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ON THE ACCELERANTS OF NON-SELF-ADJOINT DIRAC OPERATORS

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Dedicated to Professors D. Z. Arov and V. M. Adamyan, with deep appreciation of their contribution to modern functional analysis

Abstract. We prove that there is a homeomorphism between the space of accelerants and the space of potentials of non-self-adjoint Dirac operators on a finite interval.

1. Introduction and main results

The theory of accelerants was founded by M. G. Krein in the middle of the past century. The origins of this theory go back to Krein's short papers [5, 6, 7, 8], where he showed that the resolvent kernels of some integral equations generate solutions of some 2nd order differential equations and systems of 1st order differential equations. Thereby, Krein established a fundamental connection between a special class of functions called the accelerants and Sturm–Liouville and Dirac operators. A detailed presentation of some of these his results can be found in the book [3]. Krein's ideas in the theory of accelerants were continued and further developed in many papers.

Accelerants play a particular role in the theory of continuous analogues of polynomials orthogonal on the unit circle (see [5]). In this context, it is worth mentioning, e.g., remarkable lecture notes [2] by S. A. Denisov, where the detailed exposition of many aspects of the theory can be found and some new results are obtained.

Let $\mathcal{M}_r$ denote the Banach algebra of all $r \times r$ matrices with complex entries which we identify with the Banach algebra of linear operators in $\mathbb{C}^r$ endowed with the standard norm.

Definition 1.1. *We say that a function $h \in L_1((-1,1),\mathcal{M}_r)$ is an accelerant if for each $\alpha \in (0,1]$ the integral equation*

$$f(x) + \int_0^\alpha h(x-t)f(t)\,\mathrm{d}t = 0, \quad x \in (0,1), \tag{1.1}$$

has only zero solution in $L_2((0,1),\mathbb{C}^r)$.

Note that Definition 1.1 differs from the one originally introduced by Krein in that we do not require any of the conditions $h(x) = h(-x)$ or $h(x) = h(-x)^*$, $x \in (-1,1)$. Note also that if h is an accelerant, then such is also $h^\sharp$, where $h^\sharp(x) := h(-x)$, $x \in (-1,1)$ (see Remark 2.5 below).

We denote by $\mathfrak{H}_{p,r}$ the set of accelerants belonging to $L_p((-1,1),\mathcal{M}_r)$, $p \in [1,\infty)$, and endow $\mathfrak{H}_{p,r}$ with the metric of the latter. It is known (see Proposition 3.1 below)

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that for an arbitrary accelerant $h \in \mathfrak{H}_{p,r}$, the integral equation

$$r(x,t) + h(x-t) + \int_0^x r(x,s)h(s-t)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+, \tag{1.2}$$

where $\Omega_+ := \{(x,t) \mid 0 < t < x < 1\}$, has a unique solution $r_h \in L_1(\Omega_+, \mathcal{M}_r)$. If one sets $r_h(x,t) = 0$ for $(x,t) \in [0,1]^2 \setminus \Omega_+$, then $r_h \in G^+_{p,r}$ (see definition in Sect. 2.2 below). Equation (1.2) is called the *Krein equation.*

The connection between the accelerants and Dirac systems of differential equations was established by Krein in [8]. In the present paper, it is convenient to explain this connection in equivalent form using the solution of equation (1.2).

So, let $h \in \mathfrak{H}_{p,r}$. Consider the $r \times r$ matrix-valued functions

$$\varphi_1(x,\lambda) := \mathrm{e}^{\mathrm{i}\lambda x}\left(I + \int_0^x \mathrm{e}^{-2\mathrm{i}\lambda s} r_h(x, x-s)\,\mathrm{d}s\right),$$

$$\varphi_2(x,\lambda) := \mathrm{e}^{-\mathrm{i}\lambda x}\left(I + \int_0^x \mathrm{e}^{2\mathrm{i}\lambda s} r_{h^\sharp}(x, x-s)\,\mathrm{d}s\right),$$

where $x \in (0,1)$, $\lambda \in \mathbb{C}$, I is the $r \times r$ identity matrix and $r_{h^\sharp}$ is the solution of (1.2) with $h^\sharp$ instead of h. Then the $2r \times r$ matrix-valued function $\varphi := (\varphi_1, \varphi_2)^\top$ is a solution of the Cauchy problem

$$J\frac{\mathrm{d}}{\mathrm{d}x}\varphi + Q\varphi = \lambda\varphi, \quad \varphi(0,\lambda) = \begin{pmatrix} I \\ I \end{pmatrix},$$

with

$$J := \frac{1}{\mathrm{i}}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad Q(x) = [\Theta(h)](x) := \begin{pmatrix} 0 & \mathrm{i}r_h(x,0) \\ -\mathrm{i}r_{h^\sharp}(x,0) & 0 \end{pmatrix}, \quad x \in (0,1).$$

Since both functions r_h and $r_{h^\sharp}$ belong to $G^+_{p,r}$, the function $Q = \Theta(h)$ belongs to the class

$$\mathfrak{Q}_p := \{Q \in L_p((0,1), \mathcal{M}_{2r}) \mid Q(x)J = -JQ(x) \text{ a.e. on } (0,1)\}.$$

The mapping $\Theta : \mathfrak{H}_{p,r} \to \mathfrak{Q}_p$ will be called the *Krein mapping.*

The main result of this paper is the following theorem:

Theorem 1.1. *For an arbitrary $p \in [1,\infty)$, the Krein mapping is a homeomorphism between the metric spaces $\mathfrak{H}_{p,r}$ and $\mathfrak{Q}_p$. Moreover, both the Krein mapping and its inverse are locally Lipschitz.*

In his paper [8], Krein treated symmetric accelerants, i.e. the ones satisfying the condition $h(-t) = h(t) = h(t)^\top$, $t \in (-1,1)$, where $\top$ designates the transposition of matrices. Namely, he proved that there is a one-to-one correspondence between the set of all continuous symmetric accelerants and the set of all continuous symmetric potentials of the Krein systems which are closely related to Dirac operators.

The analogue of Krein's theorem was established for self-adjoint Dirac operators with continuous potentials in [1]. Therein, it was shown that there is a one-to-one correspondence between the potentials of such operators and Hermitian accelerants (i.e. such that $h(-t) = h(t)^*$) that are continuous outside the origin.

The analogous theorem for Krein systems on semi-axis was proved in [2, Theorem 5.3].

In [13, Theorem 1.9], it was proved that the Krein mapping is a homeomorphism between the space of all even accelerants h in $\mathfrak{H}_{2,r}$ and $L_2((0,1), \mathcal{M}_r)$. In [14, Theorem 1.5], the same result was established about the space of Hermitian accelerants h in $\mathfrak{H}_{p,r}$ and $L_p((0,1), \mathcal{M}_r)$, $p \in [1,\infty)$. It thus follows that the potentials of all *self-adjoint* Dirac operators on $[0,1]$ correspond to *Hermitian* accelerants $h \in \mathfrak{H}_{p,r}$. In the present paper, we actually abandon the condition of self-adjointness and show that the potentials of all (not necessarily self-adjoint) Dirac operators on $[0,1]$ correspond to (not necessarily

Hermitian) accelerants $h \in \mathfrak{H}_{p,r}$. Since $r_{h^\sharp}(\cdot,0) = [r_h(\cdot,0)]^*$ for all Hermitian accelerants, the results of the present paper correlate well with the results of [14].

2. Some facts from the theory of factorizations

2.1. **Some general facts.** Let $\boldsymbol{H}$ be a separable infinite dimensional Hilbert space and $\mathcal{B} := \mathcal{B}(\boldsymbol{H})$ be the Banach algebra of all everywhere defined bounded linear operators in $\boldsymbol{H}$. We write $\mathcal{B}_\infty$ and $\mathcal{B}_0$ for the Banach algebra of all compact operators and for the linear space of all finite dimensional operators from $\mathcal{B}$, respectively.

We say that a set $\mathfrak{P} \subset \mathcal{B}$ of orthoprojectors is a *chain* if for any $P_1, P_2 \in \mathfrak{P}$ it holds either $P_1 < P_2$ or $P_2 < P_1$. A chain is said to be *closed* if it is a closed subset of $\mathcal{B}$ in the strong operator topology. A closed chain is said to be *continuous* if for each pair $P_1, P_2 \in \mathfrak{P}$ such that $P_1 < P_2$ there is $P \in \mathfrak{P}$ such that $P_1 < P < P_2$. We say that a closed chain $\mathfrak{P}$ is *complete* if it is continuous and $0, I \in \mathfrak{P}$, where I is the identity operator in $\boldsymbol{H}$.

Let $\mathfrak{P}$ be a complete chain in $\boldsymbol{H}$. Set

$$\mathcal{B}_\infty^+ := \{B \in \mathcal{B}_\infty \mid \forall P \in \mathfrak{P} \quad (I-P)BP = 0\},$$
$$\mathcal{B}_\infty^- := \{B \in \mathcal{B}_\infty \mid \forall P \in \mathfrak{P} \quad PB(I-P) = 0\}.$$

It can be easily verified that $\mathcal{B}_\infty^+$ and $\mathcal{B}_\infty^-$ are closed Banach subalgebras in $\mathcal{B}_\infty$ and that $\mathcal{B}_\infty^+ \cap \mathcal{B}_\infty^- = \{0\}$. Furthermore, the operators from $\mathcal{B}_\infty^\pm$ are Volterra ones (see [3, Ch. I]).

Denote by $\mathcal{P}^+$ ($\mathcal{P}^-$, resp.) the projector in $\widetilde{\mathcal{B}}_\infty := \mathcal{B}_\infty^+ \dot{+} \mathcal{B}_\infty^-$ onto $\mathcal{B}_\infty^+$ ($\mathcal{B}_\infty^-$, resp.) parallel to $\mathcal{B}_\infty^-$ ($\mathcal{B}_\infty^+$, resp.). The projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ are called the *transformators of triangular truncations* (this term was suggested by I. C. Gohberg and M. G. Krein for the operators acting from one Banach algebra to another, see [3, Ch. II]).

Denote by Σ the set of all Banach algebras $\mathfrak{S} \subset \mathcal{B}_\infty$ in which the transformators $\mathcal{P}^+$ and $\mathcal{P}^-$ are continuous. For each $\mathfrak{S} \in \Sigma$ we set

$$\mathfrak{S}^\pm := \mathcal{P}^\pm \mathfrak{S}. \tag{2.1}$$

It then follows that both $\mathfrak{S}^+$ and $\mathfrak{S}^-$ are closed subalgebras in $\mathfrak{S}$ consisting of Volterra operators and that $\mathfrak{S} = \mathfrak{S}^+ \dot{+} \mathfrak{S}^-$.

Let $\mathfrak{S} \in \Sigma$. We say that the operator $I + Q$ with $Q \in \mathcal{B}_\infty$ ($Q \in \mathfrak{S}$, resp.), *admits a factorization* in $\mathcal{B}_\infty$ (in $\mathfrak{S}$, resp.) if

$$I + Q = (I + K_-)^{-1}(I + K_+)^{-1} \tag{2.2}$$

with some $K_\pm \in \mathcal{B}_\infty^\pm$ ($K_\pm \in \mathfrak{S}^\pm$, resp.).

Let Φ ($\Phi_{\mathfrak{S}}$, resp.) denote the set of all operators $Q \in \mathcal{B}_\infty$ ($Q \in \mathfrak{S}$) for which $I + Q$ admits a factorization in $\mathcal{B}_\infty$ (in $\mathfrak{S}$). It is known (see [3, Ch. IV]) that Φ is contained in the set

$$\Psi := \{Q \in \mathcal{B}_\infty \mid \forall P \in \mathfrak{P} \quad \ker(I + PQP) = \{0\}\}$$

and that for each $Q \in \Phi$ the operators $K_\pm = K_\pm(Q)$ in (2.2) are determined uniquely. The following theorem is proved in [10]:

Theorem 2.1. *Let $\mathfrak{S} \in \Sigma$. Then the set $\Phi_{\mathfrak{S}}$ is open in $\mathfrak{S}$. Moreover, the mappings $\Phi_{\mathfrak{S}} \ni Q \mapsto K_\pm(Q) \in \mathfrak{S}$ are locally Lipschitz.*

Remark 2.1. A mapping φ acting from an open set $\mathcal{O}$ in a Banach space X to a Banach space Y is said to be *locally Lipschitz* if for each $x_0 \in \mathcal{O}$ there are a neighborhood $\mathcal{U} \subset \mathcal{O}$ of x_0 and $c > 0$ such that $\|\varphi(x_1) - \varphi(x_2)\|_Y \le c\|x_1 - x_2\|_X$ for all $x_1, x_2 \in \mathcal{U}$.

Set

$$\Sigma_f := \{\mathfrak{S} \in \Sigma \mid \Phi_{\mathfrak{S}} = \Psi \cap \mathfrak{S}\}, \quad \Sigma_f^0 := \{\mathfrak{S} \in \Sigma_f \mid \mathfrak{S} \cap \mathcal{B}_0 \text{ is dense everywhere in } \mathcal{B}_\infty\}.$$

Note that as follows from the well known results in the theory of factorizations (see [3]) the Neumann–Schatten ideals $\mathcal{B}_p$, $1 < p < \infty$, belong to the class Σ_f^0.

The next two theorems follow from the results of [10, 11]:

Theorem 2.2. *Let $\mathfrak{S} \in \Sigma$ and $\mathfrak{S}_1 \in \Sigma_f^0$ be a two-sided ideal in $\mathfrak{S}$. If $\mathfrak{S}_1$ is dense everywhere in $\mathfrak{S}$, then $\mathfrak{S} \in \Sigma_f^0$.*

Theorem 2.3. *Let $Q \in \Phi$ and $Q_1 \in \mathcal{B}_0$. Then the set $\{\lambda \in \mathbb{C} \mid (Q + \lambda Q_1) \in \Psi\}$ is open and dense everywhere in $\mathbb{C}$.*

Corollary 2.1. *Let $\mathfrak{S} \in \Sigma_f^0$. Then the set $\Phi_{\mathfrak{S}}$ is dense everywhere in $\mathfrak{S}$.*

2.2. **Algebras $\mathcal{G}_{p,n}$.** For an arbitrary $p \in [1, \infty)$ and $n \in \mathbb{N}$, we denote by $G_{p,n}$ the set of all measurable functions $K : [0,1]^2 \to \mathcal{M}_n$ such that for all $x, t \in [0,1]$ the functions $K(x, \cdot)$ and $K(\cdot, t)$ belong to $L_p((0,1), \mathcal{M}_n)$ and, moreover, the mappings

$$[0,1] \ni x \mapsto K(x,\cdot) \in L_p((0,1),\mathcal{M}_n), \quad [0,1] \ni t \mapsto K(\cdot,t) \in L_p((0,1),\mathcal{M}_n)$$

are continuous. The set $G_{p,n}$ becomes a Banach space upon introducing the norm

$$\|K\|_{G_{p,n}} = \max\Big\{ \max_{x\in[0,1]} \|K(x,\cdot)\|_{L_p},\ \max_{t\in[0,1]} \|K(\cdot,t)\|_{L_p} \Big\}. \tag{2.3}$$

We denote by $\mathcal{G}_{p,n}$ the set of all integral operators in $\boldsymbol{H} := L_2((0,1), \mathbb{C}^n)$ with kernels $K \in G_{p,n}$ and endow $\mathcal{G}_{p,n}$ with the norm

$$\|\mathcal{K}\|_{\mathcal{G}_{p,n}} := \|K\|_{G_{p,n}}, \quad \mathcal{K} \in \mathcal{G}_{p,n}.$$

Note that there are continuous embeddings $\mathcal{G}_{p,n} \subset \mathcal{G}_{1,n} \subset \mathcal{B}(\boldsymbol{H})$ and that for each $\mathcal{K} \in \mathcal{G}_{p,n}$ and $\mathcal{R} \in \mathcal{G}_{1,n}$ it holds

$$\|\mathcal{K}\|_{\mathcal{G}_{1,n}} \le \|\mathcal{K}\|_{\mathcal{G}_{p,n}}, \quad \|\mathcal{R}\|_{\mathcal{B}} \le \|\mathcal{R}\|_{\mathcal{G}_{1,n}}.$$

Furthermore, it can be verified that $\mathcal{G}_{p,n}$ is a Banach algebra.

We set

$$\Omega_+ := \{(x,t) \mid 0 < t < x < 1\}, \quad \Omega_- := \{(x,t) \mid 0 < x < t < 1\}$$

and write $G_{p,n}^{\pm}$ for the sets of all functions $K \in G_{p,n}$ such that $K(x,t) = 0$ a.e. in $\overline{\Omega}_{\mp}$. We denote by $\mathcal{G}_{p,n}^{\pm}$ the subalgebras in $\mathcal{G}_{p,n}$ consisting of all operators $\mathcal{K} \in \mathcal{G}_{p,n}$ with kernels $K \in G_{p,n}^{\pm}$. It is easy to verify that $\mathcal{G}_{p,n}^{\pm}$ are closed subalgebras in $\mathcal{G}_{p,n}$ and that $\mathcal{G}_{p,n} = \mathcal{G}_{p,n}^{+} \dotplus \mathcal{G}_{p,n}^{-}$.

We denote by $\mathcal{S}_n^{\pm}$ the operator algebras consisting of all operators $\mathcal{K} \in \mathcal{G}_{1,n}^{\pm}$ with kernels that are continuous in $\overline{\Omega}_{\pm}$. The algebras $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ become Banach algebras upon introducing the norms

$$\|\mathcal{K}\|_{\mathcal{S}_n^{\pm}} := \max_{(x,t)\in\overline{\Omega}_{\pm}} \|K(x,t)\|.$$

We set $\mathcal{S}_n := \mathcal{S}_n^+ \dotplus \mathcal{S}_n^-$ and endow $\mathcal{S}_n$ with the norm

$$\|\mathcal{K}\|_{\mathcal{S}_n} := \max\Big\{\|\mathcal{K}_+\|_{\mathcal{S}_n^+},\ \|\mathcal{K}_-\|_{\mathcal{S}_n^-}\Big\}, \quad \mathcal{K} = \mathcal{K}_+ + \mathcal{K}_-, \quad \mathcal{K}_{\pm} \in \mathcal{S}_n^{\pm}.$$

It is easy to verify that $\mathcal{S}_n$ is a Banach algebra. We then denote by $\mathcal{S}_{n,0}$ a subalgebra in $\mathcal{S}_n$ consisting of all operators $\mathcal{K} \in \mathcal{S}_n$ with kernels that are continuous on $[0,1]^2$.

Lemma 2.1. *$\mathcal{S}_n$ is a two sided ideal in $\mathcal{G}_{p,n}$. Furthermore, $\mathcal{S}_n$ and $\mathcal{S}_n^{\pm}$ are continuously and densely embedded into $\mathcal{G}_{p,n}$ and $\mathcal{G}_{p,n}^{\pm}$, respectively.*

Proof. A straightforward verification shows that for each $\mathcal{K}_0 \in \mathcal{S}_n$ and $\mathcal{K}_1 \in \mathcal{G}_{p,n}$, the products $\mathcal{K}_0\mathcal{K}_1$ and $\mathcal{K}_1\mathcal{K}_0$ belong to $\mathcal{S}_{n,0}$ and that

$$\|\mathcal{K}_0\mathcal{K}_1\|_{\mathcal{S}_n}, \|\mathcal{K}_1\mathcal{K}_0\|_{\mathcal{S}_n} \le \|\mathcal{K}_0\|_{\mathcal{S}_n}\|\mathcal{K}_1\|_{\mathcal{G}_{p,n}}. \tag{2.4}$$

Therefore, $\mathcal{S}_n$ is a two sided ideal in $\mathcal{G}_{p,n}$. Since

$$\|\mathcal{K}\|_{\mathcal{G}_{p,n}} \le \|\mathcal{K}\|_{\mathcal{S}_n}, \quad \mathcal{K} \in \mathcal{S}_n, \tag{2.5}$$

one also has that $\mathcal{S}_n$ and $\mathcal{S}_n^{\pm}$ are continuously embedded into $\mathcal{G}_{p,n}$ and $\mathcal{G}_{p,n}^{\pm}$, respectively.

It was proved in [11] that $\mathcal{S}_{1,0}$ is dense everywhere in $\mathcal{G}_{1,1}$. By a straightforward modification of that proof it can be shown that $\mathcal{S}_{n,0}$ is dense everywhere in $\mathcal{G}_{p,n}$. It then follows that $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ are dense everywhere in $\mathcal{G}_{p,n}^+$ and $\mathcal{G}_{p,n}^-$, respectively. □

In particular, it follows from Lemma 2.1 that $\mathcal{G}_{p,n} \subset \mathcal{B}_\infty(\boldsymbol{H})$.

Lemma 2.2. *Let $\mathcal{K} \in \mathcal{G}_{p,n}^+ \cup \mathcal{G}_{p,n}^-$ and $\rho(\mathcal{K})$ be the spectral radius of $\mathcal{K}$ (see* [15, Ch. 10]*). Then $\rho(\mathcal{K}) = 0$.*

Proof. Since the mapping $\mathcal{K} \mapsto \mathcal{K}^*$ maps $\mathcal{G}_{p,n}^-$ onto $\mathcal{G}_{p,n}^+$ isometrically, it suffices to prove that $\rho(\mathcal{K}) = 0$ for each $\mathcal{K} \in \mathcal{G}_{p,n}^+$.

For this purpose, note that for an arbitrary sequence $(\mathcal{K}_j)_{j=1}^m$ in $\mathcal{S}_n^+$ it holds

$$\|\mathcal{K}_1 \cdots \mathcal{K}_m\|_{\mathcal{S}_n} \le \frac{1}{m!}\prod_{j=1}^m \|\mathcal{K}_j\|_{\mathcal{S}_n}. \tag{2.6}$$

Let $\mathcal{K} \in \mathcal{G}_{p,n}^+$ and $\delta \in (0,1)$. In view of Lemma 2.1, the operator $\mathcal{K}$ can be written in the form $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1$ with some $\mathcal{K}_0 \in \mathcal{S}_n^+$ and $\mathcal{K}_1 \in \mathcal{G}_{p,n}^+$ such that $\|\mathcal{K}_1\|_{\mathcal{G}_{p,n}} \le \delta$. It then holds $\mathcal{K}^s = \sum\limits_{\sigma\in U_s} \mathcal{K}_{\sigma(1)}\cdots\mathcal{K}_{\sigma(s)}$, where the sum is taken over the set U_s of all functions $\sigma : \{1,\dots,s\} \to \{0,1\}$, and thus one has

$$\|\mathcal{K}^s\|_{\mathcal{G}_{p,n}} \le 2^s \max_\sigma \|\mathcal{K}_{\sigma(1)}\cdots\mathcal{K}_{\sigma(s)}\|_{\mathcal{G}_{p,n}}.$$

Let $\sigma \in U_n$ and $m := \operatorname{card}(\sigma^{-1}(0)) > 0$. It then follows from Lemma 2.1 and from the estimates (2.4) and (2.6) that the operator $\mathcal{L} = \mathcal{K}_{\sigma(1)}\cdots\mathcal{K}_{\sigma(s)}$ belongs to $\mathcal{S}_n^+$ and, furthermore,

$$\|\mathcal{L}\|_{\mathcal{S}_n} \le \frac{1}{m!}\|\mathcal{K}_1\|_{\mathcal{G}_{p,n}}^{s-m}\|\mathcal{K}_0\|_{\mathcal{S}_n}^m \le \frac{1}{m!}\delta^{s-m}\|\mathcal{K}_0\|_{\mathcal{S}_n}^m.$$

Taking into account (2.5), we then obtain that

$$\|\mathcal{L}\|_{\mathcal{G}_{p,n}} \le \delta^s \frac{(\delta^{-1}\|\mathcal{K}_0\|_{\mathcal{S}_n})^m}{m!} \le \delta^s \exp(\delta^{-1}\|\mathcal{K}_0\|_{\mathcal{S}_n}).$$

Evidently, the latter inequality holds true also for $m = 0$. Therefore, one has

$$\|\mathcal{K}^s\|_{\mathcal{G}_{p,n}} \le (2\delta)^s \exp(\delta^{-1}\|\mathcal{K}_0\|_{\mathcal{S}_n})$$

and thus the spectral radius $\rho(\mathcal{K})$ of the operator $\mathcal{K} \in \mathcal{G}_{p,n}$ does not exceed 2δ. Since δ was arbitrary, one then has that $\rho(\mathcal{K}) = 0$. □

It follows from Lemma 2.2 that the mapping $\mathcal{K} \mapsto \gamma(\mathcal{K}) := (I+\mathcal{K})^{-1} - I$ maps both $\mathcal{G}_{p,n}^+$ and $\mathcal{G}_{p,n}^-$ into themselves. Actually even more holds true:

Lemma 2.3. *The mappings $\mathcal{G}_{p,n}^+ \ni \mathcal{K} \mapsto \gamma(\mathcal{K}) \in \mathcal{G}_{p,n}^+$ and $\mathcal{G}_{p,n}^- \ni \mathcal{K} \mapsto \gamma(\mathcal{K}) \in \mathcal{G}_{p,n}^-$ are homeomorphic and locally Lipschitz.*

Lemma 2.3 follows from the next general result:

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra with the identity e and $\mathcal{A}_0$ be its closed subalgebra such that $\rho(a) = 0$ for each $a \in \mathcal{A}_0$. Then the mapping $\mathcal{A}_0 \ni a \mapsto \gamma(a) := [(e+a)^{-1} - e] \in \mathcal{A}_0$ is homeomorphic and locally Lipschitz.*

Proof. Since $(e+a_1)^{-1}-(e+a_2)^{-1}=(e+a_1)^{-1}(a_2-a_1)(e+a_2)^{-1}$, it follows that

$$\gamma(a_1)-\gamma(a_2)=(e+\gamma(a_1))(a_2-a_1)(e+\gamma(a_2)),\quad a_1,a_2\in\mathcal{A}. \tag{2.7}$$

Let $a\in\mathcal{A}_0$. It then follows from the assumptions of the lemma that there is $m\in\mathbb{N}$ such that $\|a^m\|_{\mathcal{A}}\le 1/4$. Set $C:=\sum_{k=0}^{m-1}\|a^k\|$. Since multiplication in $\mathcal{A}_0$ is continuous, it then follows that there is a neighborhood $\mathcal{U}\subset\mathcal{A}_0$ of a such that

$$\|b^m\|_{\mathcal{A}}\le 1/2,\quad \sum_{k=0}^{m-1}\|b^k\|_{\mathcal{A}}\le 2C,\quad b\in\mathcal{U}.$$

Since

$$\gamma(b)=\sum_{s=1}^{\infty}(-b)^s=\sum_{k=0}^{m-1}(-b)^k\sum_{s=1}^{\infty}(-b)^{ms},$$

it follows that $\|\gamma(b)\|_{\mathcal{A}}\le 2C$ for all $b\in\mathcal{U}$. Taking into account (2.7) we then obtain that

$$\|\gamma(a_1)-\gamma(a_2)\|_{\mathcal{A}}\le(1+2C)^2\|a_1-a_2\|_{\mathcal{A}},\quad a_1,a_2\in\mathcal{U}.$$

Therefore the mapping γ is locally Lipschitz. Since $\gamma(\gamma(a))\equiv a$, it also follows that γ is homeomorphic. □

2.3. **Factorization of operators in $\mathcal{G}_{p,n}$.** Let $\boldsymbol{H}:=L_2((0,1),\mathbb{C}^n)$. We consider the transformators $\mathcal{P}^{\pm}$ in $\mathcal{B}_\infty(\boldsymbol{H})$ generated by a complete chain of orthoprojectors $P_\alpha:\boldsymbol{H}\to\boldsymbol{H}$, $\alpha\in[0,1]$, given by the formula

$$P_\alpha f:=\chi_{[0,\alpha]}f,\quad f\in\boldsymbol{H},$$

where $\chi_{[0,\alpha]}$ is the characteristic function of the interval $[0,\alpha]$.

Lemma 2.4. *The transformators $\mathcal{P}^{\pm}$ are continuous in $\mathcal{G}_{p,n}$ and $\mathcal{G}^{\pm}_{p,n}=\mathcal{P}^{\mp}\mathcal{G}_{p,n}$.*

Remark 2.2. The change of sign in the above formula arises due to discrepancy between definition (2.1) and the definition of algebras $\mathcal{G}^{\pm}_{p,n}$ given at the beginning of Sect. 2.2. However, the authors prefer to accept this inconvenience in order to follow both the standard notations in [3, 10, 11] and the ones used in [12, 13, 14]. This causes also sign differences between formula (2.8) below and formula (2.2).

Proof of Lemma 2.4. In the scalar case $n=1$, continuity of $\mathcal{P}^{\pm}_1:=\mathcal{P}^{\pm}$ in $\mathcal{G}_{p,1}$ follows from the results of [11]. Note that $\mathcal{G}_{p,n}$ can be considered as a tensor product of the algebras $\mathcal{G}_{p,1}$ and $\mathcal{M}_n$ and that $\mathcal{P}^{\pm}$ can be considered as tensor products of the operators $\mathcal{P}^{\pm}_1$ and $I_{\mathcal{M}_n}$. Therefore, we obtain that the transformators $\mathcal{P}^{\pm}$ act continuously in $\mathcal{G}_{p,n}$. Verification of the equalities $\mathcal{G}^{\pm}_{p,n}=\mathcal{P}^{\mp}\mathcal{G}_{p,n}$ is straightforward. □

Remark 2.3. It follows from Lemma 2.4 that $\mathcal{G}_{p,n}$ belongs to the class Σ. The algebra $\mathcal{S}_n$ belongs to the class Σ_f (see [3, Ch. IV]). Since $\mathcal{S}_n\cap\mathcal{B}_0(\boldsymbol{H})$ is dense everywhere in $\mathcal{B}_\infty(\boldsymbol{H})$, it follows that $\mathcal{S}_n$ belongs to the class Σ^0_f. Therefore, taking into account Theorem 2.2 and Lemma 2.1, we obtain that $\mathcal{G}_{p,n}$ also belongs to Σ^0_f.

We denote by $\widetilde{\mathcal{G}}_{p,n}$ the set of all operators $\mathcal{F}\in\mathcal{G}_{p,n}$ such that $\mathcal{I}+\mathcal{F}$ admits a factorization in $\mathcal{G}_{p,n}$, i.e. $\mathcal{F}\in\widetilde{\mathcal{G}}_{p,n}$ if and only if there exist $\mathcal{L}_\pm\in\mathcal{G}^{\pm}_{p,n}$ such that

$$\mathcal{I}+\mathcal{F}=(\mathcal{I}+\mathcal{L}_+)^{-1}(\mathcal{I}+\mathcal{L}_-)^{-1}. \tag{2.8}$$

In view of Theorem 2.1 and Corollary 2.1 we then arrive at the following statements:

Theorem 2.4. (i) *The set $\widetilde{\mathcal{G}}_{p,n}$ is open and dense everywhere in $\mathcal{G}_{p,n}$.*

(ii) *If $\mathcal{F}\in\widetilde{\mathcal{G}}_{p,n}$, then $\mathcal{L}_\pm$ in (2.8) are determined uniquely and $\mathcal{L}_\pm=K_\mp(\mathcal{F})$.*

(iii) *The mapping $\widetilde{\mathcal{G}}_{p,n}\ni\mathcal{F}\mapsto K_\pm(\mathcal{F})\in\mathcal{G}_{p,n}$ is locally Lipschitz.*

Theorem 2.5. *Let $\mathcal{F} \in \mathcal{G}_{p,n}$ and $F \in G_{p,n}$ be a kernel of $\mathcal{F}$. Then the following statements are equivalent:*

(i) $\mathcal{F} \in \widetilde{\mathcal{G}}_{p,n}$;

(ii) *for each $\alpha \in [0,1]$, the integral equation*

$$f(x) + \int_0^\alpha F(x,t)f(t)\,\mathrm{d}t = 0, \quad x \in (0,1), \tag{2.9}$$

has only zero solution in $\boldsymbol{H}$;

(iii) *the integral equation*

$$X(x,t) + F(x,t) + \int_0^x X(x,s)F(s,t)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+, \tag{2.10}$$

is solvable in $G^+_{p,n}$.

Remark 2.4. Equation (2.10) always has at most one solution. If $X \in G^+_{p,n}$ is a solution of (2.10), then X coincides with the kernel of the operator $\mathcal{L}_+ = K_-(\mathcal{F}) \in \mathcal{G}^+_{p,n}$.

Remark 2.5. Note that for each $\mathcal{F} \in \mathcal{G}_{p,n}$ and $P \in \mathfrak{P}$, the operators $\mathcal{I} + \mathcal{F}P$ and $\mathcal{I} + P\mathcal{F}P$ are invertible or not simultaneously. Therefore, it follows that equation (2.9) has a non-zero solution in $\boldsymbol{H}$ if and only if it has a non-zero solution in $L_2((0,\alpha),\mathbb{C}^n)$. For the same reason, we have that the functions h and $h^\sharp$ from $L_p((-1,1),\mathcal{M}_r)$ belong to $\mathfrak{H}_{p,r}$ or not simultaneously.

3. Proof of Theorem 1.1

The aim of this Section is to prove Theorem 1.1, which is the main result of this paper. Firstly, we shall use the results of the previous section to prove that the Krein mapping is locally Lipschitz. Next, we shall construct a locally Lipschitz mapping $\Upsilon : \mathfrak{Q}_p \to \mathfrak{H}_{p,r}$ and show that $\Upsilon = \Theta^{-1}$.

3.1. **The Krein mapping.** Here we shall prove that the Krein mapping is locally Lipschitz. We start with several auxiliary statements which will be useful in subsequent expositions. The first one is a corollary of Theorems 2.4 and 2.5:

Proposition 3.1. *Let $h \in L_p((-1,1),\mathcal{M}_n)$. Consider the operator $\mathcal{H} \in \mathcal{G}_{p,n}$ acting by the formula*

$$(\mathcal{H}f)(x) = \int_0^1 h(x-t)f(t)\,\mathrm{d}t, \quad f \in L_2((0,1),\mathbb{C}^n). \tag{3.1}$$

Then the following statements are equivalent:

(i) $\mathcal{H} \in \widetilde{\mathcal{G}}_{p,n}$;

(ii) *h is an accelerant, i.e. $h \in \mathfrak{H}_{p,n}$;*

(iii) *the Krein equation*

$$r(x,t) + h(x-t) + \int_0^x r(x,s)h(s-t)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+, \tag{3.2}$$

has a unique solution $r_h \in G^+_{p,n}$.

Moreover, the mapping $\mathfrak{H}_{p,n} \ni h \mapsto r_h \in G^+_{p,n}$ is locally Lipschitz and the set $\mathfrak{H}_{p,n}$ is open in $L_p((-1,1),\mathbb{C}^n)$.

Proposition 3.2. *The set $\mathfrak{H}_{p,r}$ is dense everywhere in $L_p((-1,1),\mathcal{M}_r)$.*

Proof. Let $f \in L_p((-1,1),\mathcal{M}_r)$. Then f can be written in the form $f = h + h_1$, where $h, h_1 \in L_p((-1,1),\mathcal{M}_r)$, $\|h\|_{L_p} < 1$ and h_1 is a trigonometric polynomial. Denote by $\mathcal{H}$ and $\mathcal{H}_1$ the operators constructed by formula (3.1) from functions h and h_1, respectively. It is easily seen that the operator $\mathcal{H}_1$ is finite dimensional and that the norm of the operator $\mathcal{H}$ is less than 1. Therefore, one has $\mathcal{H} \in \widetilde{\mathcal{G}}_{p,r}$. By virtue of Theorem 2.3, the set $\Lambda := \{\lambda \in \mathbb{C} \mid (\mathcal{H}+\lambda\mathcal{H}_1) \in \Psi\}$ is open and dense everywhere in $\mathbb{C}$. Since the algebra $\mathcal{G}_{p,r}$ belongs to the class Σ_f^0 (see Remark 2.3), it follows that $\Lambda = \{\lambda \in \mathbb{C} \mid (\mathcal{H}+\lambda\mathcal{H}_1) \in \widetilde{\mathcal{G}}_{p,r}\}$. In view of Proposition 3.1, this means that $\Lambda = \{\lambda \in \mathbb{C} \mid (h + \lambda h_1) \in \mathfrak{H}_{p,r}\}$. Therefore, f is a limit point of the set $\mathfrak{H}_{p,r}$. □

Lemma 3.1. *Let $h \in L_p((-1,1),\mathcal{M}_r)$,*

$$F^h(x,t) := \frac{1}{2}\begin{pmatrix} h\left(\dfrac{x-t}{2}\right) & h\left(\dfrac{x+t}{2}\right) \\ h\left(-\dfrac{x+t}{2}\right) & h\left(-\dfrac{x-t}{2}\right)\end{pmatrix}, \quad x,t \in (0,1), \tag{3.3}$$

and $\mathcal{F}^h \in \mathcal{G}_{p,2r}$ be the integral operator with kernel F^h. Then $h \in \mathfrak{H}_{p,r} \iff \mathcal{F}^h \in \widetilde{\mathcal{G}}_{p,2r}$.

Proof. In view of Proposition 3.1, the lemma will be proved if we show that

$$\mathcal{H} \in \widetilde{\mathcal{G}}_{p,r} \quad \iff \quad \mathcal{F}^h \in \widetilde{\mathcal{G}}_{p,2r}. \tag{3.4}$$

For this purpose, recall (see Theorem 2.5 and Remark 2.5) that $\mathcal{H} \in \widetilde{\mathcal{G}}_{p,r}$ if and only if the equation

$$f(x) + \int_0^\alpha h(x-t)f(t)\,\mathrm{d}t = 0, \quad x \in (0,\alpha), \tag{3.5}$$

has only zero solution in $L_2((0,\alpha),\mathbb{C}^r)$. Similarly, one has $\mathcal{F}^h \in \widetilde{\mathcal{G}}_{p,2r}$ if and only if the equation

$$g(x) + \int_0^\alpha F^h(x,t)g(t)\,\mathrm{d}t = 0, \quad x \in (0,\alpha), \tag{3.6}$$

has only zero solution in $L_2((0,\alpha),\mathbb{C}^{2r})$. Now observe that if $f \in L_2((0,\alpha),\mathbb{C}^r)$ solves (3.5), then

$$g(x) = \begin{pmatrix} f\left(\dfrac{\alpha+x}{2}\right) \\ f\left(\dfrac{\alpha-x}{2}\right)\end{pmatrix}$$

solves (3.6) and that if $g = (g_1,\ g_2)^\top$ with $g_1, g_2 \in L_2((0,\alpha),\mathbb{C}^r)$ solves (3.6), then

$$f(x) = \begin{cases} g_2(\alpha - 2x), & x \in \left(0, \frac{\alpha}{2}\right), \\ g_1(2x-\alpha), & x \in \left(\frac{\alpha}{2}, \alpha\right), \end{cases}$$

solves (3.5). Therefore, equations (3.5) and (3.6) have non-zero solutions simultaneously which proves the equivalence (3.4). □

Remark 3.1. Let $h \in \mathfrak{H}_{p,r}$. Recall that $h^\sharp(x) := h(-x)$ and set

$$H(x) := \begin{pmatrix} h(x) & 0 \\ 0 & h^\sharp(x)\end{pmatrix}, \quad x \in (-1,1). \tag{3.7}$$

It is then easily verified that for the function

$$R_H(x,t) := \begin{pmatrix} r_h(x,t) & 0 \\ 0 & r_{h^\sharp}(x,t)\end{pmatrix}, \quad (x,t) \in \overline{\Omega}_+, \tag{3.8}$$

it holds

$$R_H(x,t) + H(x-t) + \int_0^x R_H(x,s)H(s-t)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+, \tag{3.9}$$

and that for

$$L_h(x,t) := \frac{1}{2}\left\{R_H\left(x, \frac{x+t}{2}\right) + R_H\left(x, \frac{x-t}{2}\right)B\right\}, \quad B := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{3.10}$$

one has

$$F^h(x,t) + L_h(x,t) + \int_0^x L_h(s,t)F^h(x,s)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+. \tag{3.11}$$

Now we are ready to prove the following lemma which is the main purpose of this subsection:

Lemma 3.2. *For an arbitrary $p \in [1,\infty)$, the Krein mapping $\Theta : \mathfrak{H}_{p,r} \mapsto \mathfrak{Q}_p$ is locally Lipschitz.*

Proof. Let $h \in \mathfrak{H}_{p,r}$ and H be as in (3.7). It then follows from Proposition 3.1 and from (3.9) that $H \in \mathfrak{H}_{p,2r}$ and that the mapping $\mathfrak{H}_{p,2r} \ni H \mapsto R_H \in G^+_{p,2r}$ is locally Lipschitz. Note that

$$[\Theta(h)](x) = R_H(x,0)BJ, \quad x \in (0,1), \tag{3.12}$$

where B is from (3.10). In view of formulas (3.7) and (3.12), it is then easily seen that Θ is locally Lipschitz. □

3.2. **Construction of the mapping Υ.** We now construct the mapping $\Upsilon : \mathfrak{Q}_p \to \mathfrak{H}_{p,r}$ that will appear to be the inverse of the Krein mapping.

Let $Q \in \mathfrak{Q}_p$. For each $\lambda \in \mathbb{C}$, we denote by $\varphi_Q(x,\lambda)$, $x \in [0,1]$, a $2r \times r$ matrix-valued solution of the Cauchy problem

$$J\frac{\mathrm{d}}{\mathrm{d}x}\varphi + Q\varphi = \lambda\varphi, \quad \varphi(0,\lambda) = \begin{pmatrix} I \\ I \end{pmatrix}. \tag{3.13}$$

Lemma 3.3. *For each $Q \in \mathfrak{Q}_p$, there is a unique function $K_Q \in G^+_{p,2r}$ such that for all $x \in [0,1]$ and $\lambda \in \mathbb{C}$ it holds*

$$\varphi_Q(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x K_Q(x,s)\varphi_0(s,\lambda)\,\mathrm{d}s, \tag{3.14}$$

where $\varphi_0(x,\lambda)$ is a solution of the Cauchy problem (3.13) in the free case $Q = 0$. Moreover, the mapping $\mathfrak{Q}_p \ni Q \mapsto K_Q \in G^+_{p,2r}$ is locally Lipschitz.

Proof. Denote by $Y_Q(\cdot,\lambda) \in W_2^1((0,1),\mathcal{M}_{2r})$ a $2r \times 2r$ matrix-valued solution of the Cauchy problem

$$J\frac{\mathrm{d}}{\mathrm{d}x}Y + QY = \lambda Y, \quad Y(0,\lambda) = I_{2r}.$$

It then follows from [16, Theorem 2.1] that there exist unique functions $P^\pm := P^\pm_Q$ from $G^+_{p,2r}$ such that for all $x \in [0,1]$ and $\lambda \in \mathbb{C}$ it holds

$$Y_Q(x,\lambda) = \mathrm{e}^{-\lambda xJ} + \int_0^x P^+(x,t)\mathrm{e}^{-\lambda(x-2t)J}\,\mathrm{d}t + \int_0^x P^-(x,t)\mathrm{e}^{\lambda(x-2t)J}\,\mathrm{d}t. \tag{3.15}$$

Since $\varphi_Q(x,\lambda) = Y_Q(x,\lambda)a$, where $a := (I, I)^\top$, straightforward manipulations lead us to formula (3.14) with

$$K_Q(x,t) = \frac{1}{2}\left\{P^+\left(x, \frac{x-t}{2}\right) + P^+\left(x, \frac{x+t}{2}\right)B + P^-\left(x, \frac{x-t}{2}\right)B + P^-\left(x, \frac{x+t}{2}\right)\right\}, \tag{3.16}$$

where B is from (3.10).

Let us prove that the mapping $\mathfrak{Q}_p \ni Q \mapsto K_Q \in G^+_{p,2r}$ is locally Lipschitz. It follows from the proof of Theorem 2.8 in [16] that with $\widetilde{P}_Q(x,t) := P^+_Q\left(x, \frac{x-t}{2}\right)$ it holds

$$\|\widetilde{P}_{Q_1}(x,\cdot) - \widetilde{P}_{Q_2}(x,\cdot)\|_{L_p} \le (1+2\varepsilon)\mathrm{e}^{2\varepsilon}\|Q_1 - Q_2\|_{L_p}, \tag{3.17}$$

$$\|\widetilde{P}_{Q_1}(\cdot,t) - \widetilde{P}_{Q_2}(\cdot,t)\|_{L_p} \le C\|Q_1 - Q_2\|_{L_p}, \quad C := 2\varepsilon\mathrm{e}^{\varepsilon} + 2\varepsilon(1+2\varepsilon)\mathrm{e}^{2\varepsilon}, \tag{3.18}$$

for every $Q_1, Q_2 \in \mathfrak{Q}_p$ such that $\|Q_1\|, \|Q_2\| < \varepsilon$ and that the same estimates hold true also with $\widetilde{P}_Q(x,t) := P^+_Q\left(x, \frac{x+t}{2}\right)$, $\widetilde{P}_Q(x,t) := P^-_Q\left(x, \frac{x+t}{2}\right)$ and $\widetilde{P}_Q(x,t) := P^-_Q\left(x, \frac{x-t}{2}\right)$. In view of (3.16), we then obtain that the mapping $\mathfrak{Q}_p \ni Q \mapsto K_Q \in G^+_{p,2r}$ is locally Lipschitz. □

Denote by $\mathcal{K}_Q \in \mathcal{G}_{p,2r}$ the integral operator with kernel K_Q and let $\mathcal{I}$ stand for the identity operator in $\mathbb{H}$. Since $\mathcal{K}_Q$ is a Volterra operator, the operator $\mathcal{I}+\mathcal{K}_Q$ is invertible in $\mathbb{H}$. Set

$$\mathcal{L}_Q := (\mathcal{I}+\mathcal{K}_Q)^{-1} - \mathcal{I}, \tag{3.19}$$

$$\mathcal{F}_Q := (\mathcal{I}+\mathcal{K}_Q)^{-1}(\mathcal{I}+\mathcal{K}^*_{Q^*})^{-1} - \mathcal{I} \tag{3.20}$$

and denote by L_Q and F_Q the kernels of the integral operators $\mathcal{L}_Q$ and $\mathcal{F}_Q$, respectively.

Theorem 3.1. *Let $Q \in \mathfrak{Q}_p$ and $F := F_Q$. Then there is a unique $h = \Upsilon(Q) \in \mathfrak{H}_{p,r}$ such that $F_Q = F^h$ (see (3.3)). Moreover, the mapping $\Upsilon : \mathfrak{Q}_p \to \mathfrak{H}_{p,r}$ is locally Lipschitz.*

Proof. Firstly, note that the mapping $\mathfrak{Q}_p \ni Q \mapsto F_Q \in G_{p,2r}$ is locally Lipschitz. Indeed, in view of Lemma 3.3 one has that the mapping $\mathfrak{Q}_p \ni Q \mapsto K_Q \in G^+_{p,2r}$ is locally Lipschitz. Taking into account Proposition 2.1, we then easily find that the mapping $\mathfrak{Q}_p \ni Q \mapsto F_Q \in G_{p,2r}$ is locally Lipschitz as well.

Assume that for each $Q \in \mathfrak{Q}_p$ there is $h \in L_p((-1,1),\mathcal{M}_r)$ such that $F_Q = F^h$. Evidently, such h is unique and one has $h = \eta(F)$, where $\eta : G_{p,2r} \to L_p((-1,1),\mathcal{M}_r)$ is a continuous linear mapping acting by the formula

$$[\eta(F)](x) := \begin{cases} F_{21}(-2x-1,1), & -1 \le x \le -\frac{1}{2}, \\ F_{11}(2x+1,1), & -\frac{1}{2} < x \le 0, \\ F_{22}(-2x+1,1), & 0 < x \le \frac{1}{2}, \\ F_{12}(2x-1,1), & \frac{1}{2} < x \le 1, \end{cases}$$

where

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad F_{ij} \in G_{p,r}. \tag{3.21}$$

In view of (3.20), note that $\mathcal{F}_Q \in \widetilde{\mathcal{G}}_{p,2r}$. Since $\mathcal{F}_Q = \mathcal{F}^h$, it then follows from Lemma 3.1 that $h \in \mathfrak{H}_{p,r}$. Moreover, since the mapping $\mathfrak{Q}_p \ni Q \mapsto F_Q \in G_{p,2r}$ is locally Lipschitz, it follows that the mapping $\mathfrak{Q}_p \ni Q \mapsto \Upsilon(Q) := \eta(F_Q) \in \mathfrak{H}_{p,r}$ is locally Lipschitz as well.

Therefore, Theorem 3.1 will be proved if we show that for each $Q \in \mathfrak{Q}_p$ there is $h \in L_p((-1,1),\mathcal{M}_r)$ such that $F_Q = F^h$. Obviously, it suffices to prove this only for smooth functions Q.

So, let $Q \in \mathfrak{Q}_p \cap C^1([0,1],\mathcal{M}_{2r})$ and $F := F_Q$. It then follows from Proposition A.1 that $F \in C^1(\overline{\Omega}_\pm,\mathcal{M}_{2r})$ and that

$$JF'_x(x,t) + F'_t(x,t)J = 0, \quad (x,t) \in \overline{\Omega}_\pm, \tag{3.22}$$

$$F(x,0)a^* = 0, \quad aF(0,x) = 0, \quad x \in (0,1). \tag{3.23}$$

If we write F in the block form (3.21), we then obtain from (3.22) that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)F_{11} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)F_{22} = 0, \quad \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)F_{12} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)F_{21} = 0.$$

Therefore, it follows that F can be written in the form

$$F(x,t) = \frac{1}{2}\begin{pmatrix} h_1\left(\dfrac{x-t}{2}\right) & h_2\left(\dfrac{x+t}{2}\right) \\ h_3\left(-\dfrac{x+t}{2}\right) & h_4\left(-\dfrac{x-t}{2}\right)\end{pmatrix}, \quad (x,t) \in \overline{\Omega}_\pm,$$

where $h_1, h_4 \in C([-1/2,1/2],\mathcal{M}_r)$, $h_2 \in C([0,1],\mathcal{M}_r)$ and $h_3 \in C([-1,0],\mathcal{M}_r)$. Next, we find from (3.23) that $h_1 = h_4$ and that

$$h_2(x) = h_1(x), \quad x \in \left[0, \tfrac{1}{2}\right],$$
$$h_3(x) = h_1(x), \quad x \in \left[-\tfrac{1}{2}, 0\right].$$

We then arrive at $F = F^h$ with h given by the formula

$$h(x) := \begin{cases} h_1(x), & x \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ h_2(x), & x \in \left(\frac{1}{2}, 1\right), \\ h_3(x), & x \in \left(-1, -\frac{1}{2}\right). \end{cases}$$

□

3.3. **Proof of Theorem 1.1.** From Lemma 3.2 we already know that the Krein mapping Θ is locally Lipschitz. Since the mapping Υ from Theorem 3.1 is also locally Lipschitz, Theorem 1.1 will be proved if we show that $\Upsilon = \Theta^{-1}$. Since $\mathfrak{Q}_p \cap C^1([0,1],\mathcal{M}_{2r})$ is dense everywhere in $\mathfrak{Q}_p$ and $\mathfrak{H}_{p,r} \cap C^1([-1,1],\mathcal{M}_r)$ is dense everywhere in $\mathfrak{H}_{p,r}$, it suffices to prove the equalities

$$\Theta(\Upsilon(Q)) = Q, \quad Q \in \mathfrak{Q}_p \cap C^1([0,1],\mathcal{M}_{2r}), \tag{3.24}$$

$$\Upsilon(\Theta(h)) = h, \quad h \in \mathfrak{H}_{p,r} \cap C^1([-1,1],\mathcal{M}_r). \tag{3.25}$$

Let us first prove (3.24). Let $Q \in \mathfrak{Q}_p \cap C^1([0,1],\mathcal{M}_{2r})$ and $h := \Upsilon(Q)$. Since by virtue of the definition of the mapping Υ one has $F_Q = F^h$, in view of formula (3.20) one has

$$\mathcal{I} + \mathcal{F}^h = (\mathcal{I} + \mathcal{K}_Q)^{-1}(\mathcal{I} + \mathcal{K}^*_{Q^*})^{-1}.$$

From the other hand, we obtain from Remark 3.1 that

$$F^h(x,t) + L_h(x,t) + \int_0^x L_h(s,t)F^h(x,s)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+,$$

where $L_h \in G^+_{p,2r}$ is of (3.10). In view of Remark 2.4 we then find that $K_Q = L_h$. By virtue of formulas (3.10) and (A.5), it then holds

$$[\Theta(h)](x) = K_Q(x,x)J - JK_Q(x,x) = Q(x), \quad x \in (0,1),$$

as desired.

It thus only remains to prove (3.25). Let $h \in \mathfrak{H}_{p,r} \cap C^1([-1,1],\mathcal{M}_r)$ and $Q := \Theta(h)$. Then (3.25) will be proved if we show that

$$F_Q = F^h. \tag{3.26}$$

In turn, since

$$\mathcal{I} + \mathcal{F}_Q = (\mathcal{I} + \mathcal{K}_Q)^{-1}(\mathcal{I} + \mathcal{K}^*_{Q^*})^{-1},$$

we find from Remarks 3.1 and 2.4 that (3.26) will be proved if we show that $K_Q = L_h$ with L_h of (3.10). For this purpose, it suffices to verify that the function

$$\varphi(x,\lambda) := \varphi_0(x,\lambda) + \int_0^x L_h(x,t)\varphi_0(t,\lambda)\,\mathrm{d}t, \quad x \in [0,1], \quad \lambda \in \mathbb{C}, \tag{3.27}$$

where $\varphi_0(x,\lambda) := (\mathrm{e}^{\mathrm{i}\lambda x},\, \mathrm{e}^{-\mathrm{i}\lambda x})^\top$, solves the Cauchy problem

$$J\frac{\mathrm{d}}{\mathrm{d}x}\varphi + Q\varphi = \lambda\varphi, \quad \varphi(0,\lambda) = \begin{pmatrix} I \\ I \end{pmatrix}. \tag{3.28}$$

The verification of this claim repeats the proof of Theorem 3.1 in [12].

Indeed, let H and R_H be as in (3.7) and (3.8), respectively. In view of Remark 3.1, it then holds

$$R_H(x,t) + H(x-t) + \int_0^x R_H(x,s)H(s-t)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+. \tag{3.29}$$

Moreover, it follows from [13, Lemma 3.4] that in the case of the smooth h as chosen one has $R_H \in C^1(\overline{\Omega}_+, \mathcal{M}_{2r})$.

Taking into account formulas (3.10) and $B\varphi_0(x,\lambda) = \varphi_0(-x,\lambda)$, we can rewrite (3.27) in the form

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x R_H(x,x-t)\varphi_0(x-2t,\lambda)\,\mathrm{d}t.$$

From this equality, taking into account that $J\frac{\mathrm{d}}{\mathrm{d}x}\varphi_0(x,\lambda) - \lambda\varphi_0(x,\lambda) = 0$, we find that

$$\begin{aligned} J\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x,\lambda) + Q(x)\varphi(x,\lambda) - \lambda\varphi(x,\lambda) &= \{JR_H(x,0)B\varphi_0(x,\lambda) + Q(x)\varphi_0(x,\lambda)\} \\ &+ \int_0^x \left\{J\frac{\partial}{\partial x}R_H(x,x-t) + Q(x)R_H(x,x-t)\right\}\varphi_0(x-2t,\lambda)\,\mathrm{d}t. \end{aligned} \tag{3.30}$$

Since $Q(x) = -JR_H(x,0)B$, (3.30) is reduced to

$$\begin{aligned} &J\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x,\lambda) + Q(x)\varphi(x,\lambda) - \lambda\varphi(x,\lambda) \\ &\qquad = J\int_0^x \left\{\frac{\partial}{\partial x}R_H(x,x-t) - R_H(x,0)BR_H(x,t)B\right\}\varphi_0(x-2t,\lambda)\,\mathrm{d}t. \end{aligned}$$

Therefore, (3.28) will be verified if we show that

$$\frac{\partial}{\partial x}R_H(x,x-t) - R_H(x,0)BR_H(x,t)B = 0, \quad (x,t) \in \overline{\Omega}_+. \tag{3.31}$$

Let us prove (3.31). For this purpose, we obtain from (3.29) that

$$R_H(x,x-t) + H(t) + \int_0^x R_H(x,x-s)H(t-s)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+.$$

Differentiating this expression in x we find that

$$\begin{aligned} &\frac{\partial}{\partial x}R_H(x,x-t) + R_H(x,0)H(t-x) \\ &\qquad + \int_0^x \frac{\partial}{\partial x}[R_H(x,x-s)]H(t-s)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+. \end{aligned} \tag{3.32}$$

Multiplying now (3.29) by $R_H(x,0)B$ from the left and by B from the right and subtracting it from (3.32), in view also of the relation $H(x)B = BH(-x)$, we find that the function

$$X(x,t) := \frac{\partial}{\partial x}R_H(x,x-t) - R_H(x,0)BR_H(x,t)B, \quad (x,t) \in \overline{\Omega}_+,$$

solves the equation

$$X(x,t) + \int_0^x X(x,s)H(t-s)\,\mathrm{d}s = 0, \quad (x,t) \in \overline{\Omega}_+.$$

Since $R_H \in C^1(\overline{\Omega}_+, \mathcal{M}_{2r})$, one has $X \in C(\overline{\Omega}_+, \mathcal{M}_{2r})$ and thus by virtue of Proposition 3.1 we find that $X(x,t) = 0$, $(x,t) \in \overline{\Omega}_+$. Therefore, (3.31) follows and the proof is complete.

Appendix A

The aim of this appendix is to prove equalities (3.22) and (3.23) which were used in the proof of Theorem 3.1. The proof is technical and goes back to the well known fact that kernels of transformation operators satisfy some differential equations.

Let $\mathcal{A}$ be the differential operator acting on functions $X : (x,t) \mapsto \mathcal{M}_{2r}$ from the class $C^1(\overline{\Omega}_\pm, \mathcal{M}_{2r})$ by the formula

$$\mathcal{A}X := JX'_x + X'_t J, \tag{A.1}$$

where X'_x and X'_t denote the derivatives in variables x and t, respectively. We shall prove the following proposition:

Proposition A.1. *Let $Q \in \mathfrak{Q}_p \cap C^1([0,1], \mathcal{M}_{2r})$ and $F := F_Q$. Then $F \in C^1(\overline{\Omega}_\pm, \mathcal{M}_{2r})$ and*

$$(\mathcal{A}F)(x,t) = 0, \quad (x,t) \in \overline{\Omega}_\pm, \tag{A.2}$$

$$F(x,0)a^* = 0, \quad aF(0,x) = 0, \quad x \in (0,1), \tag{A.3}$$

where $a := \begin{pmatrix} I, & -I \end{pmatrix}$.

The proof of Proposition A.1 will be based on two auxiliary lemmas:

Lemma A.1. *Let $Q \in \mathfrak{Q}_p \cap C^1([0,1], \mathcal{M}_{2r})$ and $K := K_Q$. Then $K \in C^1(\overline{\Omega}_+, \mathcal{M}_{2r})$ and*

$$(\mathcal{A}K)(x,t) = -Q(x)K(x,t), \quad (x,t) \in \Omega_+. \tag{A.4}$$

$$(KJ - JK)(x,x) = Q(x), \quad K(x,0)a^* = 0, \quad x \in (0,1). \tag{A.5}$$

Proof. Let Q and K be as in the statement of the lemma. Recall (see (3.16)) that

$$K(x,t) = \frac{1}{2}\left\{P^+\left(x, \frac{x-t}{2}\right) + P^+\left(x, \frac{x+t}{2}\right)B + P^-\left(x, \frac{x-t}{2}\right)B + P^-\left(x, \frac{x+t}{2}\right)\right\}, \tag{A.6}$$

where $P^\pm$ are from (3.15) and B is from (3.10). It follows from the results of [16] that if $Q \in \mathfrak{Q}_p \cap C^1([0,1], \mathcal{M}_{2r})$, then $P^\pm \in C^1(\overline{\Omega}_+, \mathcal{M}_{2r})$ and, moreover,

$$P^+(x,t) = \int_t^x JQ(s)P^-(s, s-t)\,\mathrm{d}s, \tag{A.7}$$

$$P^-(x,t) = \int_t^x JQ(s)P^+(s, s-t)\,\mathrm{d}s + JQ(t), \tag{A.8}$$

$$P^+(x,t)J = JP^+(x,t), \quad P^-(x,t)J = -JP^-(x,t). \tag{A.9}$$

Using now (A.6)–(A.9) and the equalities

$$J^2 = -I_{2r}, \quad JB = -BJ, \quad JQ(x) = -Q(x)J, \tag{A.10}$$

by virtue of straightforward (but quite extensive) verification we then arrive at (A.4).

Now let us prove (A.5). It follows from (A.7) and (A.8) that $P^+(x,x) = 0$ and $P^-(x,x) = JQ(x)$. Therefore, in view of (A.6), we find that

$$K(x,x) = \frac{1}{2}\left\{P^+(x,0) + P^-(x,0)B + JQ(x)\right\}.$$

Taking into account (A.9) and (A.10) we then obtain that

$$K(x,x)J - JK(x,x) = Q(x).$$

Since $(I_{2r} + B)a^* = 0$, in view of formula (A.6) we then arrive at

$$K(x,0)a^* = \frac{1}{2}\left\{P^+\left(x, \tfrac{x}{2}\right) + P^-\left(x, \tfrac{x}{2}\right)\right\}(I_{2r} + B)a^* = 0$$

and thus (A.5) is proved. ∎

Lemma A.2. *Let $Q \in \mathfrak{Q}_p \cap C^1([0,1],\mathcal{M}_{2r})$ and $L := L_Q$. Then $L \in C^1(\overline{\Omega}_+,\mathcal{M}_{2r})$ and*

$$(\mathcal{A}L)(x,t) = L(x,t)Q(t), \quad (x,t) \in \overline{\Omega}_+, \tag{A.11}$$

$$(JL - LJ)(x,x) = Q(x), \quad L(x,0)a^* = 0, \quad x \in [0,1]. \tag{A.12}$$

Proof. Let $Q \in \mathfrak{Q}_p \cap C^1([0,1],\mathcal{M}_{2r})$, $K := K_Q$ and $L := L_Q$. In view of (3.19), it follows that $(\mathcal{I}+\mathcal{K}_Q)(\mathcal{I}+\mathcal{L}_Q) = (\mathcal{I}+\mathcal{L}_Q)(\mathcal{I}+\mathcal{K}_Q) = \mathcal{I}$ and thus for $(x,t) \in \overline{\Omega}_+$ it holds

$$\begin{aligned} K(x,t) + L(x,t) + \int_t^x K(x,s)L(s,t)\,\mathrm{d}s = 0,\\ K(x,t) + L(x,t) + \int_t^x L(x,s)K(s,t)\,\mathrm{d}s = 0. \end{aligned} \tag{A.13}$$

In view also of (A.5), these equalities easily lead us to (A.12).

To prove (A.11), set

$$S(x,t) := \int_t^x K(x,s)L(s,t)\,\mathrm{d}s, \quad (x,t) \in \overline{\Omega}_+. \tag{A.14}$$

Taking into account (A.4), it can be verified that

$$\begin{aligned} (\mathcal{A}S)(x,t) + Q(x)S(x,t) &= JK(x,x)L(x,t) - K(x,t)L(t,t)J\\ &- \int_t^x K_s'(x,s)JL(s,t)\,\mathrm{d}s + \int_t^x K(x,s)L_t'(s,t)J\,\mathrm{d}s. \end{aligned} \tag{A.15}$$

Integrating by parts then leads to

$$\begin{aligned} &\int_t^x K_s'(x,s)JL(s,t)\,\mathrm{d}s\\ &\qquad = K(x,x)JL(x,t) - K(x,t)JL(t,t) - \int_t^x K(x,s)JL_s'(s,t)\,\mathrm{d}s. \end{aligned}$$

Therefore, taking into account (A.5) and (A.12), we can rewrite (A.15) in the form

$$\begin{aligned} &(\mathcal{A}S)(x,t) + Q(x)S(x,t)\\ &\qquad = -Q(x)L(x,t) + K(x,t)Q(t) + \int_t^x K(x,s)(\mathcal{A}L)(s,t)\,\mathrm{d}s. \end{aligned} \tag{A.16}$$

Now let

$$X(x,t) := (\mathcal{A}L)(x,t) - L(x,t)Q(t), \quad (x,t) \in \overline{\Omega}_+,$$

and $X(x,t) := 0$, $(x,t) \in \Omega_-$. Since $S(x,t) = -K(x,t) - L(x,t)$, from (A.16) and (A.4) we then find that

$$X(x,t) + \int_0^x K(x,s)X(s,t)\,\mathrm{d}s = 0.$$

Since the operator $\mathcal{I}+\mathcal{K}$ is invertible in $\mathbb{H}$, we then obtain that $X(x,t) = 0$ for all $(x,t) \in \overline{\Omega}_+$ which proves (A.11). □

Now we are ready to prove Proposition A.1:

Proof of Proposition A.1. Let $Q \in \mathfrak{Q}_p \cap C^1([0,1],\mathcal{M}_{2r})$, $L := L_Q$, $L_* := L_{Q^*}$ and $F := F_Q$. It then follows from (3.19) and (3.20) that

$$F(x,t) = L(x,t) + L_*(t,x)^* + \int_0^1 L(x,s)L_*(t,s)^*\,\mathrm{d}s, \quad x,t \in [0,1].$$

Since $L(x,t) = L_*(x,t) = 0$ as $x < t$, we then obtain that

$$\begin{aligned} F(x,t) =& L(x,t) + \int_0^t L(x,s)L_*(t,s)^* \, \mathrm{d}s, \quad (x,t) \in \Omega_+, \\ F(x,t) =& L_*(t,x)^* + \int_0^x L(x,s)L_*(t,s)^* \, \mathrm{d}s, \quad (x,t) \in \Omega_-, \end{aligned} \tag{A.17}$$

which immediately implies (A.3). Furthermore, it follows from (A.17) and Lemma A.2 that $F \in C^1(\overline{\Omega}_\pm, \mathcal{M}_{2r})$.

To prove also (A.2), take into account (3.20) and observe that $\mathcal{F}^* := \mathcal{F}_Q^* = \mathcal{F}_{Q^*}$. Therefore, it suffices to prove (A.2) only for $(x,t) \in \Omega_+$. Taking into account (A.11), we obtain from the first equality in (A.17) that

$$\begin{aligned} (\mathcal{A}F)(x,t) = (\mathcal{A}L)(x,t) &+ L(x,t)L_*(t,t)^* J \\ &- \int_0^t L_s'(x,s)JL_*(t,s)^* \, \mathrm{d}s - \int_0^t L(x,s)Q(s)L_*(t,s)^* \, \mathrm{d}s \\ &+ \int_0^t L(x,s)Q(s)L_*(t,s)^* \, \mathrm{d}s + \int_0^t L(x,s)[(L_*(t,s))_s' J]^* \, \mathrm{d}s. \end{aligned} \tag{A.18}$$

Integrating by parts leads to

$$\begin{aligned} \int_0^t L_s'(x,s)JL_*(t,s)^* \, \mathrm{d}s &= L(x,t)JL_*(t,t)^* \\ &- L(x,0)JL_*(t,0)^* - \int_0^t L(x,s)J(L_*(t,s)^*)_s' \, \mathrm{d}s. \end{aligned} \tag{A.19}$$

Furthermore, in view of Lemma A.2 it holds

$$JL_*(t,t) - L_*(t,t)J = Q(t)^*, \quad t \in [0,1]. \tag{A.20}$$

Taking into account (A.19), (A.20) and (A.11), we then obtain from (A.18) that

$$(\mathcal{A}F)(x,t) = (\mathcal{A}L)(x,t) - L(x,t)Q(t) + L(x,0)JL_*(t,0)^* = L(x,0)JL_*(t,0)^*.$$

Finally, noting that $J = a^*aJ + Ja^*a$ and, in view of (A.12),

$$L(x,0)a^* = 0 = L_*(x,0)a^*, \quad x \in [0,1],$$

we then find that $L(x,0)JL_*(t,0)^* = 0$, $(x,t) \in \Omega_+$, which completes the proof of the proposition. □

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