}(T^*)$ is dense. We also suppose that the restriction $\dot{A}=T|_{\operatorname{Dom}(T)\cap\operatorname{Dom}(T^*)}$ is a closed symmetric operator with finite equal deficiency indices and that $\mathcal{H}_+\subset\mathcal{H}\subset\mathcal{H}_-$ is the rigged Hilbert space associated with $\dot{A}$ (see Appendix A for a detailed discussion of a concept of rigged Hilbert spaces).

One of the main objectives of the current paper is the study of the *L-system*

$$\Theta=\begin{pmatrix} \mathbb{A} & K & J\\ \mathcal{H}_+\subset\mathcal{H}\subset\mathcal{H}_- & & E\end{pmatrix}, \tag{1}$$

where the *state-space operator* $\mathbb{A}$ is a bounded linear operator from $\mathcal{H}_+$ into $\mathcal{H}_-$ such that $\dot{A}\subset T\subset\mathbb{A}$, $\dot{A}\subset T^*\subset\mathbb{A}^*$, E is a finite-dimensional Hilbert space, K is a bounded linear operator from the space E into $\mathcal{H}_-$, and $J=J^*=J^{-1}$ is a self-adjoint isometry on E such that the imaginary part of $\mathbb{A}$ has a representation $\operatorname{Im}\mathbb{A}=KJK^*$. Due to the facts that $\mathcal{H}_\pm$ is dual to $\mathcal{H}_\mp$ and that $\mathbb{A}^*$ is a bounded linear operator from $\mathcal{H}_+$ into $\mathcal{H}_-$, $\operatorname{Im}\mathbb{A}=(\mathbb{A}-\mathbb{A}^*)/2i$ is a well defined bounded operator from $\mathcal{H}_+$ into $\mathcal{H}_-$. Note that the main operator T associated with the system Θ is uniquely determined by the state-space operator $\mathbb{A}$ as its restriction on the domain $\operatorname{Dom}(T)=\{f\in\mathcal{H}_+\mid \mathbb{A}f\in\mathcal{H}\}$.

Recall that the operator-valued function given by

$$W_\Theta(z)=I-2iK^*(\mathbb{A}-zI)^{-1}KJ,\quad z\in\rho(T),$$

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is called the *transfer function* of the L-system Θ and

$$V_\Theta(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I] = K^*(\mathrm{Re}\,\mathbb{A} - zI)^{-1}K, \quad z \in \rho(T) \cap \mathbb{C}_\pm,$$

is called the *impedance function* of Θ.

We remark that under the hypothesis $\mathrm{Im}\,\mathbb{A} = KJK^*$, the linear sets $\mathrm{Ran}(\mathbb{A} - zI)$ and $\mathrm{Ran}(\mathrm{Re}\,\mathbb{A} - zI)$ contain $\mathrm{Ran}(K)$ for $z \in \rho(T)$ and $z \in \rho(T) \cap \mathbb{C}_\pm$, respectively, and therefore, both the transfer and impedance functions are well defined (see Section 2 for more details).

Note that if $\varphi_+ = W_\Theta(z)\varphi_-$, where $\varphi_\pm \in E$, with φ_- the input and φ_+ the output, then L-system (1) can be associated with the system of equations

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases}. \tag{2}$$

(To recover $W_\Theta(z)\varphi_-$ from (2) for a given φ_-, one needs to find x and then determine φ_+.)

We remark that the concept of L-systems (1)–(2) generalizes the one of the Livšic systems in the case of a bounded main operator. It is also worth mentioning that those systems are conservative in the sense that a certain metric conservation law holds (for more details see [3, Preface]). An overview of the history of the subject and a detailed description of the L-systems can be found in [3].

Another important object of interest in this context is the *Livšic function*. Recall that in [15] M. Livšic introduced a fundamental concept of a characteristic function of a densely defined symmetric operator $\dot{A}$ with deficiency indices $(1, 1)$ as well as of its maximal non-self-adjoint extension T. Introducing an auxiliary self-adjoint (reference) extension A of $\dot{A}$, in [18] two of the authors (K.A.M. and E.T.) suggested to define a characteristic function of a symmetric operator as well of its dissipative extension as the one associated with the pairs $(\dot{A}, A)$ and (T, A), rather than with the single operators $\dot{A}$ and T, respectively. Following [18] and [19] we call the characteristic function associated with the pair $(\dot{A}, A)$ the *Livšic function*. For a detailed treatment of the aforementioned concepts of the Livšic and the characteristic functions we refer to [18] (see also [2], [10], [14], [21], [23]).

The main goal of the present paper is the following.

First, we establish a connection between the classes of: (i) the Livšic functions $s(z)$, the characteristic functions of a densely defined symmetric operators $\dot{A}$ with deficiency indices $(1, 1)$; (ii) the characteristic functions $S(z)$ of a maximal dissipative extension T of $\dot{A}$, the Möbius transform of $s(z)$ determined by the von Neumann parameter κ; and (iii) the transfer functions $W_\Theta(z)$ of an L-system Θ with the main operator T. It is shown (see Theorem 7) that under some natural assumptions the functions $S(z)$ and $W_\Theta(z)$ are reciprocal to each other. In particular, when $\kappa = 0$, we have $W_\Theta(z) = \frac{1}{S(z)} = -\frac{1}{s(z)}$.

Second, in Theorem 11, we show that the impedance function of a conservative L-system with the main operator T coincides with a function from the Donoghue class $\mathfrak{M}$ if and only if the von Neumann parameter vanishes that is $\kappa = 0$. For $0 \le \kappa < 1$ we introduce the generalized Donoghue class $\mathfrak{M}_\kappa$ and establish a criterion (see Theorem 12) for an impedance function to belong to $\mathfrak{M}_\kappa$. In particular, when $\kappa = 0$ the class $\mathfrak{M}_\kappa$ coincides with the Donoghue class $\mathfrak{M} = \mathfrak{M}_0$. Also, in Theorem 14, we obtain the representation of a function from the class $\mathfrak{M}_\kappa$ via the Weyl-Titchmarsh function.

We conclude our paper by providing several examples that illustrate the main results and concepts.

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\dot{A}$ be a closed, densely defined, symmetric operator with finite equal deficiency indices acting on a Hilbert space $\mathcal{H}$ with inner product $(f, g), f, g \in \mathcal{H}$. Any operator T in $\mathcal{H}$ such that

$$\dot{A} \subset T \subset \dot{A}^*$$

is called a *quasi-self-adjoint extension* of $\dot{A}$.

Consider the rigged Hilbert space (see [6], [7], [5]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$ and

$$(f, g)_+ = (f, g) + (\dot{A}^* f, \dot{A}^* g), \quad f, g \in \mathrm{Dom}(\dot{A}^*). \tag{3}$$

Let $\mathcal{R}$ be the *Riesz-Berezansky operator* $\mathcal{R}$ (see [6], [7], [5]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ so that $(f, g) = (f, \mathcal{R}g)_+$ ($\forall f \in \mathcal{H}_+$, $\forall g \in \mathcal{H}_-$) and $\|\mathcal{R}g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$, then $\mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$.

Next we proceed with several definitions.

An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a *self-adjoint bi-extension* of a symmetric operator $\dot{A}$ if $\mathbb{A} = \mathbb{A}^*$ and $\mathbb{A} \supset \dot{A}$.

Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot{A}$ and let the operator $\hat{A}$ in $\mathcal{H}$ be defined as follows:

$$\mathrm{Dom}(\hat{A}) = \{f \in \mathcal{H}_+ : \mathbb{A} f \in \mathcal{H}\}, \quad \hat{A} = \mathbb{A}\upharpoonright \mathrm{Dom}(\hat{A}).$$

The operator $\hat{A}$ is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [23], [3, Section 2.1]).

A self-adjoint bi-extension $\mathbb{A}$ of a symmetric operator $\dot{A}$ is called twice-self-adjoint or *t-self-adjoint* (see [3, Definition 3.3.5]) if its quasi-kernel $\hat{A}$ is a self-adjoint operator in $\mathcal{H}$. In this case, according to the von Neumann Theorem (see [3, Theorem 1.3.1]) the domain of $\hat{A}$, which is a self-adjoint extension of $\dot{A}$, can be represented as

$$\mathrm{Dom}(\hat{A}) = \mathrm{Dom}(\dot{A}) \oplus (I + U)\mathfrak{N}_i, \tag{4}$$

where U is both a $(\cdot)$-isometric as well as $(+)$)-isometric operator from $\mathfrak{N}_i$ into $\mathfrak{N}_{-i}$. Here

$$\mathfrak{N}_{\pm i} = \mathrm{Ker}\,(\dot{A}^* \mp iI)$$

are the deficiency subspaces of $\dot{A}$.

An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a *quasi-self-adjoint bi-extension* of an operator T if $\dot{A} \subset T \subset \mathbb{A}$ and $\dot{A} \subset T^* \subset \mathbb{A}^*$.

In what follows we will be mostly interested in the following type of quasi-self-adjoint bi-extensions.

Definition 1. ([3]). *Let T be a quasi-self-adjoint extension of $\dot{A}$ with nonempty resolvent set $\rho(T)$. A quasi-self-adjoint bi-extension $\mathbb{A}$ of an operator T is called a $(*)$-extension of T if* $\mathrm{Re}\,\mathbb{A}$ *is a t-self-adjoint bi-extension of $\dot{A}$.*

We assume that $\dot{A}$ has equal finite deficiency indices and will say that a quasi-self-adjoint extension T of $\dot{A}$ belongs to the *class* $\Lambda(\dot{A})$ if $\rho(T) \neq \emptyset$, $\mathrm{Dom}(\dot{A}) = \mathrm{Dom}(T) \cap \mathrm{Dom}(T^*)$, and hence T admits $(*)$-extensions. The description of all $(*)$-extensions via Riesz-Berezansky operator $\mathcal{R}$ can be found in [3, Section 4.3].

Definition 2. *A system of equations*

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^* x \end{cases},$$

or an array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix} \tag{5}$$

is called an ***L-system*** *if:*

(1) $\mathbb{A}$ *is a* $(*)$*-extension of an operator* T *of the class* $\Lambda(\dot{A})$*;*
(2) $J = J^* = J^{-1} \in [E, E]$, $\dim E < \infty$*;*
(3) $\operatorname{Im} \mathbb{A} = KJK^*$, *where* $K \in [E, \mathcal{H}_-]$, $K^* \in [\mathcal{H}_+, E]$, *and* $\operatorname{Ran}(K) = \operatorname{Ran}(\operatorname{Im} \mathbb{A})$.

In what follows we assume the following terminology. In the definition above $\varphi_- \in E$ stands for an input vector, $\varphi_+ \in E$ is an output vector, and x is a state space vector in $\mathcal{H}$. The operator $\mathbb{A}$ is called the *state-space operator* of the system Θ, T is the *main operator*, J is the *direction operator*, and K is the *channel operator*. A system Θ (5) is called *minimal* if the operator $\dot{A}$ is a prime operator in $\mathcal{H}$, i.e., there exists no non-trivial subspace invariant for $\dot{A}$ such that the restriction of $\dot{A}$ to this subspace is self-adjoint.

We associate with an L-system Θ the operator-valued function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ, \quad z \in \rho(T), \tag{6}$$

which is called the **transfer function** of the L-system Θ. We also consider the operator-valued function

$$V_\Theta(z) = K^*(\operatorname{Re} \mathbb{A} - zI)^{-1}K, \quad z \in \rho(\hat{A}). \tag{7}$$

It was shown in [5], [3, Section 6.3] that both (6) and (7) are well defined. In particular, $\operatorname{Ran}(\mathbb{A} - zI)$ does not depend on $z \in \rho(T)$ while $\operatorname{Ran}(\operatorname{Re} \mathbb{A} - zI)$ does not depend on $z \in \rho(\hat{A})$. Also, $\operatorname{Ran}(\mathbb{A} - zI) \supset \operatorname{Ran}(K)$ and $\operatorname{Ran}(\operatorname{Re} \mathbb{A} - zI) \supset \operatorname{Ran}(K)$ (see [3, Theorem 4.3.2]). The transfer operator-function $W_\Theta(z)$ of the system Θ and an operator-function $V_\Theta(z)$ of the form (7) are connected by the following relations valid for $\operatorname{Im} z \neq 0$, $z \in \rho(T)$,

$$\begin{aligned} V_\Theta(z) &= i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]J, \\ W_\Theta(z) &= (I + iV_\Theta(z)J)^{-1}(I - iV_\Theta(z)J). \end{aligned} \tag{8}$$

The function $V_\Theta(z)$ defined by (7) is called the **impedance function** of the L-system Θ. The class of all Herglotz-Nevanlinna functions in a finite-dimensional Hilbert space E, that can be realized as impedance functions of an L-system, was described in [5] (see also [3, Definition 6.4.1]).

Two minimal L-systems

$$\Theta_j = \begin{pmatrix} \mathbb{A}_j & K_j & J \\ \mathcal{H}_{+j} \subset \mathcal{H}_j \subset \mathcal{H}_{-j} & & E \end{pmatrix}, \quad j = 1, 2,$$

are called **bi-unitarily equivalent** [3, Section 6.6] if there exists a triplet of operators (U_+, U, U_-) that isometrically maps the triplet $\mathcal{H}_{+1} \subset \mathcal{H}_1 \subset \mathcal{H}_{-1}$ onto the triplet $\mathcal{H}_{+2} \subset \mathcal{H}_2 \subset \mathcal{H}_{-2}$ such that $U_+ = U|_{\mathcal{H}_{+1}}$ is an isometry from $\mathcal{H}_{+1}$ onto $\mathcal{H}_{+2}$, $U_- = (U_+^*)^{-1}$ is an isometry from $\mathcal{H}_{-1}$ onto $\mathcal{H}_{-2}$, and

$$UT_1 = T_2U, \quad U_-\mathbb{A}_1 = \mathbb{A}_2U_+, \quad U_-K_1 = K_2. \tag{9}$$

It is shown in [3, Theorem 6.6.10] that if the transfer functions $W_{\Theta_1}(z)$ and $W_{\Theta_2}(z)$ of the minimal systems Θ_1 and Θ_2 coincide for $z \in (\rho(T_1) \cap \rho(T_2)) \cap \mathbb{C}_\pm \neq \emptyset$, then Θ_1 and Θ_2 are bi-unitarily equivalent.

3. On $(*)$-extension parametrization

Let $\dot{A}$ be a densely defined, closed, symmetric operator with finite deficiency indices (n,n). The von Neumann formula (see also [3, Section 2.3]) yields

$$\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*) = \mathrm{Dom}(\dot{A}) \oplus \mathfrak{N}_i \oplus \mathfrak{N}_{-i},$$

where $\oplus$ stands for the $(+)$-orthogonal sum. Moreover, all operators T from the class $\Lambda(\dot{A})$ with $-i \in \rho(T)$ are of the form (see [3, Theorem 4.1.9], [23])

$$\begin{aligned} \mathrm{Dom}(T) &= \mathrm{Dom}(\dot{A}) \oplus (\mathcal{K}+I)\mathfrak{N}_i, \quad T = \dot{A}^* \upharpoonright \mathrm{Dom}(T), \\ \mathrm{Dom}(T^*) &= \mathrm{Dom}(\dot{A}) \oplus (\mathcal{K}^*+I)\mathfrak{N}_{-i}, \quad T^* = \dot{A}^* \upharpoonright \mathrm{Dom}(T^*), \end{aligned} \tag{10}$$

where $\mathcal{K} \in [\mathfrak{N}_i, \mathfrak{N}_{-i}]$.

Let $\mathcal{M} = \mathfrak{N}_i \oplus \mathfrak{N}_{-i}$ and $P^+_{\mathfrak{N}}$ be a $(+)$-orthogonal projection onto a subspace $\mathfrak{N}$. In this case (see [23]) all quasi-self-adjoint bi-extensions of $T \in \Lambda(\dot{A})$ can be parameterized via an operator $H \in [\mathfrak{N}_{-i}, \mathfrak{N}_i]$ as follows

$$\mathbb{A} = \dot{A}^* + \mathcal{R}^{-1}(S - \frac{i}{2}\mathfrak{J})P^+_{\mathcal{M}}, \quad \mathbb{A}^* = \dot{A}^* + \mathcal{R}^{-1}(S^* - \frac{i}{2}\mathfrak{J})P^+_{\mathcal{M}}, \tag{11}$$

where $\mathfrak{J} = P^+_{\mathfrak{N}_i} - P^+_{\mathfrak{N}_{-i}}$ and $S : \mathfrak{N}_i \oplus \mathfrak{N}_{-i} \to \mathfrak{N}_i \oplus \mathfrak{N}_{-i}$, satisfies the condition

$$S = \begin{pmatrix} \frac{i}{2}I - H\mathcal{K} & H \\ -(iI - \mathcal{K}H)\mathcal{K} & \frac{i}{2}I - \mathcal{K}H \end{pmatrix}. \tag{12}$$

Introduce $(2n \times 2n)$–block-operator matrices $S_{\mathbb{A}}$ and $S_{\mathbb{A}^*}$ by

$$\begin{aligned} S_{\mathbb{A}} &= S - \frac{i}{2}\mathfrak{J} = \begin{pmatrix} -H\mathcal{K} & H \\ \mathcal{K}(H\mathcal{K} - iI) & iI - \mathcal{K}H \end{pmatrix}, \\ S_{\mathbb{A}^*} &= S^* - \frac{i}{2}\mathfrak{J} = \begin{pmatrix} -\mathcal{K}^*H^* - iI & (\mathcal{K}^*H^* - iI)\mathcal{K}^* \\ H^* & -H^*\mathcal{K}^* \end{pmatrix}. \end{aligned} \tag{13}$$

By direct calculations one finds that

$$\frac{1}{2}(S_{\mathbb{A}} + S_{\mathbb{A}^*}) = \frac{1}{2}\begin{pmatrix} -H\mathcal{K} - \mathcal{K}^*H^* - iI & H + (\mathcal{K}^*H^* + iI)\mathcal{K}^* \\ \mathcal{K}(H\mathcal{K} - iI) + H^* & iI - \mathcal{K}H - H^*\mathcal{K}^* \end{pmatrix}, \tag{14}$$

and that

$$\frac{1}{2i}(S_{\mathbb{A}} - S_{\mathbb{A}^*}) = \frac{1}{2i}\begin{pmatrix} -H\mathcal{K} + \mathcal{K}^*H^* + iI & H - (\mathcal{K}^*H^* + iI)\mathcal{K}^* \\ \mathcal{K}(H\mathcal{K} - iI) - H^* & iI - \mathcal{K}H + H^*\mathcal{K}^* \end{pmatrix}. \tag{15}$$

In the case when the deficiency indices of $\dot{A}$ are $(1,1)$, the block-operator matrices $S_{\mathbb{A}}$ and $S_{\mathbb{A}^*}$ in (13) become $(2 \times 2)-$matrices with scalar entries. As it was announced in [22], (see also [3, Section 3.4] and [23]), in this case any quasi-self-adjoint bi-extension $\mathbb{A}$ of T is of the form

$$\mathbb{A} = \dot{A}^* + [p(\cdot,\varphi) + q(\cdot,\psi)]\,\varphi + [v(\cdot,\varphi) + w(\cdot,\psi)]\,\psi, \tag{16}$$

where $S_{\mathbb{A}} = \begin{pmatrix} p & q \\ v & w \end{pmatrix}$ is a (2×2)–matrix with scalar entries such that $p = -H\mathcal{K}$, $q = H$, $v = \mathcal{K}(H\mathcal{K} - i)$, and $w = i - \mathcal{K}H$. Also, φ and ψ are $(-)$-normalized elements in $\mathcal{R}^{-1}(\mathfrak{N}_i)$ and $\mathcal{R}^{-1}(\mathfrak{N}_{-i})$, respectively. Both the parameters H and $\mathcal{K}$ are complex numbers in this case and $|\mathcal{K}| < 1$. Similarly we write

$$\mathbb{A}^* = \dot{A}^* + \left[p^\times(\cdot,\varphi) + q^\times(\cdot,\psi)\right]\varphi + \left[v^\times(\cdot,\varphi) + w^\times(\cdot,\psi)\right]\psi, \tag{17}$$

where $S_{\mathbb{A}^*} = \begin{pmatrix} p^\times & q^\times \\ v^\times & w^\times \end{pmatrix}$ is such that $p^\times = -\bar{\mathcal{K}}\bar{H} - i$, $q^\times = (\bar{\mathcal{K}}\bar{H} - i)\bar{\mathcal{K}}$, $v^\times = \bar{H}$, and $w^\times = -\bar{H}\bar{\mathcal{K}}$. A direct check confirms that $\dot{A} \subset T \subset \mathbb{A}$ and we make the corresponding calculations below for the reader's convenience.

Indeed, recall that $\|\varphi\|_- = \|\psi\|_- = 1$. Using formulas (121) and (122) from Appendix A we get

$$1 = (\varphi, \varphi)_- = (\mathcal{R}\varphi, \mathcal{R}\varphi)_+ = \|\mathcal{R}\varphi\|_+^2 = 2\|\mathcal{R}\varphi\|^2 = (\sqrt{2}\mathcal{R}\varphi, \sqrt{2}\mathcal{R}\varphi).$$

Set $g_+ = \sqrt{2}\mathcal{R}\varphi \in \mathfrak{N}_i$ and $g_- = \sqrt{2}\mathcal{R}\psi \in \mathfrak{N}_{-i}$ and note that g_+ and g_- form normalized vectors in $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$, respectively. Now let $f \in \operatorname{Dom}(T)$, where $\operatorname{Dom}(T)$ is defined in (10). Then,

$$f = f_0 + (\mathcal{K}+1)f_1 = f_0 + Cg_+ + \mathcal{K}Cg_-, \quad f_0 \in \operatorname{Dom}(\dot{A}), \quad f_1 \in \mathfrak{N}_i, \tag{18}$$

for some choice of the constant C that is specific to $f \in \operatorname{Dom}(T)$. Moreover,

$$\mathbb{A}f = Tf + [p(f,\varphi) + q(f,\psi)]\,\varphi + [v(f,\varphi) + w(f,\psi)]\,\psi, \quad f \in \operatorname{Dom}(T).$$

Let us show that the last two terms in the sum above vanish. Consider (f,φ) where f is decomposed into the (+)-orthogonal sum (18). Using (+)-orthogonality of $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$ we have

$$\begin{aligned}(f,\varphi) &= (f_0 + Cg_+ + \mathcal{K}Cg_-, \varphi) = (f_0,\varphi) + (Cg_+,\varphi) + (\mathcal{K}Cg_-,\varphi)\\ &= 0 + (Cg_+, \mathcal{R}\varphi)_+ + (\mathcal{K}Cg_-, \mathcal{R}\varphi)_+\\ &= (Cg_+, (1/\sqrt{2})g_+)_+ + (\mathcal{K}Cg_-, (1/\sqrt{2})g_+)_+\\ &= \frac{C}{\sqrt{2}}(g_+,g_+)_+ = \frac{C}{\sqrt{2}}\|g_+\|_+^2 = \sqrt{2}C\|g_+\|^2 = \sqrt{2}C.\end{aligned}$$

Similarly,

$$\begin{aligned}(f,\psi) &= (f_0 + Cg_+ + \mathcal{K}Cg_-, \psi) = (f_0,\psi) + (Cg_+,\psi) + (\mathcal{K}Cg_-,\psi)\\ &= 0 + (Cg_+, \mathcal{R}\psi)_+ + (\mathcal{K}Cg_-, \mathcal{R}\psi)_+\\ &= (Cg_+, (1/\sqrt{2})g_-)_+ + (\mathcal{K}Cg_-, (1/\sqrt{2})g_-)_+\\ &= \frac{\mathcal{K}C}{\sqrt{2}}(g_-,g_-)_+ = \frac{\mathcal{K}C}{\sqrt{2}}\|g_-\|_+^2 = \sqrt{2}\mathcal{K}C\|g_-\|^2 = \sqrt{2}\mathcal{K}C.\end{aligned}$$

Consequently,

$$p(f,\varphi) + q(f,\psi) = -H\mathcal{K}(f,\varphi) + H(f,\psi) = H[-\mathcal{K}\sqrt{2}C + \sqrt{2}\mathcal{K}C] = 0.$$

Applying similar argument for the last bracketed term in (16) we show that

$$v(f,\varphi) + w(f,\psi) = 0$$

as well. Thus, $\dot{A} \subset T \subset \mathbb{A}$. Likewise, using (17) one shows that $\dot{A} \subset T^* \subset \mathbb{A}^*$.

The following proposition was announced by one of the authors (E.T.) in [23] and we present its proof below for convenience of the reader.

Proposition 3. *Let $T \in \Lambda(\dot{A})$ and A be a self-adjoint extension of $\dot{A}$ such that U defines* $\operatorname{Dom}(A)$ *via* (4) *and $\mathcal{K}$ defines T via* (10). *Then $\mathbb{A}$ is a $(*)$-extension of T whose real part* $\operatorname{Re}\mathbb{A}$ *has the quasi-kernel A if and only if $U\mathcal{K}^* - I$ is a homeomorphism and the operator parameter H in* (12)-(13) *takes the form*

$$H = i(I - \mathcal{K}^*\mathcal{K})^{-1}[(I - \mathcal{K}^*U)(I - U^*\mathcal{K})^{-1} - \mathcal{K}^*U]U^*. \tag{19}$$

Proof. First, we are going to show that $\operatorname{Re}\mathbb{A}$ has the quasi-kernel A if and only if the system of operator equations

$$\begin{aligned} X^*(I - \tilde{\mathcal{K}}^*) + \tilde{\mathcal{K}}X(\tilde{\mathcal{K}} - I) &= i(\tilde{\mathcal{K}} - I),\\ \tilde{\mathcal{K}}^*X^*(\tilde{\mathcal{K}}^* - I) + X(I - \tilde{\mathcal{K}}) &= i(I - \tilde{\mathcal{K}}^*)\end{aligned} \tag{20}$$

has a solution. Here $\tilde{\mathcal{K}} = U^*\mathcal{K}$. Suppose $\operatorname{Re}\mathbb{A}$ has the quasi-kernel A and U defines $\operatorname{Dom}(A)$ via (4). Then there exists a self-adjoint operator $H \in [\mathfrak{N}_{-i}, \mathfrak{N}_i]$ such that $\mathbb{A}$ and $\mathbb{A}^*$ are defined via (11) where $S_{\mathbb{A}}$ and $S_{\mathbb{A}^*}$ are of the form (13). Then $\frac{1}{2}(S_{\mathbb{A}} + S_{\mathbb{A}^*})$ is given by (14). According to [3, Theorem 3.4.10] the entries of the operator matrix (14) are related by the following

$$-H\mathcal{K} - \mathcal{K}^*H^* - iI = -(H + (\mathcal{K}^*H^* + iI)\mathcal{K}^*)U,$$
$$\mathcal{K}(H\mathcal{K} - iI) + H^* = -(iI - \mathcal{K}H - H^*\mathcal{K}^*)U.$$

Denoting $\tilde{\mathcal{K}} = U^*\mathcal{K}$ and $\tilde{H} = HU$, we obtain

$$\tilde{H}^*(I - \tilde{\mathcal{K}}^*) + \tilde{\mathcal{K}}\tilde{H}(\tilde{\mathcal{K}} - I) = i(\tilde{\mathcal{K}} - I),$$
$$\tilde{\mathcal{K}}^*\tilde{H}^*(\tilde{\mathcal{K}}^* - I) + \tilde{H}(I - \tilde{\mathcal{K}}) = i(I - \tilde{\mathcal{K}}^*),$$

and hence $\tilde{H}$ is the solution to the system (20). To show the converse we simply reverse the argument.

Now assume that $U\mathcal{K}^* - I$ is a homeomorphism. We are going to prove that the operator T from the statement of the theorem has a unique $(*)$-extension $\mathbb{A}$ whose real part $\operatorname{Re}\mathbb{A}$ has the quasi-kernel A that is a self-adjoint extension of $\dot{A}$ parameterized via U. Consider the system (20). If we multiply the first equation of (20) by $\tilde{\mathcal{K}}^*$ and add it to the second, we obtain

$$(I - \tilde{\mathcal{K}}^*\tilde{\mathcal{K}})X(I - \tilde{\mathcal{K}}) = i(\tilde{\mathcal{K}}^*(\tilde{\mathcal{K}} - I) + (I - \tilde{\mathcal{K}}^*)).$$

Since $I - \tilde{\mathcal{K}}^*\tilde{\mathcal{K}} = I - \mathcal{K}^*\mathcal{K}$, $I - \tilde{\mathcal{K}}^* = I - U^*\mathcal{K}$, and $T \in \Lambda(\dot{A})$, then the operators $I - \tilde{\mathcal{K}}^*\tilde{\mathcal{K}}$ and $I - \tilde{\mathcal{K}}$ are boundedly invertible. Therefore,

$$X = i(I - \tilde{\mathcal{K}}^*\tilde{\mathcal{K}})^{-1}[(I - \tilde{\mathcal{K}}^*)(I - \tilde{\mathcal{K}})^{-1} - \tilde{\mathcal{K}}^*)]. \tag{21}$$

By the direct substitution one confirms that the operator X in (21) is a solution to the system (20). Applying the uniqueness result [3, Theorem 4.4.6] and the above reasoning we conclude that our operator T has a unique $(*)$-extension $\mathbb{A}$ whose real part $\operatorname{Re}\mathbb{A}$ has the quasi-kernel A. If, on the other hand, $\mathbb{A}$ is a $(*)$-extension whose real part $\operatorname{Re}\mathbb{A}$ has the quasi-kernel A that is a self-adjoint extension of $\dot{A}$ parameterized via U, then $U\mathcal{K}^* - I$ is a homeomorphism (see [3, Remark 4.3.4]).

Combining the two parts of the proof, replacing $\tilde{\mathcal{K}}$ with $U^*\mathcal{K}$, and X with $\tilde{H} = HU$ in (21) we complete the proof of the theorem. ☐

Suppose that for the case of deficiency indices $(1, 1)$ we have $\mathcal{K} = \mathcal{K}^* = \bar{\mathcal{K}} = \kappa$[1] and $U = 1$. Then formula (19) becomes

$$H = \frac{i}{1 - \kappa^2}[(1 - \kappa)(1 - \kappa)^{-1} - \kappa] = \frac{i}{1 + \kappa}.$$

Consequently, applying this value of H to (13) yields

$$S_{\mathbb{A}} = \begin{pmatrix} -\frac{i\kappa}{1+\kappa} & \frac{i}{1+\kappa} \\ \frac{i\kappa^2}{1+\kappa} - i\kappa & i - \frac{i\kappa}{1+\kappa} \end{pmatrix}, \quad S_{\mathbb{A}^*} = \begin{pmatrix} \frac{i\kappa}{1+\kappa} - i & -\frac{i\kappa^2}{1+\kappa} + i\kappa \\ -\frac{i}{1+\kappa} & \frac{i\kappa}{1+\kappa} \end{pmatrix}. \tag{22}$$

Performing direct calculations gives

$$\frac{1}{2i}(S_{\mathbb{A}} - S_{\mathbb{A}^*}) = \frac{1 - \kappa}{2 + 2\kappa}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{23}$$

Using (23) with (16) one obtains

$$\begin{aligned} \operatorname{Im}\mathbb{A} &= \frac{1 - \kappa}{2 + 2\kappa}\Big([(\cdot, \varphi) + (\cdot, \psi)]\varphi + [(\cdot, \varphi) + (\cdot, \psi)]\psi\Big) \\ &= \frac{1 - \kappa}{2 + 2\kappa}(\cdot, \varphi + \psi)(\varphi + \psi) \\ &= (\cdot, \chi)\chi, \end{aligned} \tag{24}$$

[1] Throughout this paper κ will be called the von Neumann parameter.

where

$$\chi = \sqrt{\frac{1-\kappa}{2+2\kappa}}\,(\varphi+\psi) = \sqrt{\frac{1-\kappa}{1+\kappa}}\left(\frac{1}{\sqrt{2}}\,\varphi + \frac{1}{\sqrt{2}}\,\psi\right). \tag{25}$$

Consider a special case when $\kappa = 0$. Then the corresponding $(*)$-extension $\mathbb{A}_0$ is such that

$$\operatorname{Im}\mathbb{A}_0 = \frac{1}{2}(\cdot,\varphi+\psi)(\varphi+\psi) = (\cdot,\chi_0)\chi_0, \tag{26}$$

where

$$\chi_0 = \frac{1}{\sqrt{2}}\,(\varphi+\psi)\,. \tag{27}$$

4. The Livšic function

Suppose that $\dot{A}$ is closed, prime, densely defined symmetric operator with deficiency indices $(1,1)$. In [15, a part of Theorem 13] (for a textbook exposition see [1]) M. Livšic suggested to call the function

$$s(z) = \frac{z-i}{z+i}\cdot\frac{(g_z,g_-)}{(g_z,g_+)},\quad z\in\mathbb{C}_+, \tag{28}$$

the characteristic function of the symmetric operator $\dot{A}$. Here $g_\pm \in \operatorname{Ker}(\dot{A}^* \mp iI)$ are normalized appropriately chosen deficiency elements and $g_z \neq 0$ are arbitrary deficiency elements of the symmetric operators $\dot{A}$. The Livšic result identifies the function $s(z)$ (modulo z-independent unimodular factor) with a complete unitary invariant of a prime symmetric operator with deficiency indices $(1,1)$ that determines the operator uniquely up to unitary equivalence. He also gave the following criterion [15, Theorem 15] (also see [1]) for a contractive analytic mapping from the upper half-plane $\mathbb{C}_+$ to the unit disk $\mathbb{D}$ to be the characteristic function of a densely defined symmetric operator with deficiency indices $(1,1)$.

Theorem 4. ([15]). *For an analytic mapping s from the upper half-plane to the unit disk to be the characteristic function of a densely defined symmetric operator with deficiency indices $(1,1)$ it is necessary and sufficient that*

$$s(i)=0 \quad \textit{and} \quad \lim_{z\to\infty} z(s(z)-e^{2i\alpha}) = \infty \quad \textit{for all} \quad \alpha\in[0,\pi), \tag{29}$$
$$0<\varepsilon\le \mathit{arg}(z) \le \pi-\varepsilon.$$

The **Livšic class** of functions described by Theorem 4 will be denoted by $\mathfrak{L}$.

In the same article, Livšic put forward a concept of a characteristic function of a quasi-self-adjoint dissipative extension of a symmetric operator with deficiency indices $(1,1)$.

Let us recall Livšic's construction. Suppose that $\dot{A}$ is a symmetric operator with deficiency indices $(1,1)$ and that $g_\pm$ are its normalized deficiency elements,

$$g_\pm \in \operatorname{Ker}(\dot{A}^* \mp iI),\quad \|g_\pm\| = 1.$$

Suppose that $T \neq (T)^*$ is a maximal dissipative extension of $\dot{A}$,

$$\operatorname{Im}(Tf,f)\ge 0,\quad f\in\operatorname{Dom}(T).$$

Since $\dot{A}$ is symmetric, its dissipative extension T is automatically quasi-self-adjoint [3], [21], that is,

$$\dot{A}\subset T\subset \dot{A}^*,$$

and hence, according to (10) with $\mathcal{K}=\kappa$,

$$g_+ - \kappa g_- \in \operatorname{Dom}(T) \quad \text{for some} \quad |\kappa|<1. \tag{30}$$

Based on the parametrization (30) of the domain of the extension T, Livšic suggested to call the Möbius transformation

$$S(z) = \frac{s(z) - \kappa}{\overline{\kappa}\, s(z) - 1}, \quad z \in \mathbb{C}_+, \tag{31}$$

where s is given by (28), the **characteristic function** of the dissipative extension T [15]. All functions that satisfy (31) for some function $s(z) \in \mathfrak{L}$ will form the **Livšic class** $\mathfrak{L}_\kappa$. Clearly, $\mathfrak{L}_0 = \mathfrak{L}$.

A culminating point of Livšic's considerations was the discovery that the characteristic function $S(z)$ (up to a unimodular factor) of a dissipative (maximal) extension T of a densely defined prime symmetric operator $\dot{A}$ is a complete unitary invariant of T (see [15, the remaining part of Theorem 13]).

In 1965 Donoghue [11] introduced a concept of the Weyl-Titchmarsh function $M(\dot{A}, A)$ associated with a pair $(\dot{A}, A)$ by

$$M(\dot{A}, A)(z) = ((Az + I)(A - zI)^{-1} g_+, g_+), \quad z \in \mathbb{C}_+,$$
$$g_+ \in \operatorname{Ker}(\dot{A}^* - iI), \quad \|g_+\| = 1,$$

where $\dot{A}$ is a symmetric operator with deficiency indices $(1,1)$, $\operatorname{def}(\dot{A}) = (1,1)$, and A is its self-adjoint extension.

Denote by $\mathfrak{M}$ the **Donoghue class** of all analytic mappings M from $\mathbb{C}_+$ into itself that admits the representation

$$M(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu, \tag{32}$$

where μ is an infinite Borel measure and

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1\,, \quad \text{equivalently,} \quad M(i) = i. \tag{33}$$

It is known [11], [12], [13], [18] that $M \in \mathfrak{M}$ if and only if M can be realized as the Weyl-Titchmarsh function $M(\dot{A}, A)$ associated with a pair $(\dot{A}, A)$. The Weyl-Titchmarsh function M is a (complete) unitary invariant of the pair of a symmetric operator with deficiency indices $(1,1)$ and its self-adjoint extension and determines the pair of operators uniquely up to unitary equivalence.

Livšic's definition of a characteristic function of a symmetric operator (see eq. (28)) has some ambiguity related to the choice of the deficiency elements $g_\pm$. To avoid this ambiguity we proceed as follows. Suppose that A is a self-adjoint extension of a symmetric operator $\dot{A}$ with deficiency indices $(1,1)$. Let $g_\pm$ be deficiency elements $g_\pm \in \operatorname{Ker}((\dot{A})^* \mp iI)$, $\|g_+\| = 1$. Assume, in addition, that

$$g_+ - g_- \in \operatorname{Dom}(A). \tag{34}$$

Following [18] we introduce the *Livšic function $s(\dot{A}, A)$ associated with the pair $(\dot{A}, A)$* by

$$s(z) = \frac{z - i}{z + i} \cdot \frac{(g_z, g_-)}{(g_z, g_+)}, \quad z \in \mathbb{C}_+, \tag{35}$$

where $0 \neq g_z \in \operatorname{Ker}((\dot{A})^* - zI)$ is an arbitrary (deficiency) element.

A standard relationship between the Weyl-Titchmarsh and the Livšic functions associated with the pair $(\dot{A}, A)$ was described in [18]. In particular, if we denote by $M = M(\dot{A}, A)$ and by $s = s(\dot{A}, A)$ the Weyl-Titchmarsh function and the Livšic function associated with the pair $(\dot{A}, A)$, respectively, then

$$s(z) = \frac{M(z) - i}{M(z) + i}, \quad z \in \mathbb{C}_+. \tag{36}$$

Hypothesis 5. *Suppose that $T \neq T^*$ is a maximal dissipative extension of a symmetric operator $\dot{A}$ with deficiency indices $(1,1)$. Assume, in addition, that A is a self-adjoint (reference) extension of $\dot{A}$. Suppose, that the deficiency elements $g_\pm \in \operatorname{Ker}(\dot{A}^* \mp iI)$ are normalized, $\|g_\pm\| = 1$, and chosen in such a way that*

$$g_+ - g_- \in \operatorname{Dom}(A) \quad \textit{and} \quad g_+ - \kappa g_- \in \operatorname{Dom}(T) \quad \textit{for some} \quad |\kappa| < 1. \tag{37}$$

Under Hypothesis 5, we introduce the characteristic function $S = S(\dot{A}, T, A)$ associated with the triple of operators $(\dot{A}, T, A)$ as the Möbius transformation

$$S(z) = \frac{s(z) - \kappa}{\overline{\kappa}\, s(z) - 1}, \quad z \in \mathbb{C}_+, \tag{38}$$

of the Livšic function $s = s(\dot{A}, A)$ associated with the pair $(\dot{A}, A)$.

We remark that given a triple $(\dot{A}, T, A)$, one can always find a basis $g_\pm$ in the deficiency subspace $\operatorname{Ker}(\dot{A}^* - iI) \dot{+} \operatorname{Ker}(\dot{A}^* + iI)$,

$$\|g_\pm\| = 1, \quad g_\pm \in \operatorname{Ker}(\dot{A}^* \mp iI),$$

such that

$$g_+ - g_- \in \operatorname{Dom}(A) \quad \text{and} \quad g_+ - \kappa g_- \in \operatorname{Dom}(T),$$

and then, in this case,

$$\kappa = S(\dot{A}, T, A)(i). \tag{39}$$

Our next goal is to provide a *functional model* of a prime dissipative triple[2] parameterized by the characteristic function and obtained in [18].

Given a contractive analytic map S,

$$S(z) = \frac{s(z) - \kappa}{\overline{\kappa}\, s(z) - 1}, \quad z \in \mathbb{C}_+, \tag{40}$$

where $|\kappa| < 1$ and s is an analytic, contractive function in $\mathbb{C}_+$ satisfying the Livšic criterion (29), we use (36) to introduce the function

$$M(z) = \frac{1}{i} \cdot \frac{s(z) + 1}{s(z) - 1}, \quad z \in \mathbb{C}_+,$$

so that

$$M(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu(\lambda), \quad z \in \mathbb{C}_+,$$

for some infinite Borel measure with

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1.$$

In the Hilbert space $L^2(\mathbb{R}; d\mu)$ introduce the multiplication (self-adjoint) operator by the independent variable $\mathcal{B}$ on

$$\operatorname{Dom}(\mathcal{B}) = \left\{ f \in L^2(\mathbb{R}; d\mu) \,\middle|\, \int_{\mathbb{R}} \lambda^2 |f(\lambda)|^2 d\mu(\lambda) < \infty \right\}, \tag{41}$$

denote by $\dot{\mathcal{B}}$ its restriction on

$$\operatorname{Dom}(\dot{\mathcal{B}}) = \left\{ f \in \operatorname{Dom}(\mathcal{B}) \,\middle|\, \int_{\mathbb{R}} f(\lambda) d\mu(\lambda) = 0 \right\}, \tag{42}$$

and let $T_{\mathcal{B}}$ be the dissipative restriction of the operator $(\dot{\mathcal{B}})^*$ on

$$\operatorname{Dom}(T_{\mathcal{B}}) = \operatorname{Dom}(\dot{\mathcal{B}}) \dot{+} \operatorname{lin\ span} \left\{ \frac{1}{\cdot - i} - S(i) \frac{1}{\cdot + i} \right\}. \tag{43}$$

[2]We call a triple $(\dot{A}, T, A)$ a prime triple if $\dot{A}$ is a prime symmetric operator.

We will refer to the triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$ as *the model triple* in the Hilbert space $L^2(\mathbb{R}; d\mu)$.

It was established in [18] that a triple $(\dot{A}, T, A)$ with the characteristic function S is unitarily equivalent to the model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$ in the Hilbert space $L^2(\mathbb{R}; d\mu)$ whenever the underlying symmetric operator $\dot{A}$ is prime. The triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$ will therefore be called *the functional model* for $(\dot{A}, T, A)$.

It was pointed out in [18], if $\kappa = 0$, the quasi-self-adjoint extension T coincides with the restriction of the adjoint operator $(\dot{A})^*$ on

$$\mathrm{Dom}(T) = \mathrm{Dom}(\dot{A}) \dot{+} \mathrm{Ker}\,(\dot{A}^* - iI)$$

and the prime triples $(\dot{A}, T, A)$ with $\kappa = 0$ are in a one-to-one correspondence with the set of prime symmetric operators. In this case, the characteristic function S and the Livšic function s coincide (up to a sign),

$$S(z) = -s(z), \quad z \in \mathbb{C}_+.$$

For the resolvents of the model dissipative operator $T_{\mathcal{B}}$ and the self-adjoint (reference) operator $\mathcal{B}$ from the model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$ one gets the following resolvent formula.

Proposition 6. ([18]). *Suppose that $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$ is the model triple in the Hilbert space $L^2(\mathbb{R}; d\mu)$. Then the resolvent of the model dissipative operator $T_{\mathcal{B}}$ in $L^2(\mathbb{R}; d\mu)$ has the form*

$$(T_{\mathcal{B}} - zI)^{-1} = (\mathcal{B} - zI)^{-1} - p(z)(\cdot\,, g_{\overline{z}})g_z,$$

with

$$p(z) = \left(M(\dot{\mathcal{B}}, \mathcal{B})(z) + i\frac{\kappa+1}{\kappa-1}\right)^{-1}, \quad z \in \rho(T_{\mathcal{B}}) \cap \rho(\mathcal{B}).$$

Here $M(\dot{\mathcal{B}}, \mathcal{B})$ is the Weyl-Titchmarsh function associated with the pair $(\dot{\mathcal{B}}, \mathcal{B})$ continued to the lower half-plane by the Schwarz reflection principle, and the deficiency elements g_z are given by

$$g_z(\lambda) = \frac{1}{\lambda - z}, \quad \mu\text{-a.e.}\ .$$

5. Transfer function vs Livšic function

In this section and below, without loss of generality, we can assume that κ is real and that $0 \le \kappa < 1$. Indeed, if $\kappa = |\kappa| e^{i\theta}$, then change (the basis) g_- to $e^{i\theta} g_-$ in the deficiency subspace $\mathrm{Ker}\,(\dot{A}^* + iI)$, say. Thus, for the remainder of this paper we suppose that the von Neumann parameter κ is real and $0 \le \kappa < 1$.

The theorem below is the principal result of the current paper.

Theorem 7. *Let*

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{44}$$

be an L-system whose main operator T and the quasi-kernel $\hat{A}$ of $\mathrm{Re}\,\mathbb{A}$ satisfy Hypothesis 5 with the reference operator $A = \hat{A}$ and the von Neumann parameter κ. Then the transfer function $W_\Theta(z)$ and the characteristic function $S(z)$ of the triple $(\dot{A}, T, \hat{A})$ are reciprocals of each other, i.e.,

$$W_\Theta(z) = \frac{1}{S(z)}, \quad z \in \mathbb{C}_+ \cap \rho(T), \tag{45}$$

and $\frac{1}{W_\Theta(z)} \in \mathfrak{L}_\kappa$.

Proof. We are going to break the proof into three major steps.

Step 1. Let us consider the model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}_0}, \mathcal{B})$ developed in Section 4 and described via formulas (41)-(43) with $\kappa = 0$. Let $\mathbb{B}_0 \in [\mathcal{H}_+, \mathcal{H}_-]$ be a $(*)$-extension of $T_{\mathcal{B}_0}$ such that $\mathrm{Re}\,\mathbb{B}_0 \supset \mathcal{B} = \mathcal{B}^*$. Clearly, $T_{\mathcal{B}_0} \in \Lambda(\dot{\mathcal{B}})$ and $\mathcal{B}$ is the quasi-kernel of $\mathrm{Re}\,\mathbb{B}_0$. It was shown in [3, Theorem 4.4.6] that $\mathbb{B}_0$ exists and unique. We also note that by the construction of the model triple the von Neumann parameter $\mathcal{K} = \kappa$ that parameterizes $T_{\mathcal{B}_0}$ via (10) equals zero, i.e., $\mathcal{K} = \kappa = 0$. At the same time the parameter U that parameterizes the quasi-kernel $\mathcal{B}$ of $\mathrm{Re}\,\mathbb{B}_0$ is equal to 1, i.e., $U = 1$. Consequently, we can use the derivations of the end of Section 3 on $\mathbb{B}_0$, use formulas (26), (27) to conclude that

$$\mathrm{Im}\,\mathbb{B}_0 = (\cdot, \chi_0)\chi_0, \quad \chi_0 = \frac{1}{\sqrt{2}}\,(\varphi + \psi) \in \mathcal{H}_-, \tag{46}$$

where $\varphi \in \mathcal{H}_-$ and $\psi \in \mathcal{H}_-$ are basis vectors in $\mathcal{R}^{-1}(\mathfrak{N}_i)$ and $\mathcal{R}^{-1}(\mathfrak{N}_{-i})$, respectively. Now we can construct (see [3]) an L-system of the form

$$\Theta_0 = \begin{pmatrix} \mathbb{B}_0 & K_0 & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{47}$$

where $K_0 c = c \cdot \chi_0$, $K_0^* f = (f, \chi_0)$, $(f \in \mathcal{H}_+)$. The transfer function of this L-system can be written (see (6), (50) and [3]) as

$$W_{\Theta_0}(z) = 1 - 2i((\mathbb{B}_0 - zI)^{-1}\chi_0, \chi_0), \quad z \in \rho(T_{\mathcal{B}_0}), \tag{48}$$

and the impedance function is[3]

$$V_{\Theta_0}(z) = ((\mathrm{Re}\,\mathbb{B}_0 - zI)^{-1}\chi_0, \chi_0) = ((\mathcal{B} - zI)^{-1}\chi_0, \chi_0), \quad z \in \mathbb{C}_\pm. \tag{49}$$

At this point we apply Proposition 6 and obtain the following resolvent formula

$$(T_{\mathcal{B}_0} - zI)^{-1} = (\mathcal{B} - zI)^{-1} - \frac{1}{M(\dot{\mathcal{B}}, \mathcal{B})(z) - i}(\cdot, g_{\bar{z}})g_z, \quad z \in \rho(T_{\mathcal{B}_0}) \cap \mathbb{C}_\pm, \tag{50}$$

where $g_z = 1/(t - z)$ and $M(\dot{\mathcal{B}}, \mathcal{B})(z)$ is the Weyl-Titchmarsh function associated with the pair $(\dot{\mathcal{B}}, \mathcal{B})$. Moreover,

$$\begin{aligned} W_{\Theta_0}(z) &= 1 - 2i((\mathbb{B}_0 - zI)^{-1}\chi_0, \chi_0) \\ &= 1 - 2i((T_{\mathcal{B}_0} - zI)^{-1}\chi_0, \chi_0) \\ &= 1 - 2i\left[((\mathcal{B} - zI)^{-1}\chi_0, \chi_0) - \left(\frac{1}{M(\dot{\mathcal{B}}, \mathcal{B})(z) - i}(\chi_0, g_{\bar{z}})g_z, \chi_0\right)\right]. \end{aligned}$$

Without loss of generality we can assume that

$$g_z = (\mathcal{B} - zI)^{-1}\chi_0 = (\mathrm{Re}\,\mathbb{B}_0 - zI)^{-1}\chi_0 = \frac{1}{t - z}, \quad z \in \mathbb{C}_\pm. \tag{51}$$

Indeed, clearly $(\mathrm{Re}\,\mathbb{B}_0 - zI)^{-1}\chi_0 \in \mathfrak{N}_z$, where $\mathfrak{N}_z$ is the deficiency subspace of $\dot{\mathcal{B}}$, and thus

$$(\mathrm{Re}\,\mathbb{B}_0 - zI)^{-1}\chi_0 = \frac{\xi}{t - z}, \quad z \in \mathbb{C}_\pm,$$

[3] Here and below when we write $(\mathcal{B} - zI)^{-1}\chi_0$ for $\chi_0 \in \mathcal{H}_-$ we mean that the resolvent $(\mathcal{B} - zI)^{-1}$ is considered as extended to $\mathcal{H}_-$ (see [3]).

for some $\xi \in \mathbb{C}$. Let us show that $|\xi| = 1$. For the impedance function $V_{\Theta_0}(z)$ in (49) we have

$$
\begin{aligned}
\operatorname{Im} V_{\Theta_0}(z) &= \frac{1}{2i}\left[((\operatorname{Re}\mathbb{B}_0 - zI)^{-1}\chi_0, \chi_0) - ((\operatorname{Re}\mathbb{B}_0 - \bar{z}I)^{-1}\chi_0, \chi_0)\right] \\
&= \frac{1}{2i}\left[(z-\bar{z})((\operatorname{Re}\mathbb{B}_0 - zI)^{-1}(\operatorname{Re}\mathbb{B}_0 - \bar{z}I)^{-1}\chi_0, \chi_0)\right] \\
&= \operatorname{Im} z((\operatorname{Re}\mathbb{B}_0 - \bar{z}I)^{-1}\chi_0, (\operatorname{Re}\mathbb{B}_0 - \bar{z}I)^{-1}\chi_0) \\
&= \operatorname{Im} z\left(\frac{\xi}{t-\bar{z}}, \frac{\xi}{t-\bar{z}}\right)_{L^2(\mathbb{R};d\mu)} = (\operatorname{Im} z)|\xi|^2\int_{\mathbb{R}}\frac{d\mu}{|t-z|^2}.
\end{aligned} \tag{52}
$$

On the other hand, we know [3] that $V_{\Theta_0}(z)$ is a Herglotz-Nevanlinna function that has integral representation

$$
V_{\Theta_0}(z) = Q + \int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)d\mu, \quad Q = \bar{Q}.
$$

Using the above representation, the property $\overline{V_{\Theta_0}}(z) = V_{\Theta_0}(\bar{z})$, and straightforward calculations we find that

$$
\operatorname{Im} V_{\Theta_0}(z) = (\operatorname{Im} z)\int_{\mathbb{R}}\frac{d\mu}{|t-z|^2}. \tag{53}
$$

Considering that $\int_{\mathbb{R}}\frac{d\mu}{|t-z|^2} > 0$, we compare (52) with (53) and conclude that $|\xi| = 1$. Since $|\xi| = 1$, $\bar{\xi}$ can be scaled into χ_0 and we obtain (51).

Taking into account (51) and denoting $M_0 = M(\dot{\mathcal{B}}, \mathcal{B})(z)$ for the sake of simplicity, we continue

$$
\begin{aligned}
W_{\Theta_0}(z) &= 1 - 2i\left(V_{\Theta_0}(z) - \frac{1}{M_0 - i}V_{\Theta_0}^2(z)\right) \\
&= 1 - 2i\left(i\frac{W_{\Theta_0}(z)-1}{W_{\Theta_0}(z)+1} + \frac{1}{M_0-i}\left(\frac{W_{\Theta_0}(z)-1}{W_{\Theta_0}(z)+1}\right)^2\right).
\end{aligned}
$$

Thus,

$$
W_{\Theta_0}(z) - 1 = 2\frac{W_{\Theta_0}(z)-1}{W_{\Theta_0}(z)+1} - \frac{2i}{M_0-i}\left(\frac{W_{\Theta_0}(z)-1}{W_{\Theta_0}(z)+1}\right)^2,
$$

or

$$
1 = \frac{2}{W_{\Theta_0}(z)+1} - \frac{2i}{M_0-i}\cdot\frac{W_{\Theta_0}(z)-1}{(W_{\Theta_0}(z)+1)^2}.
$$

Solving this equation for $W_{\Theta_0}(z)+1$ yields

$$
W_{\Theta_0}(z) + 1 = \frac{(M_0 - 2i) \pm M_0}{M_0 - i}. \tag{54}
$$

Assume that $M_0(z) \neq i$ for $z \in \mathbb{C}_+$ and consider the two outcomes for formula (54). First case leads to $W_{\Theta_0}(z)+1 = 2$ or $W_{\Theta_0}(z) = 1$ which is impossible because it would lead (via (8)) to $V_{\Theta_0}(z) = 0$ that contradicts (53). The second case is

$$
W_{\Theta_0}(z) + 1 = -\frac{2i}{M_0 - i},
$$

leading to (see (36))

$$
W_{\Theta_0}(z) = -\frac{2i}{M_0-i} - 1 = -\frac{M_0+i}{M_0-i} = -\frac{1}{s(z)}, \quad z \in \mathbb{C}_+ \cap \rho(T_{\mathcal{B}_0}),
$$

where $s(z)$ is the Livšic function associated with the pair $(\dot{\mathcal{B}}, \mathcal{B})$. As we mentioned in Section 3, in the case when $\kappa = 0$ the characteristic function S and the Livšic function s

coincide (up to a sign), or $S(z) = -s(z)$. Hence,

$$W_{\Theta_0}(z) = -\frac{1}{s(z)} = \frac{1}{S(z)}, \quad z \in \mathbb{C}_+ \cap \rho(T_{\mathcal{B}_0}), \tag{55}$$

where $S(z)$ is the characteristic function of the model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}_0}, \mathcal{B})$.

In the case when $M_0(z) = i$ for all $z \in \mathbb{C}_+$, formula (36) would imply that $s(z) \equiv 0$ in the upper half-plane. Then, as it was shown in [18, Lemma 5.1], all the points $z \in \mathbb{C}_+$ are eigenvalues for $T_{\mathcal{B}_0}$ and the function $W_{\Theta_0}(z)$ is simply undefined in $\mathbb{C}$ making (54) irrelevant.

As we mentioned above, if $M_0(z) = i$ for all $z \in \mathbb{C}_+$, the function $W_{\Theta_0}(z)$ is ill-defined and (54) does not make sense in $\mathbb{C}_+$. One can, however, in this case re-write (54) in $\mathbb{C}_-$. Using the symmetry of $M_0(z)$ we get that $M_0(z) = -i$ for all $z \in \mathbb{C}_-$. Then (54) yields that $W_{\Theta_0}(z) = 0$. On the other hand, (36) extended to $\mathbb{C}_-$ in this case implies that $s(z) = \infty$ for all $z \in \mathbb{C}_-$ and hence (55) still formally holds true here for $z \in \mathbb{C}_+ \cap \rho(T_{\mathcal{B}_0})$.

Let us also make one more observation. Using formulas (8) and (55) yields

$$W_{\Theta_0}(z) = \frac{1 - iV_{\Theta_0}(z)}{1 + iV_{\Theta_0}(z)} = -\frac{V_{\Theta_0}(z) + i}{V_{\Theta_0}(z) - i} = -\frac{M_0(z) + i}{M_0(z) - i},$$

and hence

$$V_{\Theta_0}(z) = M_0(z), \quad z \in \mathbb{C}_+. \tag{56}$$

Step 2. Now we are ready to treat the case when $\kappa = \bar{\kappa} \neq 0$. Assume Hypothesis 5 and consider the model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$ described by formulas (41)-(43) with some κ, $0 \le \kappa < 1$. Let $\mathbb{B} \in [\mathcal{H}_+, \mathcal{H}_-]$ be a $(*)$-extension of $T_{\mathcal{B}}$ such that $\operatorname{Re}\mathbb{B} \supset \mathcal{B} = \mathcal{B}^*$. Below we describe the construction of $\mathbb{B}$. Equation (37) of Hypothesis 5 implies that

$$g_+ - g_- \in \operatorname{Dom}(\mathcal{B}) \quad \text{or} \quad g_+ + (-g_-) \in \operatorname{Dom}(\mathcal{B})$$

and

$$g_+ - \kappa g_- \in \operatorname{Dom}(T_{\mathcal{B}}) \quad \text{or} \quad g_+ + \kappa(-g_-) \in \operatorname{Dom}(T_{\mathcal{B}}).$$

Thus the von Neumann parameter $\mathcal{K}$ that parameterizes $T_{\mathcal{B}}$ via (10) is κ but the basis vector in $\mathfrak{N}_{-i}$ is $-g_-$. Consequently, $\mathcal{R}^{-1}g_+ = \varphi$ and $\mathcal{R}^{-1}(-g_-) = -\psi$. Using (24) and (25) and replacing ψ with $-\psi$, one obtains

$$\operatorname{Im}\mathbb{B} = (\cdot, \chi)\chi, \quad \chi = \sqrt{\frac{1-\kappa}{1+\kappa}}\left(\frac{1}{\sqrt{2}}\,\varphi - \frac{1}{\sqrt{2}}\,\psi\right). \tag{57}$$

We notice that if we followed the same basis pattern for the $(*)$-extension $\mathbb{B}_0$ (when $\kappa = 0$) then (46) would become slightly modified as follows

$$\operatorname{Im}\mathbb{B}_0 = (\cdot, \chi_0)\chi_0, \quad \chi_0 = \frac{1}{\sqrt{2}}\,(\varphi - \psi)\,. \tag{58}$$

As before we use $\mathbb{B}$ to construct a model L-system of the form

$$\Theta' = \begin{pmatrix} \mathbb{B} & K' & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{59}$$

where $K'c = c \cdot \chi$, $K'^*f = (f, \chi)$, $(f \in \mathcal{H}_+)$.

The impedance function of Θ' is

$$\begin{aligned} V_{\Theta'}(z) &= ((\operatorname{Re}\mathbb{B} - zI)^{-1}\chi, \chi) = ((\mathcal{B} - zI)^{-1}\chi, \chi) \\ &= \left((\mathcal{B} - zI)^{-1}\sqrt{\frac{1-\kappa}{1+\kappa}}\left(\frac{1}{\sqrt{2}}\,\varphi - \frac{1}{\sqrt{2}}\,\psi\right), \sqrt{\frac{1-\kappa}{1+\kappa}}\left(\frac{1}{\sqrt{2}}\,\varphi - \frac{1}{\sqrt{2}}\,\psi\right)\right) \\ &= \frac{1-\kappa}{1+\kappa}((\mathcal{B} - zI)^{-1}\chi_0, \chi_0) = \frac{1-\kappa}{1+\kappa}\,V_{\Theta_0}(z) = \frac{1-\kappa}{1+\kappa}\,M_0(z), \quad z \in \mathbb{C}_+. \end{aligned} \tag{60}$$

Here we used relations (56) and (58). On the other hand, using (38), (55), and (56) yields

$$\begin{aligned} S(z) &= \frac{s(z)-\kappa}{\kappa\, s(z)-1} = \frac{\frac{M_0-i}{M_0+i}-\kappa}{\kappa\,\frac{M_0-i}{M_0+i}-1} = \frac{(1-\kappa)M_0 - i(\kappa+1)}{(\kappa-1)M_0-(\kappa+1)i} \\ &= -\frac{\frac{1-\kappa}{1+\kappa}M_0 - i}{\frac{1-\kappa}{1+\kappa}M_0+i} = -\frac{V_\Theta(z)-i}{V_\Theta(z)+i} = \frac{1}{W_\Theta(z)}. \end{aligned}$$

Thus,

$$W_{\Theta'}(z) = \frac{1}{S(z)}, \quad z\in\mathbb{C}_+\cap\rho(T_{\mathcal{B}}), \tag{61}$$

where $S(z)$ is the characteristic function of the model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$.

Step 3. Now we are ready to treat the general case. Let

$$\Theta = \begin{pmatrix} & \mathbb{A} & & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & & & \mathbb{C} \end{pmatrix}$$

be an L-system from the statement of our theorem. Without loss of generality we can consider our L-system Θ to be minimal. If it is not minimal, we can use its so called "principal part", which is an L-system that has the same transfer and impedance functions (see [3, Section 6.6]). We use the von Neumann parameter κ of T and the conditions of Hypothesis 5 to construct a model system Θ' given by (59). By construction $W_\Theta(z) = W_{\Theta'(z)}$ and the characteristic functions of $(\dot{A}, T, \hat{A})$ and the model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$ coincide. The conclusion of the theorem then follows from Step 2 and formula (61). $\square$

Corollary 8. *If under conditions of Theorem 7 we also have that the von Neumann parameter κ of T equals zero, then $W_\Theta(z) = -1/s(z)$, where $s(z)$ is the Livšic function associated with the pair $(\dot{A}, \hat{A})$.*

Corollary 9. *Let Θ be an arbitrary L-system of the form (44). Then the transfer function of $W_\Theta(z)$ and the characteristic function $S(z)$ of a triple $(\dot{A}, T, \hat{A}_1)$ satisfying Hypothesis 5 with reference operator $A = \hat{A}_1$ are related via*

$$W_\Theta(z) = \frac{\nu}{S(z)}, \quad z\in\mathbb{C}_+\cap\rho(T), \tag{62}$$

where $\nu\in\mathbb{C}$ and $|\nu| = 1$.

Proof. The only difference between the L-system Θ here and the one described in Theorem 7 is that the set of conditions of Hypothesis 5 is satisfied for the latter. Moreover, there is an L-system Θ_1 of the form (44) with the same main operator T that complies with Hypothesis 5. Then according to the theorem about a constant J-unitary factor [3, Theorem 8.2.1], [4], $W_\Theta(z) = \nu W_{\Theta_1}(z)$, where ν is a unimodular complex number. Applying Theorem 7 to the L-system Θ_1 yields $W_{\Theta_1}(z) = 1/S(z)$, where $S(z)$ is the characteristic function of the triplet $(\dot{A}, T, \hat{A}_1)$ and $\hat{A}_1$ is the quasi-kernel of the real part of the operator $\mathbb{A}_1$ in Θ_1. Consequently,

$$W_\Theta(z) = \nu W_{\Theta_1}(z) = \frac{\nu}{S(z)},$$

where $|\nu| = 1$. $\square$

6. Impedance functions of the classes $\mathfrak{M}$ and $\mathfrak{M}_\kappa$

We say that an analytic function V from $\mathbb{C}_+$ into itself belongs to the **generalized Donoghue class** $\mathfrak{M}_\kappa$, $(0\le\kappa<1)$ if it admits the representation (32) with an infinite Borel measure μ and

$$\int_{\mathbb{R}}\frac{d\mu(\lambda)}{1+\lambda^2}=\frac{1-\kappa}{1+\kappa},\quad\text{equivalently,}\quad V(i)=i\,\frac{1-\kappa}{1+\kappa}. \tag{63}$$

Clearly, $\mathfrak{M}_0=\mathfrak{M}$.

We proceed by stating and proving the following important lemma.

Lemma 10. *Let Θ_κ of the form (44) be an L-system whose main operator T (with the von Neumann parameter κ, $0\le\kappa<1$) and the quasi-kernel $\hat{A}$ of $\mathrm{Re}\,\mathbb{A}$ satisfy the conditions of Hypothesis 5 with the reference operator $A=\hat{A}$. Then the impedance function $V_{\Theta_\kappa}(z)$ admits the representation*

$$V_{\Theta_\kappa}(z)=\frac{1-\kappa}{1+\kappa}\,V_{\Theta_0}(z),\quad z\in\mathbb{C}_+, \tag{64}$$

where $V_{\Theta_0}(z)$ is the impedance function of an L-system Θ_0 with the same set of conditions but with $\kappa_0=0$, where κ_0 is the von Neumann parameter of the main operator T_0 of Θ_0.

Proof. Once again we rely on our derivations above. We use the von Neumann parameter κ of T and the conditions of Hypothesis 5 to construct a model system Θ' given by (59). By construction $V_{\Theta_\kappa}(z)=V_{\Theta'}(z)$. Similarly, the impedance function $V_{\Theta_0}(z)$ coincides with the impedance function of a model system (47). The conclusion of the lemma then follows from (56) and (60). □

Theorem 11. *Let Θ of the form (44) be an L-system whose main operator T has the von Neumann parameter κ, $0\le\kappa<1$. Then its impedance function $V_\Theta(z)$ belongs to the Donoghue class $\mathfrak{M}$ if and only if $\kappa=0$.*

Proof. First of all, we note that in our system Θ the quasi-kernel $\hat{A}$ of $\mathrm{Re}\,\mathbb{A}$ does not necessarily satisfy the conditions of Hypothesis 5. However, if Θ_κ is a system from the statement of Lemma 10 with the same κ and Hypothesis 5 requirements, then

$$W_\Theta(z)=\nu W_{\Theta_\kappa}(z), \tag{65}$$

where ν is a complex number such that $|\nu|=1$. This follows from the theorem about a constant J-unitary factor [3, Theorem 8.2.1], [4].

To prove the Theorem in one direction we assume that $V_\Theta(z)\in\mathfrak{M}$ and $\kappa\ne0$. We know that Theorem 7 applies to the L-system Θ_κ and hence formula (45) takes place. Combining (45) with (65) and using the normalization condition (39) we obtain

$$W_\Theta(i)=\frac{\nu}{\kappa}. \tag{66}$$

We also know that according to [3, Theorem 6.4.3] the impedance function $V_\Theta(z)$ admits the following integral representation

$$V_\Theta(z)=Q+\int_{\mathbb{R}}\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)d\mu, \tag{67}$$

where Q is a real number and μ is an infinite Borel measure such that

$$\int_{\mathbb{R}}\frac{d\mu(\lambda)}{1+\lambda^2}=L<\infty.$$

It follows directly from (67) that $V_\Theta(i)=Q+iL$. Therefore, applying (8) directly to $W_\Theta(z)$ and using (66) yields

$$W_\Theta(i)=\frac{1-iV_\Theta(i)}{1+iV_\Theta(i)}=\frac{1-i(Q+iL)}{1+i(Q+iL)}=\frac{1+L-iQ}{1-L+iQ}=\frac{\nu}{\kappa}.$$

Cross multiplying yields

$$\kappa + \kappa L - i\kappa Q = \nu - \nu L + i\nu Q. \tag{68}$$

Solving this relation for Q gives us

$$Q = i\frac{\nu(1-L) - \kappa(1+L)}{\nu + \kappa}. \tag{69}$$

Taking into account that $\nu\bar{\nu} = 1$ and recalling our agreement in Section 3 to consider real κ only, we get

$$\bar{Q} = -i\frac{\bar{\nu}(1-L) - \kappa(1+L)}{\bar{\nu} + \kappa} = -i\frac{(1-L) - \kappa\nu(1+L)}{1 + \nu\kappa}. \tag{70}$$

But $Q = \bar{Q}$ and hence equating (69) and (70) and solving for L yields

$$L = \frac{\nu - \kappa^2\nu}{(\nu+\kappa)(1+\kappa\nu)}. \tag{71}$$

Clearly, $V_\Theta(z) \in \mathfrak{M}$ if and only if $Q = 0$ and $L = 1$. Setting the right hand side of (71) to 1 and solving for κ gives $\kappa = 0$ or $\kappa = -(\nu^2+1)/(2\nu)$, but only $\kappa = 0$ makes $Q = 0$ in (69). Consequently, our assumption that $\kappa \neq 0$ leads to a contradiction. Therefore, $V_\Theta(z) \in \mathfrak{M}$ implies $\kappa = 0$.

In order to prove the converse we assume that $\kappa = 0$. Let Θ_0 be the L-system Θ_κ described in the beginning of the proof with $\kappa = 0$. Let also $\hat{A}_0$ be the reference operator in Θ_0 that is the quasi-kernel of the real part of the state-space operator in Θ_0. Then the fact that $S(\dot{A}, T, \hat{A}_0)(z) = -s(\dot{A}, \hat{A}_0)(z)$ for $\kappa = 0$ (see Section 4) and (36) yield

$$W_\Theta(z) = \nu W_{\Theta_0}(z) = \frac{\nu}{S(\dot{A}, \hat{A}_0)(z)} = -\frac{\nu}{s(\dot{A}, \hat{A}_0)(z)} = \frac{\nu(M(\dot{A}, \hat{A}_0)(z) + i)}{i - M(\dot{A}, \hat{A}_0)(z)}.$$

Moreover, applying (8) to the above formula for $W_\Theta(z)$ we obtain

$$V_\Theta(z) = i\frac{W_\Theta(z) - 1}{W_\Theta(z) + 1} = i\frac{\frac{\nu(M(\dot{A},\hat{A}_0)(z)+i)}{i-M(\dot{A},\hat{A}_0)(z)} - 1}{\frac{\nu(M(\dot{A},\hat{A}_0)(z)+i)}{i-M(\dot{A},\hat{A}_0)(z)} + 1} = i\frac{(1+\bar{\nu})M(\dot{A}, \hat{A}_0)(z) + (1-\bar{\nu})i}{(1-\bar{\nu})M(\dot{A}, \hat{A}_0)(z) + (1+\bar{\nu})i}. \tag{72}$$

Substituting $z = -i$ to (72) yields $V_\Theta(-i) = -i$ and thus, by symmetry property of $V_\Theta(z)$, we have that $V_\Theta(i) = i$ and hence $V_\Theta(z) \in \mathfrak{M}$.

□

Consider the L-system Θ of the form (44) that was used in the statement of Theorem 11. This L-system does not necessarily comply with the conditions of Hypothesis 5 and hence the quasi-kernel $\hat{A}$ of $\operatorname{Re}\mathbb{A}$ is parameterized via (4) by some complex number U, $|U| = 1$. Then $U = e^{2i\beta}$, where $\beta \in [0, \pi)$. This representation allows us to introduce a one-parametric family of L-systems $\Theta_0(\beta)$ that all have $\kappa = 0$. That is,

$$\Theta_0(\beta) = \begin{pmatrix} \mathbb{A}_0(\beta) & K_0(\beta) & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}. \tag{73}$$

We note that $\Theta_0(\beta)$ satisfies the conditions of Hypothesis 5 only for the case when $\beta = 0$. Hence, the L-system Θ_0 from Lemma 10 can be written as $\Theta_0 = \Theta_0(0)$ using (73). Moreover, it directly follows from Theorem 11 that all the impedance functions $V_{\Theta_0(\beta)}(z)$ belong to the Donoghue class $\mathfrak{M}$ regardless of the value of $\beta \in [0, \pi)$.

The next theorem gives criteria on when the impedance function of an L-system belongs to the generalized Donoghue class $\mathfrak{M}_\kappa$.

Theorem 12. *Let Θ_κ, $0 < \kappa < 1$, of the form (44) be a minimal L-system with the main operator T and the impedance function $V_{\Theta_\kappa}(z)$ which is not an identical constant in $\mathbb{C}_+$. Then $V_{\Theta_\kappa}(z)$ belongs to the generalized Donoghue class $\mathfrak{M}_\kappa$ and (64) holds if and only if the triple $(\dot{A}, T, \hat{A})$ satisfies Hypothesis 5 with $A = \hat{A}$, the quasi-kernel of $\operatorname{Re}\mathbb{A}$.*

Proof. We prove the necessity first. Suppose the triple $(\dot{A}, T, \hat{A})$ in Θ satisfies the conditions of Hypothesis 5. Then, according to Lemma 10, formula (64) holds and consequently $V_{\Theta_\kappa}(z)$ belongs to the generalized Donoghue class $\mathfrak{M}_\kappa$.

In order to prove the Theorem in the other direction we assume that $V_{\Theta_\kappa}(z) \in \mathfrak{M}_\kappa$ satisfies equation (64) for some L-system Θ_0. Then according to Theorem 11 $V_{\Theta_0}(z)$ belongs to the Donoghue class $\mathfrak{M}$. Clearly then (64) implies that $V_{\Theta_\kappa}(z)$ has $Q = 0$ in its integral representation (69). Moreover,

$$V_{\Theta_\kappa}(i) = \frac{1-\kappa}{1+\kappa} V_{\Theta_0}(i) = i\frac{1-\kappa}{1+\kappa} = i\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1+\lambda^2},$$

where $\mu(\lambda)$ is the measure from the integral representation (69) of $V_{\Theta_\kappa}(z)$. Thus,

$$L = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1+\lambda^2} = \frac{1-\kappa}{1+\kappa}.$$

Assume the contrary, i.e., suppose that the quasi-kernel $\hat{A}$ of $\mathrm{Re}\,\mathbb{A}$ of Θ_κ does not satisfy the conditions of Hypothesis 5. Then, consider another L-system Θ' of the form (44) which is only different from Θ by that its quasi-kernel $\hat{A}'$ of $\mathrm{Re}\,\mathbb{A}'$ satisfies the conditions of Hypothesis 5 for the same value of κ. Applying the theorem about a constant J-unitary factor [3, Theorem 8.2.1] then yields

$$W_{\Theta_\kappa}(z) = \nu W_{\Theta'}(z),$$

where ν is a complex number such that $|\nu| = 1$. Our goal is to show that $\nu = 1$. Since we know the values of Q and L in the integral representation (69) of $V_{\Theta_\kappa}(z)$, we can use this information to find ν from (69). We have then

$$0 = i\frac{\nu(1-L) - \kappa(1+L)}{\nu+\kappa}, \qquad \text{where} \qquad L = \frac{1-\kappa}{1+\kappa}.$$

Consequently, $\nu(1-L) - \kappa(1+L) = 0$ or

$$\nu = \kappa\,\frac{1+L}{1-L} = \kappa\,\frac{1+\frac{1-\kappa}{1+\kappa}}{1-\frac{1-\kappa}{1+\kappa}} = \kappa\cdot\frac{2}{2\kappa} = 1.$$

Thus, $\nu = 1$ and hence

$$W_{\Theta_\kappa}(z) = W_{\Theta'}(z). \tag{74}$$

Our L-system Θ_κ is minimal and hence we can apply the Theorem on bi-unitary equivalence [3, Theorem 6.6.10] for L-systems Θ_κ and Θ' and obtain that the pairs $(\dot{A}, \hat{A})$ and $(\dot{A}, \hat{A}')$ are unitarily equivalent. Consequently, the Weyl-Titchmarsh functions $M(\dot{A}, \hat{A})$ and $M(\dot{A}, \hat{A}')$ coincide. At the same time, both $\hat{A}$ and $\hat{A}'$ are self-adjoint extensions of the symmetric operator $\dot{A}$ giving us the following relation between $M(\dot{A}, \hat{A})$ and $M(\dot{A}, \hat{A}')$ (see [19, Subsection 2.2])

$$M(\dot{A}, \hat{A}) = \frac{\cos\alpha\, M(\dot{A}, \hat{A}') - \sin\alpha}{\cos\alpha + \sin\alpha\, M(\dot{A}, \hat{A}')} \quad \text{for some} \quad \alpha \in [0, \pi). \tag{75}$$

Using $M(\dot{A}, \hat{A}')(z) = M(\dot{A}, \hat{A})(z)$ for $z \in \mathbb{C}_+$ on (75) and solving for $M(\dot{A}, \hat{A})(z)$ gives us that either $\alpha = 0$ or $M(\dot{A}, \hat{A})(z) = i$ for all $z \in \mathbb{C}_+$. The former case of $\alpha = 0$ gives $\hat{A} = \hat{A}'$, and thus $\hat{A}$ satisfies the conditions of Hypothesis 5 which contradicts our assumption. The latter case would imply (via (36)) that $s(z) = s(\dot{A}, \hat{A})(z) \equiv 0$ and consequently $S(z) = S(\dot{A}, \hat{A}, T)(z) \equiv \kappa$ in the upper half-plane. Then (45) and (62) yield $W_{\Theta_\kappa}(z) = \theta/\kappa$ for some θ such that $|\theta| = 1$ and hence

$$V_{\Theta_\kappa}(z) = i\frac{\theta/\kappa - 1}{\theta/\kappa + 1} = i\frac{\theta - \kappa}{\theta + \kappa}, \quad z \in \mathbb{C}_+. \tag{76}$$

Thus, in particular,

$$V_{\Theta_\kappa}(i) = i\frac{\theta-\kappa}{\theta+\kappa}.$$

On the other hand, we know that $V_{\Theta_\kappa}(z)$ satisfies equation (64) and hence (taking into account that $V_{\Theta_0}(i)=i$), plugging $z=i$ in (64) gives

$$V_{\Theta_\kappa}(i) = i\frac{1-\kappa}{1+\kappa}.$$

Combining the two equations above we get $\theta = 1$. Therefore, (76) yields

$$V_{\Theta_\kappa}(z) = i\frac{1-\kappa}{1+\kappa},\quad z\in\mathbb{C}_+, \tag{77}$$

which brings us back to a contradiction with a condition of the Theorem that $V_{\Theta_\kappa}(z)$ is not an identical constant. Consequently, $\alpha = 0$ is the only feasible choice and hence $\hat{A} = \hat{A}'$ implying that $\hat{A}$ satisfies the conditions of Hypothesis 5. □

Remark 13. *Let us consider the case when the condition of $V_{\Theta_\kappa}(z)$ not being an identical constant in $\mathbb{C}_+$ is omitted in the statement of Theorem 12. Then, as we have shown in the proof of the theorem, $V_{\Theta_\kappa}(z)$ may take a form* (77)*. We will show that in this case the L-system Θ from the statement of Theorem 12 is bi-unitarily equivalent to an L-system Θ' that satisfies the conditions of Hypothesis 5.*

Let $V_{\Theta_\kappa}(z)$ from Theorem 12 takes a form (77)*. Let also $\mu(\lambda)$ be a Borel measure on $\mathbb{R}$ given by the simple formula*

$$\mu(\lambda) = \frac{\lambda}{\pi},\quad \lambda\in\mathbb{R}, \tag{78}$$

and let $V_0(z)$ be a function with integral representation (32) *with the measure μ, i.e.,*

$$V_0(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\right) d\mu.$$

Then by direct calculations one immediately finds that $V_0(i)=i$ and that $V_0(z_1)-V_0(z_2)=0$ for any $z_1\neq z_2$ in $\mathbb{C}_+$. Therefore, $V_0(z)\equiv i$ in $\mathbb{C}_+$ and hence using (77) *we obtain* (64) *or*

$$V_{\Theta_\kappa}(z) = i\frac{1-\kappa}{1+\kappa} = \frac{1-\kappa}{1+\kappa}\,V_0(z),\quad z\in\mathbb{C}_+. \tag{79}$$

Let us construct a model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$ defined by (41)–(43) *in the Hilbert space $L^2(\mathbb{R};d\mu)$ using the measure μ from* (78) *and our value of κ. Using the formula for the deficiency elements $g_z(\lambda)$ of $\dot{B}$ (see Proposition 6) and the definition of $s(\dot{B},\mathcal{B})(z)$ in* (35) *we evaluate that $s(\dot{B},\mathcal{B})(z)\equiv 0$ in $\mathbb{C}_+$. Then,* (40) *yields $S(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})(z)\equiv\kappa$ in $\mathbb{C}_+$. Moreover, applying Proposition 6 to the operator $T_{\mathcal{B}}$ in our triple we obtain*

$$(T_{\mathcal{B}} - zI)^{-1} = (\mathcal{B}-zI)^{-1} + i\left(\frac{\kappa-1}{2\kappa}\right)(\cdot\,, g_{\overline{z}})g_z. \tag{80}$$

Let us now follow Step 2 of the proof of Theorem 7 to construct a model L-system Θ' of the form (59) *corresponding to our model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}}, \mathcal{B})$. Note, that this L-system Θ' is minimal by construction, its main operator $T_{\mathcal{B}}$ has regular points in $\mathbb{C}_+$ due to* (80)*, and, according to* (45)*, $W_{\Theta'}(z)\equiv 1/\kappa$. But formulas* (8) *yield that in the case under consideration $W_{\Theta_\kappa}(z)\equiv 1/\kappa$. Therefore $W_{\Theta_\kappa}(z) = W_{\Theta'}(z)$ and we can (taking into account the properties of Θ' we mentioned) apply the Theorem on bi-unitary equivalence* [3, Theorem 6.6.10] *for L-systems Θ_κ and Θ'. Thus we have successfully constructed an L-system Θ' that is bi-unitarily equivalent to the L-system Θ_κ and satisfies the conditions of Hypothesis 5.*

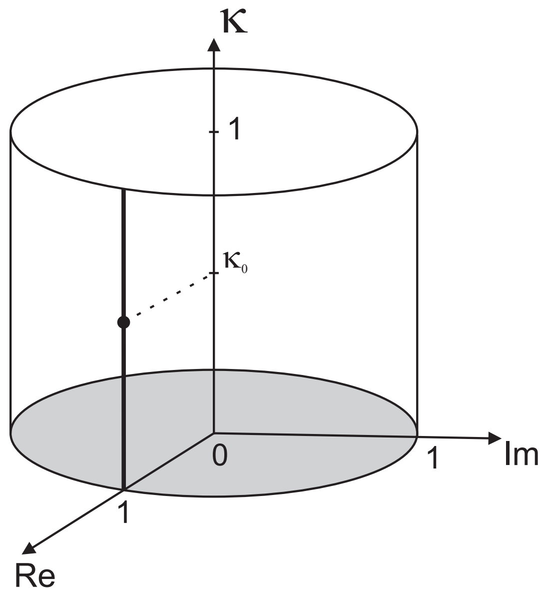

FIGURE 1. Parametric region $0 \le \kappa < 1$, $0 \le \beta < \pi$

Using similar reasoning as above we introduce another one parametric family of L-systems

$$\Theta_\kappa(\beta) = \begin{pmatrix} \mathbb{A}_\kappa(\beta) & K_\kappa(\beta) & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{81}$$

which is different from the family in (73) by the fact that all the members of the family have the same operator T with the fixed von Neumann parameter $\kappa \neq 0$. It easily follows from Theorem 12 that for all $\beta \in [0, \pi)$ there is only one non-constant in $\mathbb{C}_+$ impedance function $V_{\Theta_\kappa(\beta)}(z)$ that belongs to the class $\mathfrak{M}_\kappa$. This happens when $\beta = 0$ and consequently the L-system $\Theta_\kappa(0)$ complies with the conditions of Hypothesis 5. The results of Theorems 11 and 12 can be illustrated with the help of Figure 1 describing the parametric region for the family of L-systems $\Theta(\beta)$. When $\kappa = 0$ and β changes from 0 to π, every point on the unit circle with cylindrical coordinates $(1, \beta, 0)$, $\beta \in [0, \pi)$ describes an L-system $\Theta_0(\beta)$ and Theorem 11 guarantees that $V_{\Theta_0(\beta)}(z)$ belongs to the class $\mathfrak{M}$. On the other hand, for any κ_0 such that $0 < \kappa_0 < 1$ we apply Theorem 12 to conclude that only the point $(1, 0, \kappa_0)$ on the wall of the cylinder is responsible for an L-system $\Theta_{\kappa_0}(0)$ such that $V_{\Theta_{\kappa_0}(0)}(z)$ belongs to the class $\mathfrak{M}_{\kappa_0}$.

Theorem 14. *Let $V(z)$ belong to the generalized Donoghue class $\mathfrak{M}_\kappa$, $0 \le \kappa < 1$. Then $V(z)$ can be realized as the impedance function $V_{\Theta_\kappa}(z)$ of an L-system Θ_κ of the form* (44) *with the triple $(\dot{A}, T, \hat{A})$ that satisfies Hypothesis 5 with $A = \hat{A}$, the quasi-kernel of* $\mathrm{Re}\,\mathbb{A}$. *Moreover,*

$$V(z) = V_{\Theta_\kappa}(z) = \frac{1-\kappa}{1+\kappa} M(\dot{A}, \hat{A})(z), \quad z \in \mathbb{C}_+, \tag{82}$$

where $M(\dot{A}, \hat{A})(z)$ is the Weyl-Titchmarsh function associated with the pair $(\dot{A}, \hat{A})$.

Proof. Since $V(z) \in \mathfrak{M}_\kappa$, then it admits the integral representation (32) with normalization condition (63) on the measure μ. Set

$$c = \frac{1+\kappa}{1-\kappa}.$$

It follows directly from definitions of classes $\mathfrak{M}$ and $\mathfrak{M}_\kappa$ that the function $cV \in \mathfrak{M}$ and thus has the integral representation (32) with the measure $\mu_0 = c\mu$ and normalization condition (33) on the measure μ_0. We use the measure μ_0 to construct a model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}_0}, \mathcal{B})$ described by (41)-(43) with $S(i) = 0$. Note that the model triple $(\dot{\mathcal{B}}, T_{\mathcal{B}_0}, \mathcal{B})$ satisfies Hypothesis 5. Then we follow Step 1 of the proof of Theorem 7 to build an L-system Θ_0 given by (47). According to (56) $V_{\Theta_0}(z) = M(\dot{\mathcal{B}}, \mathcal{B})(z)$. On the other hand, since $M(\dot{\mathcal{B}}, \mathcal{B})(z)$ is the Weyl-Titchmarsh function associated with the pair $(\dot{\mathcal{B}}, \mathcal{B})$, then it also admits a representation

$$M(\dot{\mathcal{B}}, \mathcal{B})(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\mu_0, \quad z \in \mathbb{C}_+,$$

with the same measure μ_0 as in the representation for cV. Therefore,

$$cV(z) = M(\dot{\mathcal{B}}, \mathcal{B})(z) = V_{\Theta_0}(z), \quad z \in \mathbb{C}_+,$$

or $V(z) = (1/c)V_{\Theta_0}(z)$. Then we proceed with Step 2 of the proof of Theorem 7 to construct an L-system Θ' given by (59). It is shown in (60) that

$$V_{\Theta'}(z) = \frac{1-\kappa}{1+\kappa} M(\dot{\mathcal{B}}, \mathcal{B})(z), \quad z \in \mathbb{C}_+, \tag{83}$$

and hence

$$V_{\Theta'}(z) = \frac{1-\kappa}{1+\kappa} M(\dot{\mathcal{B}}, \mathcal{B})(z) = \frac{1-\kappa}{1+\kappa} cV(z) = V(z).$$

Therefore, we have constructed an L-system $\Theta_\kappa = \Theta'$ such that $V(z) = V_{\Theta_\kappa}(z)$. The remaining part of (82) follows from (83). □

7. Examples

Example 1. Following [1] we consider the prime symmetric operator

$$\dot{A}x = i\frac{dx}{dt}, \quad \mathrm{Dom}(\dot{A}) = \left\{ x(t) \,\Big|\, x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]},\, x(0) = x(\ell) = 0 \right\}. \tag{84}$$

Its (normalized) deficiency vectors of $\dot{A}$ are

$$g_+ = \frac{\sqrt{2}}{\sqrt{e^{2\ell}-1}} e^t \in \mathfrak{N}_i, \quad g_- = \frac{\sqrt{2}}{\sqrt{1-e^{-2\ell}}} e^{-t} \in \mathfrak{N}_{-i}. \tag{85}$$

If we set $C = \frac{\sqrt{2}}{\sqrt{e^{2\ell}-1}}$, then (85) can be re-written as

$$g_+ = Ce^t, \quad g_- = Ce^\ell e^{-t}.$$

Let

$$Ax = i\frac{dx}{dt}, \quad \mathrm{Dom}(A) = \left\{ x(t) \,\Big|\, x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]},\, x(0) = -x(\ell) \right\} \tag{86}$$

be a self-adjoint extension of $\dot{A}$. Clearly, $g_+(0) - g_-(0) = C - Ce^\ell$ and $g_+(\ell) - g_-(\ell) = Ce^\ell - C$ and hence (34) is satisfied, i.e., $g_+ - g_- \in \mathrm{Dom}(A)$.

Then the Livšic characteristic function $s(z)$ for the pair $(\dot{A}, A)$ ha stye form (see [1])

$$s(z) = \frac{e^\ell - e^{-i\ell z}}{1 - e^\ell e^{-i\ell z}}. \tag{87}$$

We introduce the operator

$$Tx = i\frac{dx}{dt}, \quad \mathrm{Dom}(T) = \left\{ x(t) \,\Big|\, x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]},\, x(0) = 0 \right\}. \tag{88}$$

By construction, T is a dissipative extension of $\dot{A}$ parameterized by a von Neumann parameter κ. To find κ we use (85) with (30) to obtain

$$x(t) = Ce^t - \kappa\, Ce^\ell e^{-t} \in \mathrm{Dom}(T), \quad x(0) = 0, \tag{89}$$

yielding

$$\kappa = e^{-\ell}. \tag{90}$$

Obviously, the triple of operators $(\dot{A}, T, A)$ satisfy the conditions of Hypothesis 5 since $|\kappa| = e^{-\ell} < 1$. Therefore, we can use (38) to write out the characteristic function $S(z)$ for the triple $(\dot{A}, T, A)$

$$S(z) = \frac{s(z) - \kappa}{\bar{\kappa} s(z) - 1} = \frac{e^{\ell} - \kappa + e^{-i\ell z}(\kappa e^{\ell} - 1)}{\bar{\kappa} e^{\ell} - 1 + e^{-i\ell z}(e^{\ell} - \bar{\kappa})}, \tag{91}$$

and apply the value of $\kappa = e^{-\ell}$ to get

$$S(z) = e^{i\ell z}. \tag{92}$$

Now we shall use the triple $(\dot{A}, T, A)$ for an L-system Θ that we about to construct. First, we note that by the direct check one gets

$$T^* x = i\frac{dx}{dt}, \quad \mathrm{Dom}(T) = \left\{ x(t) \,\middle|\, x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]},\, x(\ell) = 0 \right\}. \tag{93}$$

Following the steps of Example 7.6 of [3] we have

$$\dot{A}^* x = i\frac{dx}{dt}, \quad \mathrm{Dom}(\dot{A}^*) \quad = \left\{ x(t) \,\middle|\, x(t) - \text{ abs. cont.}, x'(t) \in L^2_{[0,\ell]} \right\}. \tag{94}$$

Then $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*) = W^1_2$ is the Sobolev space with scalar product

$$(x, y)_+ = \int_0^\ell x(t)\overline{y(t)}\, dt + \int_0^\ell x'(t)\overline{y'(t)}\, dt. \tag{95}$$

Construct rigged Hilbert space $W^1_2 \subset L^2_{[0,\ell]} \subset (W^1_2)_-$ and consider operators

$$\mathbb{A} x = i\frac{dx}{dt} + ix(0)\left[\delta(t) - \delta(t - \ell)\right], \quad \mathbb{A}^* x = i\frac{dx}{dt} + ix(l)\left[\delta(t) - \delta(t - \ell)\right], \tag{96}$$

where $x(t) \in W^1_2$, $\delta(t)$, $\delta(t-\ell)$ are delta-functions and elements of $(W^1_2)_-$ that generate functionals by the formulas $(x, \delta(t)) = x(0)$ and $(x, \delta(t-\ell)) = x(\ell)$. It is easy to see that $\mathbb{A} \supset T \supset \dot{A}$, $\mathbb{A}^* \supset T^* \supset \dot{A}$, and that

$$\mathrm{Re}\,\mathbb{A} x = i\frac{dx}{dt} + \frac{i}{2}(x(0) + x(\ell))\left[\delta(t) - \delta(t - \ell)\right].$$

Clearly, $\mathrm{Re}\,\mathbb{A}$ has its quasi-kernel equal to A in (86). Moreover,

$$\mathrm{Im}\,\mathbb{A} x = \left(\cdot, \frac{1}{\sqrt{2}}[\delta(t) - \delta(t - \ell)]\right) \frac{1}{\sqrt{2}}[\delta(t) - \delta(t - \ell)] = (\cdot, \chi)\chi,$$

where $\chi = \frac{1}{\sqrt{2}}[\delta(t) - \delta(t - \ell)]$. Now we can build

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ W^1_2 \subset L^2_{[0,\ell]} \subset (W^1_2)_- & & \mathbb{C} \end{pmatrix}, \tag{97}$$

that is a minimal L-system with

$$\begin{aligned} Kc &= c \cdot \chi = c \cdot \frac{1}{\sqrt{2}}[\delta(t) - \delta(t - l)], \quad (c \in \mathbb{C}), \\ K^* x &= (x, \chi) = \left(x, \frac{1}{\sqrt{2}}[\delta(t) - \delta(t - l)]\right) = \frac{1}{\sqrt{2}}[x(0) - x(l)], \end{aligned} \tag{98}$$

and $x(t) \in W_2^1$. In order to find the transfer function of Θ we begin by evaluating the resolvent of operator T in (88). Solving the linear differential equation of the first order with the initial condition from (88) yields

$$R_z(T)f = (T - zI)^{-1}f = -ie^{-izt}\int_0^t f(s)e^{izs}\,ds, \quad f \in L^2_{[0,\ell]}. \tag{99}$$

Similarly, one finds that

$$R_z(T^*)f = (T^* - zI)^{-1}f = ie^{-izt}\int_t^\ell f(s)e^{izs}\,ds, \quad f \in L^2_{[0,\ell]}. \tag{100}$$

We need to extend $R_z(T)$ to $(W_2^1)_-$ to apply it to the vector g. We can accomplish this via finding the values of $\hat{R}_z(T)\delta(t)$ and $\hat{R}_z(T)\delta(t-l)$ (here $\hat{R}_z(T)$ is the extended resolvent). We have

$$\begin{aligned}(\hat{R}_z(T)\delta(t), f) &= (\delta(t), R_{\bar{z}}(T^*)f) = \overline{R_{\bar{z}}(T^*)f\Big|_{t=0}} = -i\int_0^\ell e^{-izs}\overline{f(s)}ds\\ &= (-ie^{-izt}, f), \quad f \in L^2_{[0,\ell]},\end{aligned}$$

and hence $\hat{R}_z(T)\delta(t) = -ie^{-izt}$. Similarly, we determine that $\hat{R}_z(T)\delta(t-l) = 0$. Consequently,

$$\hat{R}_z(T)g = -\frac{i}{\sqrt{2}}e^{-izt}.$$

Therefore,

$$\begin{aligned}W_\Theta(z) &= 1 - 2i((T - zI)^{-1}\chi, \chi) = 1 - 2i\left(-\frac{i}{\sqrt{2}}e^{-izt}, \frac{1}{\sqrt{2}}[\delta(t) - \delta(t-\ell)]\right)\\ &= 1 - (e^{-izt}, \delta(t) - \delta(t-\ell)) = 1 - 1 + e^{-i\ell z} = e^{-i\ell z}.\end{aligned} \tag{101}$$

This confirms the result of Theorem 7 and formula (55) by showing that $W_\Theta(z) = 1/S(z)$. The corresponding impedance function is found via (8) and is

$$V_\Theta(z) = i\frac{e^{-i\ell z} - 1}{e^{-i\ell z} + 1}.$$

Direct substitution yields

$$V_\Theta(i) = i\frac{e^\ell - 1}{e^\ell + 1} = i\frac{1 - e^{-\ell}}{1 + e^{-\ell}} = i\frac{1-\kappa}{1+\kappa},$$

and thus $V_\Theta(z) \in \mathfrak{M}_\kappa$ with $\kappa = e^{-\ell}$.

Example 2. In this Example we will rely on the main elements of the construction presented in Example 1 but with some changes. Let $\dot{A}$ and A be still defined by formulas (84) and (86), respectively and let $s(z)$ be the Livšic characteristic function $s(z)$ for the pair $(\dot{A}, A)$ given by (87). We introduce the operator

$$T_0 x = i\frac{dx}{dt}, \tag{102}$$

$$\mathrm{Dom}(T_0) = \left\{x(t)\,\Big|\, x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]},\ x(\ell) = e^\ell x(0)\right\}.$$

It turns out that T_0 is a dissipative extension of $\dot{A}$ parameterized by a von Neumann parameter $\kappa = 0$. Indeed, using (85) with (30) again we obtain

$$x(t) = Ce^t - \kappa\, Ce^\ell e^{-t} \in \mathrm{Dom}(T), \quad x(\ell) = e^\ell x(0), \tag{103}$$

yielding $\kappa = 0$. Clearly, the triple of operators $(\dot{A}, T_0, A)$ satisfy the conditions of Hypothesis 5 but this time, since $\kappa = 0$, we have that $S(z) = -s(z)$.

Following the steps of Example 1 we are going to use the triple $(\dot{A}, T_0, A)$ in the construction of an L-system Θ_0. By the direct check one gets

$$T_0^* x = i\frac{dx}{dt}, \tag{104}$$

$$\operatorname{Dom}(T) = \left\{ x(t) \,\Big|\, x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]},\, x(\ell) = e^{-\ell} x(0) \right\}.$$

Once again, we have $\dot{A}^*$ defined by (94) and $\mathcal{H}_+ = \operatorname{Dom}(\dot{A}^*) = W_2^1$ is a space with scalar product (95). Consider the operators

$$\begin{aligned} \mathbb{A}_0 x &= i\frac{dx}{dt} + i\frac{x(\ell) - e^{\ell} x(0)}{e^{\ell} - 1}\left[\delta(t-\ell) - \delta(t)\right], \\ \mathbb{A}_0^* x &= i\frac{dx}{dt} + i\frac{x(0) - e^{\ell} x(\ell)}{e^{\ell} - 1}\left[\delta(t-\ell) - \delta(t)\right], \end{aligned} \tag{105}$$

where $x(t) \in W_2^1$. It is easy to see that $\mathbb{A} \supset T_0 \supset \dot{A}$, $\mathbb{A}^* \supset T_0^* \supset \dot{A}$, and

$$\operatorname{Re} \mathbb{A}_0 x = i\frac{dx}{dt} - \frac{i}{2}(x(0) + x(\ell))\left[\delta(t-\ell) - \delta(t)\right].$$

Thus $\operatorname{Re} \mathbb{A}_0$ has its quasi-kernel equal to A in (86). Similarly,

$$\operatorname{Im} \mathbb{A}_0 x = \left(\frac{1}{2}\right)\frac{e^{\ell} + 1}{e^{\ell} - 1}(x(\ell) - x(0))\left[\delta(t-\ell) - \delta(t)\right].$$

Therefore,

$$\begin{aligned} \operatorname{Im} \mathbb{A}_0 &= \left(\cdot, \sqrt{\frac{e^{\ell}+1}{2(e^{\ell}-1)}}\left[\delta(t-\ell) - \delta(t)\right]\right)\sqrt{\frac{e^{\ell}+1}{2(e^{\ell}-1)}}\left[\delta(t-\ell) - \delta(t)\right] \\ &= (\cdot, \chi_0)\chi_0, \end{aligned}$$

where $\chi_0 = \sqrt{\frac{e^{\ell}+1}{2(e^{\ell}-1)}}\left[\delta(t-\ell) - \delta(t)\right]$. Now we can build

$$\Theta_0 = \begin{pmatrix} \mathbb{A}_0 & K_0 & 1 \\ W_2^1 \subset L^2_{[0,l]} \subset (W_2^1)_- & & \mathbb{C} \end{pmatrix},$$

which is a minimal L-system with $K_0 c = c \cdot \chi_0$, $(c \in \mathbb{C})$, $K_0^* x = (x, \chi_0)$ and $x(t) \in W_2^1$. Following Example 1 we derive

$$\begin{aligned} R_z(T_0) &= (T_0 - zI)^{-1} f \\ &= -ie^{-izt}\left(\int_0^t f(s)e^{izs}\,ds + \frac{e^{-i\ell z}}{e^{\ell} - e^{-i\ell z}}\int_0^l f(s)e^{izs}\,ds\right) \end{aligned} \tag{106}$$

and

$$\begin{aligned} R_z(T_0^*) &= (T_0^* - zI)^{-1} f \\ &= -ie^{-izt}\left(\int_0^t f(s)e^{izs}\,ds + \frac{e^{-i\ell z}}{e^{-\ell} - e^{-i\ell z}}\int_0^l f(s)e^{izs}\,ds\right) \end{aligned} \tag{107}$$

for $f \in L^2_{[0,\ell]}$. Then again

$$\begin{aligned} (\hat{R}_z(T_0)\delta(t), f) &= (\delta(t), R_{\bar{z}}(T_0^*) f) = \overline{R_{\bar{z}}(T_0^*) f\Big|_{t=0}} = \frac{ie^{i\ell z}}{e^{-\ell} - e^{i\ell z}}\int_0^{\ell} e^{-izs}\overline{f(s)}ds \\ &= \frac{ie^{\ell}}{e^{-i\ell z} - e^{\ell}}(e^{-izt}, f), \quad f \in L^2_{[0,\ell]}. \end{aligned}$$

Similarly,

$$(\hat{R}_z(T_0)\delta(t-\ell), f) = (\delta(t-\ell), R_{\bar{z}}(T_0^*)f) = \overline{R_{\bar{z}}(T_0^*)f\Big|_{t=\ell}}$$
$$= \frac{ie^{i\ell z}e^{-\ell}}{e^{-\ell}-e^{i\ell z}}\int_0^\ell e^{-izs}\overline{f(s)}ds = \frac{i}{e^{-i\ell z}-e^{\ell}}(e^{-izt}, f), \quad f \in L^2_{[0,\ell]}.$$

Hence,

$$\hat{R}_z(T_0)\delta(t) = \frac{ie^{\ell}}{e^{-i\ell z}-e^{\ell}}\,e^{-izt}, \quad \hat{R}_z(T_0)\delta(t-\ell) = \frac{i}{e^{-i\ell z}-e^{\ell}}\,e^{-izt}, \tag{108}$$

and

$$\hat{R}_z(T_0)\chi_0 = \hat{R}_z(T_0)\sqrt{\frac{e^\ell+1}{2(e^\ell-1)}}\,[\delta(t-\ell)-\delta(t)] = \sqrt{\frac{e^\ell+1}{2(e^\ell-1)}}\frac{i-ie^\ell}{e^{-i\ell z}-e^\ell}\,e^{-izt}.$$

Using techniques of Example 1 one finds the transfer function of Θ_0 to be

$$\begin{aligned}
W_{\Theta_0}(z) &= 1-2i(\hat{R}_z(T_0)\chi_0,\chi_0)\\
&= 1-2i\left(\sqrt{\frac{e^\ell+1}{2(e^\ell-1)}}\frac{i-ie^\ell}{e^{-i\ell z}-e^\ell}\,e^{-izt}, \sqrt{\frac{e^\ell+1}{2(e^\ell-1)}}[\delta(t-\ell)-\delta(t)]\right)\\
&= 1+\frac{e^\ell+1}{e^\ell-1}\left(\frac{e^\ell-1}{e^{-i\ell z}-e^\ell}\,e^{-izt}, \delta(t-\ell)-\delta(t)\right)\\
&= 1-\frac{e^\ell+1}{e^\ell-1}\left(\frac{e^\ell-1}{e^{-i\ell z}-e^\ell}-\frac{(e^\ell-1)e^{-iz\ell}}{e^{-i\ell z}-e^\ell}\right)\\
&= 1+(e^\ell+1)\left(\frac{1-e^{-iz\ell}}{e^{-i\ell z}-e^\ell}\right)\\
&= \frac{e^\ell e^{-i\ell z}-1}{e^\ell-e^{-i\ell z}}.
\end{aligned}$$

This confirms the result of Corollary 8 and formula (55) by showing that $W_{\Theta_0}(z) = -1/s(z)$. The corresponding impedance function is

$$V_{\Theta_0}(z) = i\frac{e^\ell+1}{e^\ell-1}\cdot\frac{e^{-i\ell z}-1}{e^{-i\ell z}+1}.$$

A quick inspection confirms that $V_{\Theta_0}(i) = i$ and hence $V_{\Theta_0}(z) \in \mathfrak{M}$.

Remark. We can use Examples 1 and 2 to illustrate Lemma 10 and Theorem 12. As one can easily tell that the impedance function $V_{\Theta_0}(z)$ from Example 2 above and the impedance function $V_\Theta(z)$ from Example 1 are related via (64) with $\kappa = e^{-\ell}$, that is,

$$V_\Theta(z) = \frac{1-e^{-\ell}}{1+e^{-\ell}}\,V_{\Theta_0}(z).$$

Let Θ be the L-system of the form (97) described in Example 1 with the transfer function $W_\Theta(z)$ given by (101). It was shown in [3, Theorem 8.3.1] that if one takes a function $W(z) = -W_\Theta(z)$, then $W(z)$ can be realized as a transfer function of another L-system Θ_1 that shares the same main operator T with Θ and in this case

$$V_{\Theta_1}(z) = -1/V_\Theta(z) = i\frac{e^{-i\ell z}+1}{e^{-i\ell z}-1}.$$

Clearly, $V_{\Theta_1}(z)$ and $V_{\Theta_0}(z)$ are not related via (64) even though Θ_1 has the same operator T with the same parameter $\kappa = e^{-\ell}$ as in Θ. The reason for that is the fact that the quasi-kernel of the real part of $\mathbb{A}_1$ of the L-system Θ_1 does not satisfy the conditions of Hypothesis 5 as indicated by Theorem 12.

Example 3. In this Example we are going to extend the construction of Example 2 to obtain a family of L-systems $\Theta_0(\beta)$ described in (73). Let $\dot A$ be defined by formula (84) but the operator A be an arbitrary self-adjoint extension of $\dot A$. It is known then [1] that all such operators A are described with the help of a unimodular parameter μ as follows

$$Ax = i\frac{dx}{dt}, \tag{109}$$

$$\mathrm{Dom}(A) = \left\{x(t)\,\Big|\, x(t)\in \mathrm{Dom}(\dot A^*),\ \mu x(\ell)+x(0)=0,\ |\mu|=1\right\}.$$

In order to establish the connection between the boundary value μ in (109) and the von Neumann parameter U in (4) we follow the steps similar to Example 1 to guarantee that $g_+ + Ug_- \in \mathrm{Dom}(A)$, where $g_\pm$ are given by (85). Quick set of calculations yields

$$U = -\frac{1+\mu e^{\ell}}{\mu+e^{\ell}}. \tag{110}$$

For this value of U we set the value of β so that $U = e^{2i\beta}$, where $\beta\in[0,\pi)$ and thus establish the link between the parameters μ and β that will be used to construct the family $\Theta_0(\beta)$. In particular, we note that $\beta = 0$ if and only if $\mu = -1$.

Once again, having $\dot A^*$ defined by (94) and $\mathcal{H}_+ = \mathrm{Dom}(\dot A^*) = W_2^1$ a space with scalar product (95), consider the following operators

$$\begin{aligned}\mathbb{A}_0(\beta)x &= i\frac{dx}{dt} + i\frac{\bar\mu}{\bar\mu+e^{-\ell}}(x(0)-e^{-\ell}x(\ell))\left[\mu\delta(t-\ell)+\delta(t)\right],\\ \mathbb{A}_0^*(\beta)x &= i\frac{dx}{dt} + i\frac{1}{\mu+e^{-\ell}}(e^{-\ell}x(0)-x(\ell))\left[\mu\delta(t-\ell)+\delta(t)\right],\end{aligned} \tag{111}$$

where $x(t)\in W_2^1$. It is immediate that $\mathbb{A}\supset T_0\supset \dot A$, $\mathbb{A}^*\supset T_0^*\supset \dot A$, where T_0 and T_0^* are given by (102) and (104). Also, as one can easily see, when $\beta=0$ and consequently $\mu=-1$, the operators $\mathbb{A}_0(0)$ and $\mathbb{A}_0^*(0)$ in (111) match the corresponding pair $\mathbb{A}_0$ and $\mathbb{A}_0^*$ in (105). By performing direct calculations we obtain

$$\mathrm{Re}\,\mathbb{A}_0(\beta)x = i\frac{dx}{dt} + \frac{i}{2}(\nu x(\ell)+x(0))\left[\mu\delta(t-\ell)+\delta(t)\right],$$

where

$$\nu = \frac{2\mu e^{-\ell}+e^{-2\ell}+1}{\mu+2e^{-\ell}+\mu e^{-2\ell}} \tag{112}$$

and $|\nu|=1$. Consequently, $\mathrm{Re}\,\mathbb{A}_0$ has its quasi-kernel

$$\hat A_0(\beta) = i\frac{dx}{dt},\quad \mathrm{Dom}(A) = \left\{x(t)\,\Big|\, x(t)\in\mathrm{Dom}(\dot A^*),\ \nu x(\ell)+x(0)=0\right\}. \tag{113}$$

Moreover,

$$\mathrm{Im}\,\mathbb{A}_0(\beta)x = \left(\frac{1}{2}\right)\left(\frac{1-e^{-2\ell}}{|\mu+e^{-2\ell}|}\right)(\bar\mu x(\ell)+x(0))\left[\mu\delta(t-\ell)+\delta(t)\right].$$

Therefore,

$$\begin{aligned}\mathrm{Im}\,\mathbb{A}_0(\beta) &= \left(\cdot,\frac{\sqrt{1-e^{-2\ell}}}{\sqrt{2}|\mu+e^{-2\ell}|}\left[\mu\delta(t-\ell)+\delta(t)\right]\right)\frac{\sqrt{1-e^{-2\ell}}}{\sqrt{2}|\mu+e^{-2\ell}|}\left[\mu\delta(t-\ell)+\delta(t)\right]\\ &= (\cdot,\chi_0(\beta))\chi_0(\beta),\end{aligned}$$

where $\chi_0(\beta) = \sqrt{\frac{e^\ell+1}{2(e^\ell-1)}}\,[\delta(t-\ell)-\delta(t)]$. Now we can compose our one-parametric L-system family

$$\Theta_0(\beta) = \begin{pmatrix} \mathbb{A}_0(\beta) & K_0(\beta) & 1 \\ W_2^1 \subset L^2_{[0,l]} \subset (W_2^1)_- & & \mathbb{C} \end{pmatrix},$$

where $K_0(\beta)c = c\cdot\chi_0(\beta)$, $(c\in\mathbb{C})$, $K_0^*(\beta)x = (x,\chi_0(\beta))$ and $x(t)\in W_2^1$. Using techniques of Example 2 one finds the transfer function of $\Theta_0(\beta)$ to be

$$W_{\Theta_0(\beta)}(z) = 1 - 2i(\hat{R}_z(T_0)\chi_0(\beta),\chi_0(\beta)) = \left(\frac{e^\ell+\mu}{\mu e^\ell+1}\right)\frac{e^\ell e^{-i\ell z}-1}{e^\ell - e^{-i\ell z}}.$$

The corresponding impedance function is again found via (8)

$$V_{\Theta_0(\beta)}(z) = i\,\frac{(\bar\mu e^{-i\ell z}-1)(e^{2\ell}+1)+2e^\ell e^{-i\ell z}-2\bar\mu e^\ell}{(\bar\mu e^{-i\ell z}+1)(e^{2\ell}-1)}.$$

A quick inspection confirms that $V_{\Theta_0(\beta)}(i) = i$ and hence $V_{\Theta_0(\beta)}(z)$ belongs to the Donoghue class $\mathfrak{M}$ for all $\beta\in[0,\pi)$ (equivalently $|\mu|=1$). Also, one can see that if $\beta=0$ and consequently $\mu=-1$ the conditions of Hypothesis 5 are satisfied and the L-system $\Theta_0(0)$ coincides with the L-system Θ_0 of Example 2 and so do its transfer and impedance functions.

Example 4. In this Example we will generalize the results obtained in Examples 1 and 2. Once again, let $\dot A$ and A be defined by formulas (84) and (86), respectively and let $s(z)$ be the Livšic characteristic function $s(z)$ for the pair $(\dot A, A)$ given by (87). We introduce a one-parametric family of operators

$$(114)\quad T_\rho x = i\frac{dx}{dt},\quad \operatorname{Dom}(T_\rho) = \left\{x(t)\,\Big|\,x(t)-\text{abs. cont.}, x'(t)\in L^2_{[0,\ell]},\ x(\ell)=\rho x(0)\right\}.$$

We are going to select the values of boundary parameter ρ in a way that will make T_ρ compliant with Hypothesis 5. By performing the direct check we conclude that $\operatorname{Im}(T_\rho f,f)\ge 0$ for $f\in\operatorname{Dom}(T_\rho)$ if $|\rho|>1$. This will guarantee that T_ρ is a dissipative extension of $\dot A$ parameterized by a von Neumann parameter κ. For further convenience we assume that $\rho\in\mathbb{R}$. To find the connection between κ and ρ we use (85) with (30) again to obtain

$$(115)\qquad x(t) = Ce^t - \kappa\, Ce^\ell e^{-t}\in\operatorname{Dom}(T),\quad x(\ell)=\rho x(0).$$

Solving (115) in two ways yields

$$(116)\qquad \kappa = \frac{\rho - e^\ell}{\rho e^\ell - 1}\quad\text{and}\quad \rho = \frac{\kappa - e^\ell}{\kappa e^\ell - 1}.$$

Using the first of relations (116) to find which values of ρ provide us with $0\le\kappa<1$ we obtain

$$(117)\qquad \rho\in(-\infty,-1)\cup[e^\ell,+\infty).$$

Now assuming (117) we can acknowledge that the triplet of operators $(\dot A, T_\rho, A)$ satisfy the conditions of Hypothesis 5. Following Examples 1 and 2, we are going to use the triplet $(\dot A, T_\rho, A)$ in the construction of an L-system Θ_ρ. By the direct check we have

$$(118)\quad T^*_\rho x = i\frac{dx}{dt},\quad \operatorname{Dom}(T_\rho) = \left\{x(t)\,\Big|\,x(t)-\text{abs. cont.}, x'(t)\in L^2_{[0,\ell]},\ \rho x(\ell)=x(0)\right\}.$$

Once again, we have $\dot A^*$ defined by (94) and $\mathcal{H}_+ = \mathrm{Dom}(\dot A^*) = W_2^1$ is a space with scalar product (95). Consider the operators

$$\begin{aligned} \mathbb{A}_\rho x &= i\frac{dx}{dt} + i\frac{x(\ell)-\rho x(0)}{\rho-1}\left[\delta(t-\ell)-\delta(t)\right], \\ \mathbb{A}_\rho^* x &= i\frac{dx}{dt} + i\frac{x(0)-\rho x(\ell)}{\rho-1}\left[\delta(t-\ell)-\delta(t)\right], \end{aligned} \tag{119}$$

where $x(t) \in W_2^1$. One easily checks that since $\mathrm{Im}\,\rho = 0$, then $\mathbb{A}_\rho^*$ is the adjoint to $\mathbb{A}_\rho$ operator. Evidently, that $\mathbb{A} \supset T_\rho \supset \dot A$, $\mathbb{A}^* \supset T_\rho^* \supset \dot A$, and

$$\mathrm{Re}\,\mathbb{A}_\rho x = i\frac{dx}{dt} - \frac{i}{2}(x(0)+x(\ell))\left[\delta(t-\ell)-\delta(t)\right].$$

Thus $\mathrm{Re}\,\mathbb{A}_\rho$ has its quasi-kernel equal to A defined in (86). Similarly,

$$\mathrm{Im}\,\mathbb{A}_\rho x = \left(\frac{1}{2}\right)\frac{\rho+1}{\rho-1}(x(\ell)-x(0))\left[\delta(t-\ell)-\delta(t)\right].$$

Therefore,

$$\begin{aligned} \mathrm{Im}\,\mathbb{A}_\rho &= \left(\cdot, \sqrt{\frac{\rho+1}{2(\rho-1)}}\left[\delta(t-\ell)-\delta(t)\right]\right)\sqrt{\frac{\rho+1}{2(\rho-1)}}\left[\delta(t-\ell)-\delta(t)\right] \\ &= (\cdot,\chi_\rho)\chi_\rho, \end{aligned}$$

where $\chi_\rho = \sqrt{\frac{\rho+1}{2(\rho-1)}}\left[\delta(t-\ell)-\delta(t)\right]$. Now we can build

$$\Theta_\rho = \begin{pmatrix} \mathbb{A}_\rho & K_\rho & 1 \\ W_2^1 \subset L^2_{[0,l]} \subset (W_2^1)_- & & \mathbb{C} \end{pmatrix},$$

which is a minimal L-system with $K_\rho c = c\cdot\chi_\rho$, $(c \in \mathbb{C})$, $K_\rho^* x = (x,\chi_\rho)$ and $x(t) \in W_2^1$. Evaluating the transfer function $W_{\Theta_\rho}(z)$ resembles the steps performed in Example 2. We have

$$\begin{aligned} R_z(T_\rho) &= (T_\rho - zI)^{-1} f \\ &= -ie^{-izt}\left(\int_0^t f(s)e^{izs}\,ds + \frac{e^{-i\ell z}}{\rho - e^{-i\ell z}}\int_0^l f(s)e^{izs}\,ds\right). \end{aligned} \tag{120}$$

This leads to

$$\hat R_z(T_\rho)\chi_\rho = \hat R_z(T_\rho)\sqrt{\frac{\rho+1}{2(\rho-1)}}\left[\delta(t-\ell)-\delta(t)\right] = i\sqrt{\frac{\rho+1}{2(\rho-1)}}\left(\frac{1-\rho}{e^{-i\ell z}-\rho}\right)e^{-izt}$$

and eventually to

$$W_{\Theta_\rho}(z) = 1 - 2i(\hat R_z(T_\rho)\chi_\rho,\chi_\rho) = \frac{\rho e^{-i\ell z}-1}{\rho - e^{-i\ell z}}.$$

Evaluating the impedance function $V_{\Theta_\rho}(z)$ results in

$$V_{\Theta_\rho}(z) = i\frac{\rho+1}{\rho-1}\cdot\frac{1-e^{-i\ell z}}{1+e^{-i\ell z}}.$$

Using direct calculations and (116) gives us

$$\frac{\rho+1}{\rho-1} = \frac{1-\kappa}{1+\kappa}\cdot\frac{e^\ell+1}{e^\ell-1},$$

and thus

$$V_{\Theta_\rho}(z) = \frac{1-\kappa}{1+\kappa}\,V_{\Theta_0}(z),$$

which confirms the result of Lemma 10.

Appendix A. Rigged Hilbert spaces

In this Appendix we are going to explain the construction and basic geometry of rigged Hilbert spaces.

We start with a Hilbert space $\mathcal{H}$ with inner product (x,y) and norm $\|\cdot\|$. Let $\mathcal{H}_+$ be a dense in $\mathcal{H}$ linear set that is a Hilbert space itself with respect to another inner product $(x,y)_+$ generating the norm $\|\cdot\|_+$. We assume that $\|x\| \le \|x\|_+$, $(x \in \mathcal{H}_+)$, i.e., the norm $\|\cdot\|_+$ generates a stronger than $\|\cdot\|$ topology in $\mathcal{H}_+$. The space $\mathcal{H}_+$ is called the *space with the positive norm.*

Now let $\mathcal{H}_-$ be a space dual to $\mathcal{H}_+$. It means that $\mathcal{H}_-$ is a space of linear functionals defined on $\mathcal{H}_+$ and continuous with respect to $\|\cdot\|_+$. By the $\|\cdot\|_-$ we denote the norm in $\mathcal{H}_-$ that has a form

$$\|h\|_- = \sup_{u\in\mathcal{H}_+} \frac{|(h,u)|}{\|u\|_+}, \quad h \in \mathcal{H}.$$

The value of a functional $f \in \mathcal{H}_-$ on a vector $u \in \mathcal{H}_+$ is denoted by (u,f). The space $\mathcal{H}_-$ is called the *space with the negative norm.*

Consider an embedding operator $\sigma : \mathcal{H}_+ \mapsto \mathcal{H}$ that embeds $\mathcal{H}_+$ into $\mathcal{H}$. Since $\|\sigma f\| \le \|f\|_+$ for all $f \in \mathcal{H}_+$, then $\sigma \in [\mathcal{H}_+,\mathcal{H}]$. The adjoint operator σ^* maps $\mathcal{H}$ into $\mathcal{H}_-$ and satisfies the condition $\|\sigma^* f\|_- \le \|f\|$ for all $f \in \mathcal{H}$. Since σ is a monomorphism with a $(\cdot)$-dense range, then σ^* is a monomorphism with $(-)$-dense range. By identifying $\sigma^* f$ with f $(f \in \mathcal{H})$ we can consider $\mathcal{H}$ embedded in $\mathcal{H}_-$ as a $(-)$-dense set and $\|f\|_- \le \|f\|$. Also, the relation

$$(\sigma f, h) = (f, \sigma^* h), \quad f \in \mathcal{H}_+, \quad h \in \mathcal{H},$$

implies that the value of the functional $\sigma^* h \in \mathcal{H}$ calculated at a vector $f \in \mathcal{H}_+$ as $(f,\sigma^* h)$ corresponds to the value (f,h) in the space $\mathcal{H}$.

It follows from the Riesz representation theorem that there exists an isometric operator $\mathcal{R}$ which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f,g) = (f,\mathcal{R}g)_+$ $(\forall f \in \mathcal{H}_+, g \in \mathcal{H}_-)$ and $\|\mathcal{R}g\|_+ = \|g\|_-$. Now we can turn $\mathcal{H}_-$ into a Hilbert space by introducing $(f,g)_- = (\mathcal{R}f,\mathcal{R}g)_+$. Thus,

$$\begin{aligned} (f,g)_- &= (f,\mathcal{R}g) = (\mathcal{R}f,g) = (\mathcal{R}f,\mathcal{R}g)_+, \quad (f,g \in \mathcal{H}_-), \\ (u,v)_+ &= (u,\mathcal{R}^{-1}v) = (\mathcal{R}^{-1}u,v) = (\mathcal{R}^{-1}u,\mathcal{R}^{-1}v)_-, \quad (u,v \in \mathcal{H}_+). \end{aligned} \tag{121}$$

The operator $\mathcal{R}$ (or $\mathcal{R}^{-1}$) will be called the *Riesz-Berezansky operator.* We note that $\mathcal{H}_+$ is also dual to $\mathcal{H}_-$. Applying the above reasoning, we define a triplet $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ to be called the *rigged Hilbert space* [6], [7].

Now we explain how to construct a rigged Hilbert space using a symmetric operator. Let $\dot A$ be a closed symmetric operator whose domain $\mathrm{Dom}(\dot A)$ is not assumed to be dense in $\mathcal{H}$. Setting $\overline{\mathrm{Dom}(\dot A)} = \mathcal{H}_0$, we can consider $\dot A$ as a densely defined operator from $\mathcal{H}_0$ into $\mathcal{H}$. Clearly, $\mathrm{Dom}(\dot A^*)$ is dense in $\mathcal{H}$ and $\mathrm{Ran}(\dot A^*) \subset \mathcal{H}_0$. We introduce a new Hilbert space $\mathcal{H}_+ = \mathrm{Dom}(\dot A^*)$ with inner product

$$(f,g)_+ = (f,g) + (\dot A^* f, \dot A^* g), \quad (f,g \in \mathcal{H}_+), \tag{122}$$

and then construct the operator generated rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$.

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