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NONZERO CAPACITY SETS AND DENSE SUBSPACES IN SCALES OF SOBOLEV SPACES

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ABSTRACT. We show that for a compact set $K \subset \mathbb{R}^n$ of nonzero α-capacity, $C_\alpha(K) > 0$, $\alpha \geq 1$, the subspace $\overset{\circ}{W}{}^{\alpha,2}(\Omega)$, $\Omega = \mathbb{R}^n \setminus K$ in $W^{\alpha,2}(\mathbb{R}^n)$ is dense in $W^{m,2}(\mathbb{R}^n)$, $m \leq \alpha - 1$, iff the m-capacity of K is zero, $C_m(K) = 0$.

1. Introduction

This paper was stimulated by the following problem.

Consider a couple of Hilbert spaces $\mathcal{H} \sqsupset \mathcal{K}$, where $\sqsupset$ denotes a dense and continuous embedding. Besides we assume that the norms in $\mathcal{H}$ and $\mathcal{K}$ satisfy the inequality

$$\|\varphi\|_{\mathcal{H}} \leq \|\varphi\|_{\mathcal{K}}, \quad \varphi \in \mathcal{K}.$$

Let $\mathcal{K}$ be decomposed into an orthogonal sum of nontrivial subspaces

$$\mathcal{K} = \mathcal{M} \oplus \mathcal{N}, \quad \mathcal{M} \neq \{0\} \neq \mathcal{N}. \tag{1}$$

Under what conditions one of these subspaces, say $\mathcal{M}$, is dense in $\mathcal{H}$? That is we ask, under what conditions on $\mathcal{N}$ in (1) the following embedding are both dense:

$$\mathcal{H} \sqsupset \mathcal{K}, \quad \mathcal{H} \sqsupset \mathcal{M}?$$

In other terms this question arises in the operator theory and various applications to mathematical physics. For example, the above problem has an important motivation in the theory of self-adjoint extensions of symmetric operators.

Indeed, let A be a self-adjoint operator in $\mathcal{H}$. Set $\mathcal{K} = \mathrm{Dom}A$ equipped with the norm

$$\|\varphi\|_{\mathcal{K}} := \|A\varphi\|_{\mathcal{H}}, \quad \varphi \in \mathrm{Dom}A.$$

Consider some restriction of A to a linear set,

$$\dot{A} := A \restriction \mathfrak{D}, \quad \mathfrak{D} \subset \mathrm{DomA}.$$

Under what conditions, $\dot{A}$ is a densely defined symmetric operator with nontrivial deficiency indices? Evidently this is so, if the closure $\mathfrak{D}$ in the Hilbert space $\mathcal{K} = \mathrm{Dom}A$ defines some subspace $\mathcal{M}$ which is dense in $\mathcal{H}$. Of course it may happen that $\mathfrak{D}$ is dense in $\mathcal{K}$. Then $\dot{A}$ is essentially self-adjoint and the above question has no sense.

We remark that in an abstract setting, the problem of density in $\mathcal{H}$ of a proper subspace $\mathcal{M}$ from $\mathcal{K}$ (under the assumption that $\mathcal{K}$ already is densely embedded in $\mathcal{H}$) has been investigated in a series of publications (see, for example, [2, 3, 6, 14]) in terms of rigged Hilbert spaces.

Let a triplet of Hilbert spaces,

$$\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+ \tag{2}$$

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constitutes a rigged Hilbert space [4]. This means (for details see [4, 5, 14]) that both embeddings $\mathcal{H}_- \sqsupset \mathcal{H}_0$, $\mathcal{H}_0 \sqsupset \mathcal{H}_+$ are dense and continuous, the norms in $\mathcal{H}_-$, $\mathcal{H}_0$, $\mathcal{H}_+$ satisfy the inequalities

$$\|\cdot\|_- \leq \|\cdot\|_0 \leq \|\cdot\|_+,$$

and the spaces $\mathcal{H}_-$, $\mathcal{H}_+$ are mutually conjugate with respect to $\mathcal{H}_0$. Due to the last condition there exists a bounded map

$$D_{-,+} : \mathcal{H}_+ \longrightarrow \mathcal{H}_-$$

such that

$$\|D_{-,+}\varphi\|_- = \|\varphi\|_+, \quad \varphi \in \mathcal{H}_+,$$
$$\langle \omega, \varphi\rangle_{-,+} = (\omega, D_{-,+}\varphi)_- = (I_{+,-}\omega, \varphi)_+, \quad \omega \in \mathcal{H}_-,$$

where $I_{+,-} := D_{-,+}^{-1}$ and $\langle \cdot, \cdot \rangle_{-,+}$ stands for the dual inner product between $\mathcal{H}_-$ and $\mathcal{H}_+$. Note that

$$\langle f, g\rangle_{-,+} = (f, g)_0, \quad f, g \in \mathcal{H}_0.$$

The operators $D_{-,+}$ and $I_{+,-}$ are called the Berezansky canonical isomorphisms.

In what follows $\mathcal{H}_+$ plays a role of $\mathcal{K}$. Assume that $\mathcal{H}_+$ is decomposed into an orthogonal sum of subspaces, $\mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+$. Our starting results read as follows (for deeper statements see [2, 3, 6, 13, 14]).

Theorem 1. *Let in (2) $\mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+$, $\mathcal{N}_+ \neq \{0\}$. Then the subspace $\mathcal{M}_+$ is dense in $\mathcal{H}_0$, if and only if the subspace $\mathcal{N}_- := D_{-,+}\mathcal{N}_+$ of $\mathcal{H}_-$, which is the image of $\mathcal{N}_+$ with respect to the Berezansky canonical isomorphism $D_{-,+}$, consists of only those vectors which do not belong to $\mathcal{H}_0$, except for zero,*

$$\mathcal{H}_0 \sqsupset \mathcal{M}_+ \Longleftrightarrow \mathcal{N}_- \cap \mathcal{H}_0 = \{0\}. \tag{3}$$

Proof. Let $\mathcal{N}_- \cap \mathcal{H}_0 = \{0\}$. Take a vector $\psi \in \mathcal{H}_0$ such that $\psi \perp \mathcal{M}_+$ in $\mathcal{H}_0$. Then

$$0 = (\psi, \mathcal{M}_+) = \langle \psi, \mathcal{M}_+\rangle_{-,+} = (D_{-,+}^{-1}\psi, \mathcal{M}_+)_+.$$

This means that $I_{+,-}\psi \in \mathcal{N}_+$ and thus $\psi \in \mathcal{N}_-$. Therefore $\psi = 0$ since we assume that $\mathcal{N}_- \cap \mathcal{H}_0 = \{0\}$.

Let us prove (3) in the other direction. Assume that $\mathcal{M}_+$ is dense in $\mathcal{H}_0$ and take $\omega \in \mathcal{N}_- \cap \mathcal{H}_0$. Then

$$\langle \omega, \mathcal{M}_+\rangle_{-,+} = (\omega, \mathcal{M}_+)_0 = (I_{+,-}\omega, \mathcal{M}_+)_+ = 0$$

due to $\omega \in \mathcal{N}_-$ and $I_{+,-}\omega \in \mathcal{N}_+$. Thus $\omega \perp \mathcal{M}_+$ in $\mathcal{H}_0$ and therefore $\omega = 0$. This proves that $\mathcal{N}_- \cap \mathcal{H}_0 = \{0\}$. □

Theorem 2. *Let us have two rigged Hilbert spaces, i.e., (2) and*

$$\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+, \tag{4}$$

such that

$$\mathcal{H}_- \sqsupset \tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+ \sqsupset \mathcal{H}_+.$$

And let $\mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+$, $\mathcal{N}_+ \neq \{0\}$. Then the subspace $\mathcal{M}_+$ is dense in $\tilde{\mathcal{H}}_+$ if and only if the subspace $\mathcal{N}_- := D_{-,+}\mathcal{N}_+$ has a trivial intersection with $\tilde{\mathcal{H}}_-$,

$$\tilde{\mathcal{H}}_+ \sqsupset \mathcal{M}_+ \Longleftrightarrow \mathcal{N}_- \cap \tilde{\mathcal{H}}_- = \{0\}. \tag{5}$$

Proof. Let $\mathcal{M}_+$ be dense in $\tilde{\mathcal{H}}_+$. Then for $\omega \in \mathcal{N}_- \cap \tilde{\mathcal{H}}_-$ we have

$$\langle \omega, \mathcal{M}_+\rangle_{-,+} = \langle \omega, \mathcal{M}_+\rangle_{-,+} = (I_{+,-}\omega, \mathcal{M}_+)_+ = 0$$

due to $\omega \in \mathcal{N}_-$ and $I_{+,-}\omega \in \mathcal{N}_+$. But with $\tilde{I}_{+,-}$, there corresponds to (4)

$$0 = \langle \omega, \mathcal{M}_+\rangle_{-,+} = (\tilde{I}_{+,-}\omega, \mathcal{M}_+)_+,$$

which implies that $\tilde{I}_{+,-}\omega = 0$ since $\mathcal{M}_+ \sqsubset \tilde{\mathcal{H}}_+$. Thus $\omega = 0$ and $\mathcal{N}_- \cap \tilde{\mathcal{H}}_- = \{0\}$.

Conversely, let $\mathcal{N}_- \cap \tilde{\mathcal{H}}_- = \{0\}$. Assume for a moment that $\mathcal{M}_+$ is not dense in $\tilde{\mathcal{H}}_+$. Then there exists a vector $0 \neq \varphi \in \tilde{\mathcal{H}}_+$ such that

$$0 = (\varphi, \mathcal{M}_+)_+ = \langle \tilde{D}_{-,+}\varphi, \mathcal{M}_+\rangle_{-,+} = \langle \omega, \mathcal{M}_+\rangle_{-,+}.$$

This means that $0 \neq \omega = \tilde{D}_{-,+}\varphi \in \mathcal{N}_- \cap \tilde{\mathcal{H}}_-$ which is a contradiction. □

In the present paper we develop and apply the above abstract results in the case where the spaces $\mathcal{H}_-$, $\tilde{\mathcal{H}}_-$, $\tilde{\mathcal{H}}_+$, $\mathcal{H}_+$ are taken from the Sobolev scale of spaces and the subspace $\mathcal{N}_-$ is generated by distributions supported on a set of zero capacity.

2. On dense subspaces in the Sobolev scale of spaces

The above general results have applications to the problem of constructing dense subspaces in scales of functional spaces, in particular, for a scale of Sobolev spaces.

Let us consider the scale of the Sobolev spaces (see [1, 4, 5, 15])

$$W^{-k,2} \sqsupset L^2(\mathbb{R}^n, dx) \sqsupset W^{k,2} \equiv W^{k,2}(\mathbb{R}^n), \quad k > 0.$$

Let $K \subset \mathbb{R}^n$ be a compact set. In what follows we denote the complement $K^c = \mathbb{R}^n \setminus K$ by Ω. We are interested in the following question corresponding to the previous abstract problem. Under what condition is the set $\overset{\circ}{W}{}^{k,2}(\Omega)$ dense in $W^{m,2}$, $m \leq k-1$? We recall that by definition (see, for example, [4]), the Sobolev space $\overset{\circ}{W}{}^{k,2}(\Omega)$ is the closure of the set $C_0^\infty(\Omega)$ in $W^{k,2}$. The answer we will give using the notion of capacity for a compact set K which is a fruitful tool in such kind of problems. So, we need some preparations.

At first we recall the notion of capacity for sets which are small in some sense (for more details and generalizations see [1, 15, 17, 16]).

Definition 1. *The positive value*

$$C_\alpha(K) := \mathrm{cap}_\alpha(K) = \inf\{\|\varphi\|^2_{W^{\alpha,2}} \mid \varphi \in C_0^\infty, \varphi \geq 1 \text{ on } K\} \tag{6}$$

is called α-capacity of a compact set $K \subset \mathbb{R}^n$.

In (6) the set C_0^∞ can be replaced by the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and $\varphi \geq 1$ can be replaced with the condition $\varphi(x) = 1$ for $x \in K$.

Denote

$$\begin{aligned} \mathcal{M}_{-k}(\Omega) :=& \{\omega \in W^{-k,2} | \langle \omega, \varphi\rangle_{-k,k} = 0, \ \forall \varphi \in W^{k,2}, \ \mathrm{supp}\varphi \subseteq K\} \\ =& \{\omega \in W^{-k,2} \mid \mathrm{supp}\omega \subset \Omega\} \end{aligned} \tag{7}$$

and define

$$\mathcal{M}_k(\Omega) = (I_{k,-k}\mathcal{M}_{-k}(\Omega))^{\mathrm{cl},k}, \tag{8}$$

where $I_{k,-k} : W^{-k,2} \longrightarrow W^{k,2}$ stands for the Berezansky canonical isomorphism (see [14]). By construction $\mathcal{M}_k(\Omega)$ is a subspace in $W^{k,2}(\mathbb{R}^n)$. Let us remark that in [11] the abstract version of the capacity concept is presented using [8, 16, 17].

As a first step of our investigation we discuss the following question. Under which condition is the subspace $\mathcal{M}_k(\Omega)$ a strict one in $W^{k,2}$, i.e., $\mathcal{M}_k(\Omega) \neq W^{k,2}$, and is dense in $L_2(\mathbb{R}^n)$? The next theorem gives a simple answer to this question.

Theorem 3. *Let $k > n/2$. Then for each compact set $K \subset \mathbb{R}^n$ of zero Lebesgue measure,*

$$\lambda(K) = 0, \tag{9}$$

such that

$$C_k(K) > 0, \tag{10}$$

the subspace $\mathcal{M}_k(\Omega)$, $\Omega = \mathbb{R}^n \setminus K$ is a strict one in $W^{k,2}(\mathbb{R}^n)$ and dense in $L_2(\mathbb{R}^n)$.

Proof. By the well-known Sobolev embedding theorem, $W^{k,2}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ if $k > n/2$. Hence for each point $y \in K$ the linear functional $l_{\delta_y}(\varphi) := \langle \delta_y, \varphi \rangle_{-k,k} = \varphi(y)$, $\varphi \in W^{k,2}$ is continuous. Thus all distributions spanned by the set $\{\omega = \delta_y | y \in K\}$ belong to the space $W^{-k,2}$ and moreover due to (9) the subspace

$$\mathcal{N}_{-k} = (\text{span}\{\omega = \delta_y \mid y \in K\})^{\text{cl},-k}$$

has a zero intersection with $L_2(\mathbb{R}^n)$ (cl,-k denotes the closure in $W^{-k,2}$). Even more, from (7) it follows that the subspace $\mathcal{N}_k(K) = I_{k,-k}\mathcal{N}_{-k}$ is orthogonal to $\mathcal{M}_k(\Omega)$,

$$\mathcal{N}_k(K) \perp \mathcal{M}_k(\Omega).$$

Therefore one can use Theorem 1. □

Let us denote

$$\Phi(K) = \{\varphi \in \mathcal{S}(\mathbb{R}^n) \mid \varphi \geq 1 \text{ on } K\}, \quad \alpha \geq 1.$$

Let $\Phi^{\text{cl},\alpha}(K)$ be the closure of $\Phi(K)$ in $W^{\alpha,2}$. The next result is well-known in potential theory (see, for example, [1], theorem 2.2.7).

Theorem 4. *Let $K \subset \mathbb{R}^n$ be compact. Assume that for some $\alpha \geq 1$ the α-capacity of K is strictly positive, $C_\alpha(K) > 0$. Then there exists a unique extremal element $\varphi_K \in \Phi^{\text{cl},\alpha}(K)$ such that*

$$C_\alpha(K) = \|\varphi_K\|^2_{W^{\alpha,2}}.$$

Moreover, there exists an α-capacity measure $\mu_K \in W^{-\alpha,2}$ supported on K such that

$$\mu_K(K) = C_\alpha(K).$$

Clearly $\mu_K = D_{-k,k}\varphi_K$, and therefore

$$\varphi_K = I_{k,-k}\mu_K = G_\alpha * (G_\alpha * \mu_K),$$

where G_α denotes the Bessel integral operator, and

$$(G_\alpha * \mu_K)(x) = \int G_\alpha(x-y)\, d\mu_K(y) \in L_2(\mathbb{R}^n).$$

It is easy to see that the extremal element φ_K is orthogonal to the subspace $\overset{\circ}{W}{}^{\alpha,2}(\Omega)$ in $W^{\alpha,2}$,

$$\varphi_K \perp \overset{\circ}{W}{}^{\alpha,2}(\Omega).$$

Indeed, let $\varphi_n \in \Phi(K)$ be a minimizing sequence in (6) which converges to φ_K. Then, obviously

$$(\varphi_K, \varphi)_{W^{\alpha,2}} = \lim_{n\to\infty}(\varphi_n, \varphi) = 0, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Therefore in our case, when $C_\alpha(K) > 0$, the orthogonal compliment to $\overset{\circ}{W}{}^{\alpha,2}(\Omega)$ in $W^{\alpha,2}$ consists of a nontrivial subspace which we denote by $\mathcal{N}_\alpha(K)$. Surely the extremal element φ_K belongs to $\mathcal{N}_\alpha(K)$. Thus, we can write

$$W^{\alpha,2} = \mathcal{M}_\alpha(\Omega) \oplus \mathcal{N}_\alpha(K), \quad \mathcal{M}_\alpha(\Omega) \equiv \overset{\circ}{W}{}^{\alpha,2}(\Omega).$$

Denote by $\mathcal{N}_{-\alpha}(K)$ the dual subspace to $\mathcal{N}_\alpha(K)$ in $W^{-\alpha,2}$. This subspace is connected with $\mathcal{N}_\alpha(K)$ by the Beresansky canonical isomorphism $D_{-\alpha,\alpha} : W^{\alpha,2} \longrightarrow W^{-\alpha,2}$,

$$\mathcal{N}_{-\alpha}(K) = D_{-\alpha,\alpha}\mathcal{N}_\alpha(K).$$

The next proposition gives a deeper description of this subspace.

Proposition 1. *Let $C_\alpha(K) > 0$. Then the subspace $\mathcal{N}_{-\alpha}(K)$ consists of the distributions $\omega \in W^{-\alpha,2}$ supported by K,*

$$\mathcal{N}_{-\alpha}(K) = \{\omega \in W^{-\alpha,2} \mid \text{supp}\omega \subseteq K\}.$$

Proof. The statement is based on and follows from Theorem 9.1.3 and its Corollary 9.1.6 in [1]. So Theorem 9.1.3 says that $\overset{\circ}{W}{}^{\alpha,2}(\Omega)$ coincides with a set of functions $\varphi \in W^{\alpha,2}$ such that

$$(D^\beta \varphi) \restriction K = 0, \quad 0 \le |\beta| \le \alpha - 1. \tag{11}$$

Hence for $\omega \in W^{-\alpha,2}$ with $\mathrm{supp}\omega \subseteq K$ we have

$$\langle \omega, \varphi \rangle_{-\alpha,\alpha} = 0, \tag{12}$$

for all $\varphi \in W^{\alpha,2}$ with condition (11) since all such functions admit an approximation with sequences $\varphi_n \in C_0^\infty(\Omega)$. Certainly, if $\omega \in W^{-\alpha,2}$ and satisfies (12), then obviously $\mathrm{supp}\omega \subseteq K$. □

For a deeper understanding of the above result we remark that the subspace $\mathcal{M}_\alpha(\Omega) = \overset{\circ}{W}{}^{\alpha,2}(\Omega)$ can be described in terms of the vector-valued operators Tr_K^α defined on $W^{\alpha,2}$ by the expression

$$\mathrm{Tr}_K^\alpha \varphi = \{D^\beta \varphi \restriction K \mid \varphi \in W^{\alpha,2} \ \text{ for } \ |\beta| \le \alpha - 1\},$$

where β is a multi-index. Clearly, from (12) it follows that

$$\overset{\circ}{W}{}^{\alpha,2}(\Omega) = \mathrm{Ker}(\mathrm{Tr}_K^\alpha). \tag{13}$$

Now we can formulate our main result, which is similar to Theorem 3.

Theorem 5. *Let $K \subset \mathbb{R}^n$ be a compact set of nonzero α-capacity, $C_\alpha(K) > 0$, where $\alpha \ge 1$ is integer. Then $\mathcal{M}_\alpha(\Omega) = \overset{\circ}{W}{}^{\alpha,2}(\Omega)$, $\Omega = \mathbb{R}^n \setminus K$ is dense in $W^{m,2}$, $m \le \alpha - 1$ if and only if the m-capacity of K is zero,*

$$W^{m,2} \sqsupset \overset{\circ}{W}{}_2^\alpha(\Omega) \Longleftrightarrow C_m(K) = 0. \tag{14}$$

Proof. At first we remark that by the Sobolev embedding theorem the m-capacity of a single point $x \in \mathbb{R}^n$ is strictly positive, $C_m(\{x\}) > 0$, if $m > n/2$. Hence, the condition $C_m(K) = 0$ is possible only if $m \le n/2$. Thus, let $m \le n/2$ and $C_m(K) = 0$. Then the following statement is true (see Theorem 9.9.1 [1]). For each $h \in W^{m,2}$, any $\varepsilon > 0$, and every neighborhood V of a compact K there exists a function

$$\varphi \in C_0^\infty(V), \quad 0 \le \varphi \le 1, \quad \varphi(x) = 1, \quad x \in K,$$

such that $\|\varphi h\|_{W^{m,2}} < \varepsilon$. Consider now a sequence of functions φ_n satisfying the above condition with $\varepsilon_n \to 0$, $n \to \infty$,

$$\|\varphi_n h\|_{W^{m,2}} < \varepsilon_n. \tag{15}$$

By (15) for the sequence $\check{h}_n := \varphi_n h$ we have that $\check{h}_n \longrightarrow 0$ in $W^{m,2}$. Then the sequence $h_n = (1 - \varphi_n) h \in W^{m,2}$ converges to h as well as for $h = h_n + \check{h}_n$. In fact, each element $h_n \in \overset{\circ}{W}{}^{m,2}(\Omega)$, since by the construction $\mathrm{Tr}_K^m h_n = 0$, i.e., $h_n \in \mathrm{Ker}\mathrm{Tr}_K^m$ (see (13)). Take now a sequence of functions $\psi_n \in C_0^\infty(\Omega)$ such that $\|\psi_n - h_n\|_{\overset{\circ}{W}{}^{m,2}} < \varepsilon_n$. Since $\psi_n \in \overset{\circ}{W}{}_\alpha^m(\Omega)$, due to $\check{h}_n \longrightarrow 0$, we obtain that $\|\psi_n - h_n\|_{W^{m,2}} \to 0$, $n \to \infty$. This proves the density $\overset{\circ}{W}{}^{\alpha,2}(\Omega)$ in $W^{m,2}$, that is, the implication from (14) in one direction is true.

Let us prove the inverse implication. Let $\overset{\circ}{W}{}^{\alpha,2}(\Omega) \sqsubset W^{m,2}$. Assume that $C_m(K) > 0$. Then by Theorem 4, in $W^{m,2}$, there exists an element φ_K belonging to the subspace $\mathcal{N}_m(K)$ which is orthogonal to $\mathcal{M}_m(K) = \overset{\circ}{W}{}^{m,2}(\Omega)$. This gives $(\varphi_K, \overset{\circ}{W}{}^{\alpha,2}(\Omega))_m = 0$ since $\overset{\circ}{W}{}^{\alpha,2}(\Omega) \subset W^{m,2}(\Omega)$. However for $\varphi_K \ne 0$ this is impossible due to density of $\overset{\circ}{W}{}^{m,2}(\Omega)$ in $W^{m,2}$. Thus we obtain a contradiction and therefore $C_m(K) = 0$. □

In applications to mathematical physics (see, for example, [12, 13]) the reduced variant of Theorem 5 is usually used.

Example. Let $K = \{y_i \in \mathbb{R}^n\}_{i=1}^{l}$, $l < \infty$. Then $C_\alpha(K) > 0$ for all $\alpha > n/2$. In this case the subspace $\overset{\circ}{W}{}^{\alpha,2}(\mathbb{R}^n \setminus K)$ is dense in $\overset{\circ}{W}{}^{m,2}(\mathbb{R}^n)$, $m \le \alpha - 1$ if $m < n/2$.

We refer to [9, 10] and also [7, 11] for similar results connected with the notion of capacity.

References

1. D. Adams, L. Hedberg, *Function Spaces and Potential Theory*, Fundamental Principles of Mathematical Sciences, Vol. 314, Springer-Verlag, Berlin, 1996.
2. S. Albeverio, W. Karwowski, V. Koshmanenko, *Square powers of singularly perturbed operators*, Math. Nachr. **173** (1995), 5–24.
3. S. Albeverio, R. Bozhok, M. Dudkin, V. Koshmanenko, *Dense subspaces in scales of Hilbert spaces*, Methods Funct. Anal. Topology **11** (2005), no. 2, 156–169.
4. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
5. Yu. M. Berezanskii, *Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables*, Amer. Math. Soc., Providence, RI, 1986. (Russian edition: Naukova Dumka, Kiev, 1978)
6. R. V. Bozhok, V. D. Koshmanenko, *Parametrization of supersingular perturbations in the method of rigged Hilbert spaces*, Russ. J. Math. Phys. **14** (2007), no. 4, 409–416.
7. J. Brasche, *On eigenvalues and eigensolutions of the Schrödinger equation on the complement of a set with classical capacity zero*, Methods Funct. Anal. Topology **9** (2003), no. 3, 189–206.
8. M. Demuth, I. McGillivray, A. Noll, *Capacity and spectral theory*, Spectral Theory, Microlocal Analysis, Singular Manifolds (Eds. M. Demuth, E. Schrohe, B.-W. Schulze, and J. Sjostrand), Akademie Verlag, Berlin, 1997, pp. 12–77.
9. M. E. Dudkin, *Invariant symmetric restrictions of a selfadjoint operator.* II, Ukrain. Mat. Zh. **50** (1998), no. 6, 781–791. (Ukrainian); English transl. Ukrainian Math. J. **50** (1999), no. 6, 888–900.
10. N. Dudkin, *The closability of singularly perturbed operators by capacity methods*, Naukovi visty NTUU KPI (1999), no. 2, 122–125. (Ukrainian)
11. N. Dudkin, *About the capacity concept of the symmetric operator domain*, Naukovi visty NTUU KPI (2005), no. 5, 140–142. (Ukrainian)
12. V. D. Koshmanenko, *Towards the rank-one singular perturbations of self-adjoint operators*, Ukrain. Mat. Zh. **43** (1991), no. 11, 1559–1566. (English); English transl. Ukrainian Math. J. **43** (1992), no. 11, 1450–1457.
13. V. D. Koshmanenko, *Singular Quadratic Forms in Perturbation Theory*, Kluwer Academic Publishers, Dordrecht—Boston—London, 1999.
14. V. D. Koshmanenko, M. E. Dudkin, *The Rigged Spaces Method in Singular Perturbation Theory of Self-Adjoint Operators*, Zb. prac' Inst. Mat. NAN Ukr., Vol. 96, Institute of Mathematics NAN of Ukraine, Kyiv, 2013.
15. V. G. Maz'ja, *Sobolev Spaces*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985.
16. A. Noll, *Domain perturbations, shift of eigenvalues and capacity*, J. Funct. Anal. **170** (2000), no. 1, 246–263.
17. A. Noll, *Domain Perturbations and Capacity in General Hilbert Spaces and Applications to Spectral Theory*, Oper. Theory Adv. Appl., Vol. 145, Birkhäuser, Basel, 2003, pp. 281–362.

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