ARENS ALGEBRAS OF MEASURABLE OPERATORS FOR MAHARAM TRACES

A. A. ALIMOV

ABSTRACT. We study order and topological properties of the non-commutative Arens algebra associated with arbitrary Maharam trace.

1. INTRODUCTION

The integration theory for traces and weights, given on von Neumann algebras, is one of the central objects for numerous investigations connected with operator algebras and their applications. Development of the non-commutative theory begun with the work by I. Segal [22], where the author introduced the $*$-algebra $S(M)$ of all measurable operators affiliated with a von Neumann algebra M being a non-commutative analog of the $*$-algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable complex functions given on a measurable space (Ω, Σ, μ). This algebra $S(M)$ has became a base for construction of the general theory of non-commutative L^p spaces associated with a faithful normal semi-finite trace μ given on M (see, for example, [25, 13, 19]. The Banach spaces $L^p(M, \tau), p \geq 1$, are ideal linear subspaces in the $*$-algebra $S(M, \tau)$ of all τ-measurable operators affiliated with M. The $*$-algebra $S(M, \tau)$, introduced in [18], is itself a $*$-subalgebra in $S(M)$ and it coincides with the completion of M with respect to the topology of convergence in measure generated by the trace τ. It should be noted, both *-algebras $S(M)$ and $S(M, \tau)$ are used for meaningful examples of EW^*-algebras of unbounded operators that are important in the general theory of algebras of unbounded operators (see, for example [3]). Interesting examples of EW^*-algebras are also the Arens *-algebras $L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau)$. The Arens algebras $L^\omega(M, \tau)$ were considered at first in [4] for the case of $M = L^\infty(0, 1)$. Non-commutative Arens algebras were introduced by A. Inoue in [15], and then properties of such algebras were studied in [1, 27].

Due to presence of center-valued traces on finite von Neumann algebras, it is natural to extend the integration theory for traces with values in a complex order-complete lattices $F_{\mathbb{C}} = F \oplus iF$.

If the original von Neumann algebra is commutative, then constructing of $F_{\mathbb{C}}$-valued integration for it is a component part of investigations of properties of order-continuous positive mappings of vector lattices. The theory of such mappings was described in details in [16], chapter 4. Operators, having the Maharam property, are important among these mappings. L^p-spaces associated with such operators are significant examples of Banach-Kantorovich vector lattices.

For non-commutative von Neumann algebras M, properties of the spaces

$$(L^p(M, \Phi), \| \cdot \|_p),$$

constructed from a $F_{\mathbb{C}}$-valued trace Φ, were considered in [14] and [6] in the case where $F_{\mathbb{C}}$ is a von Neumann subalgebra in the center of the algebra, and the trace Φ has the following modularity property: $\Phi(zx) = z\Phi(x)$ for all $z \in F_{\mathbb{C}}, x \in M$.

The modularity property implies immediately that if $0 \leq f \leq \Phi(x)$, $f \in F$, $x \in M$, then there exists $y \in M$, $0 \leq y \leq x$, such that $\Phi(y) = f$. It means that the trace Φ possesses the Maharam property (compare [16], 3.4.1).

Faithful normal traces Φ on a von Neumann algebra M with values in arbitrary complex order-complete vector lattice were considered in [9], where, in particular, a full description of such traces is given for the case when Φ is a Maharam trace. In [9], the non-commutative L^1-space $L^1(M, \Phi) \subset S(M)$ associated with a Maharam trace Φ was constructed with the help of topology of convergence in measure, and it was established that $L^1(M, \Phi)$ is a Banach-Kantorovich space. Later in [10], non-commutative L^p-spaces $L^p(M, \Phi)$ were defined for all $p > 1$. The problem of construction and description of properties of the Arens algebras $L^\omega(M, \Phi)$, associated with a Maharam trace Φ has arisen naturally. Such a problem is solved in [2] for the case where Φ takes values in $S(A)$, where A is a von Neumann subalgebra in the center $Z(M)$ of an algebra M.

In the present paper, we study order and topological properties of the Arens algebra $L^\omega(M, \Phi)$ associated with an arbitrary Maharam trace Φ.

Necessary and sufficient conditions are determined that provide local convexity of the topology $\tau_\omega(M, \Phi)$ in $L^\omega(M, \Phi)$ generated by the system of norms $\{\|\cdot\|_p\}_{p \geq 1}$. A criterion for coincidence of the topology $\tau_\omega(M, \Phi)$ with the (o)-topology in the ordered linear space $L_h^\omega(M, \Phi) = \{x \in L^\omega(M, \Phi) : x = x^*\}$ is established.

We use terminology and notations of the theory of von Neumann algebras from [23, 24], the theory of measurable operators from [22, 17] and the theory of vector lattices from [16].

2. Preliminaries

Let H be a Hilbert space over the field of complex numbers $\mathbb{C}$, $B(H)$ be a $*$-algebra of all bounded linear operators acting in H, $\mathbf{1}$ be an identical operator in H, and let M be a von Neumann subalgebra in $B(H)$. Denote by $P(M) = \{p \in M : p^2 = p = p^*\}$ the lattice of all projects from M, and by $P_{fin}(M)$ the sublattice of all finite projections from M.

A closed linear operator x, affiliated with the von Neumann algebra M, having a dense domain $D(x) \subset H$, is called *measurable* with respect to M if there exists a sequence $\{p_n\}_{n=1}^\infty \subset P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset D(x)$ and $p_n^\perp = \mathbf{1} - p_n \in P_{fin}(M)$ for each $n = 1, 2, \ldots$

The set $S(M)$ of all measurable operators with respect to M is a $*$-algebra with the unit $\mathbf{1}$ over $\mathbb{C}$ with respect to the natural involution, multiplication on a scalar, and operations of the strong addition and strong multiplication obtained by closure of the usual operations [22]. It is clear, M is a $*$-subalgebra in $S(M)$.

If $x \in S(M)$, and $x = u \mid x \mid$ is the polar decomposition of the operator x where $\mid x \mid = (x^*x)^{\frac{1}{2}}$, u is the corresponding partial isometry from $B(H)$ for which u^*u is the right support for x, then $u \in M$ and $\mid x \mid \in S(M)$. The spectral family of projectors $\{E_\lambda(x)\}_{\lambda \in \mathbb{R}}$ of the self-adjoint operator $x \in S(M)$ is always contained in $P(M)$.

For any subset $E \subset S(M)$, we denote by E_h (E_+, respectively) the set of all self-adjoint (positive, respectively) operators from E.

Let M be a commutative von Neumann algebra. In this case, there exists a faithful normal semi-finite trace τ on M, and M is *-isomorphic to the W^*-algebra $L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded complex measurable functions given on a measurable space (Ω, Σ, μ) with a locally finite measure μ having the direct sum property (almost everywhere equal functions are identified). Moreover, $\mu(A) = \tau(\tilde{\chi}_A)$, $A \in \Sigma$ where $\tilde{\chi}_A$ is

the equivalence class containing the function χ_A (recall that $\chi_A(\omega) = 1$ for $\omega \in A$, and $\chi_A(\omega) = 0$ if $\omega \notin A$). In addition, the *-algebra $S(M)$ is identified with the *-algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable complex functions given on (Ω, Σ, μ) (almost everywhere equal functions are identified) [22]. Let us consider in $L^0(\Omega, \Sigma, \mu)$ the topology $t(M)$ of convergence locally in the measure, i.e., the Hausdorff topology endowing $L^0(\Omega, \Sigma, \mu)$ with the structure of a complete topological *-algebra, the base of zero neighborhoods of which is formed by the sets in the form of

$$(2.1) \quad W(B, \varepsilon, \delta) = \Big\{ f \in L^0(\Omega, \Sigma, \mu) : \text{ there exists a set } E \in \Sigma \text{ such that}$$
$$E \subseteq B, \mu(B \setminus E) \leq \delta, f\chi_E \in L^\infty(\Omega, \Sigma, \mu), \|f\chi_E\|_{L^\infty(\Omega, \Sigma, \mu)} \leq \varepsilon \Big\},$$

where $\varepsilon, \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$, $\|\cdot\|_{L^\infty(\Omega, \Sigma, \mu)}$ is the C^*-norm in $L^\infty(\Omega, \Sigma, \mu)$. The sets $W(B, \varepsilon, \delta)$ have the following ideality property: if $g \in L^0(\Omega, \Sigma, \mu)$, $f \in W(B, \varepsilon, \delta)$, and $|g| \leq |f|$, then $g \in W(B, \varepsilon, \delta)$.

Convergence of the net f_α to f in the topology $t(M)$ (notation: $f_\alpha \xrightarrow{t(M)} f$) means that $f_\alpha \chi_B \to f\chi_B$ by the measure μ for any $B \in \Sigma$ with $\mu(B) < \infty$. Evidently, the topology $t(M)$ is not changed if the trace τ is replaced with another faithful normal semi-finite trace on M. Therefore the topology is uniquely defined by the von Neumann algebra M itself. It is clear that the topology $t(M)$ is metrizable if and only if the algebra M is σ-finite, i.e., any set of nonzero mutually orthogonal projections at most countable.

Now let M be an arbitrary finite von Neumann algebra, $Z(M)$ be the center in M, and $\Phi_M : M \to Z(M)$ be a center-valued trace on M ([23], 7.11). Identify the center $Z(M)$ with the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ and $S(Z(M))$ with the $*$-algebra $L^0(\Omega, \Sigma, \mu)$. For arbitrary numbers $\varepsilon, \delta > 0$ and arbitrary set $B \in \Sigma$ with the measure $\mu(B) < \infty$, set

$$V(B, \varepsilon, \delta) = \Big\{ x \in S(M) : \text{ there exist } p \in P(M), z \in P(Z(M)) \text{ such that}$$
$$xp \in M, \|xp\|_M \leq \varepsilon, z^\perp \in W(B, \varepsilon, \delta), \text{ and } \Phi_M(zp^\perp) \leq \varepsilon z \Big\},$$

where $\|\cdot\|_M$ is the C^*-norm in M. In [17], § 3.5, it is shown that the system of sets

$$(2.2) \quad \Big\{ \{x + V(B, \varepsilon, \delta)\} : \ x \in S(M), \ \varepsilon, \delta > 0, \ B \in \Sigma, \ \mu(B) < \infty \Big\}$$

defines in $S(M)$ a Hausdorff vector topology $t(M)$, in which the sets (2.2) form a base of neighborhoods for the operator $x \in S(M)$. In addition, $(S(M), t(M))$ is a complete topological *-algebra. The topology $t(M)$ is called the *topology of convergence locally in measure* [1]. It is clear, the topology $t(M)$ induces in $S(Z(M))$ the topology $t(Z(M))$. Moreover, if $Z(M)$ is a σ-finite algebra, then $t(M)$ is metrizable.

The following criterion for convergence of nets in the topology $t(M)$ follows from [17], § 3.5.

Proposition 2.1. *A net $\{x_\alpha\}_{\alpha \in A} \subset S(M)$ converges to zero in the topology $t(M)$ if and only if $\Phi_M(E_\lambda^\perp(|\ x_\alpha\ |)) \xrightarrow{t(M)} 0$ for any $\lambda > 0$.*

Let F be an order complete vector lattice, $F_{\mathbb{C}} = F \oplus iF$ be the complexification of F, where i is the imaginary unit. As usual, for an element $z = \alpha + i\beta \in F_{\mathbb{C}}$, $\alpha, \beta \in F$, the adjoint element is defined as $\bar{z} = \alpha - i\beta$, and the module $|\ z\ |$ is defined as $|\ z\ | := \sup\{Re(e^{i\theta}z) : 0 \leq \theta < 2\pi\}$ ([16], 1.3.13).

A linear mapping Φ from a von Neumann algebra M into $F_{\mathbb{C}}$ is said to be an $F_{\mathbb{C}}$-valued trace if $\Phi(x^*x) = \Phi(xx^*) \geq 0$ for all $x \in M$. It is clear, $\Phi(M_h) \subset F$, $\Phi(M_+) \subset F_+ = \{a \in F, \ a \geq 0\}$.

The trace Φ is said to be *faithful* if the equality $\Phi(x^*x) = 0$ implies $x = 0$. As well as for numerical traces (see, for example, ([24], chapter V, § 2), it is established that if

there is a faithful $F_{\mathbb{C}}$-valued trace on a von Neumann algebra M, then this algebra is finite. Let us list some necessary properties of faithful traces $\Phi : M \to F_{\mathbb{C}}$.

Proposition 2.2. ([8]). *For any $x, y, a, b \in M$ the following relations hold:*
1. $\Phi(x^*) = \overline{\Phi(x)}$;
2. $\Phi(xy) = \Phi(yx)$;
3. $\Phi(\mid x^* \mid) = \Phi(\mid x \mid)$;
4. $\mid \Phi(axb) \mid \leq \parallel a \parallel_M \parallel b \parallel_M \Phi(\mid x \mid)$;
5. *If x_n, $x \in M$, and $\parallel x_n - x \parallel_M \to 0$, then $\mid \Phi(x_n) - \Phi(x) \mid$ converges in F to zero with the regulator $\Phi(\mathbf{1})$;*
6. $\Phi(\mid x + y \mid) \leq \Phi(\mid x \mid) + \Phi(\mid y \mid)$.

We say that a trace $\Phi : M \to F_{\mathbb{C}}$ possesses the Maharam property if for any $x \in M_+$, $0 \leq f \leq \Phi(x)$, $f \in F$, there exists $y \in M_+$ such that $y \leq x$ and $\Phi(y) = f$.

A trace Φ is said to be *normal* if x_α, $x \in M_h$, $x_\alpha \uparrow x$, imply $\Phi(x_\alpha) \uparrow \Phi(x)$. A faithful normal $F_{\mathbb{C}}$-valued trace Φ possessing the Maharam property is said to be *the Maharam trace* ([8, 9]).

Let $\mathbf{1}_F$ be a weak unit in F. Denote by $B(F)$ the complete Boolean algebra of unit elements in F with respect to $\mathbf{1}_F$. Let Q be the Stone compact for $B(F)$, and let $C_\infty(Q)$ be an extended order complete vector lattice of all continuous functions $f : Q \to [-\infty; +\infty]$ taking the values $\pm\infty$ on nowhere dense sets from Q. Identify F with the fundament in the lattice $C_\infty(Q)$ consisting of the algebra $C(Q)$ of all continuous functions on Q, in addition $\mathbf{1}_F$ is identified with the function which is identically equal to the unit ([16], 1.4.4).

The following theorem from [9] gives description of Maharam traces.

Theorem 2.1. ([9]). *Let Φ be a $F_{\mathbb{C}}$-valued trace on a von Neumann algebra M. Then there exist a von Neumann subalgebra A in the center of $Z(M)$, an $*$-isomorphism ψ from A onto the $*$-algebra $C(Q)_{\mathbb{C}} = C(Q) \oplus iC(Q)$, a positive normal linear operator $\mathcal{E}$ from $Z(M)$ onto A with $\mathcal{E}(\mathbf{1}) = (\mathbf{1})$, $\mathcal{E}^2 = \mathcal{E}$, such that*
1. $\Phi(x) = \Phi(\mathbf{1}) \cdot \psi(\mathcal{E}(\Phi_M(x)))$ *for all $x \in M$;*
2. $\Phi(zy) = \Phi(z\mathcal{E}(y))$ *for all $z, y \in Z(M)$;*
3. $\Phi(zy) = \psi(z)\Phi(y)$ *for all $z \in A$, $y \in M$.*

Theorem 2.1 implies that the $*$-algebra $B = C(Q)_{\mathbb{C}}$ is a commutative von Neumann algebra and the $*$-algebra $(C_\infty(Q))_{\mathbb{C}}$ is identified with the $*$-algebra $S(B)$. We denote the unit of the algebra B by $\mathbf{1}_B$ (it coincides with the weak unit $\mathbf{1}_F$).

Let Φ be $S(B)$-valued Maharam trace on the von Neumann algebra M. Recall definition of the space $L^1(M, \Phi)$ [11]. We say that a net $\{x_\alpha\} \subset S(M)$ converges to $x \in S(M)$ by a trace Φ (notation: $x_\alpha \xrightarrow{\Phi} x$) if

$$\Phi(E_\lambda^\perp(\mid x_\alpha - x \mid)) \xrightarrow{t(B)} 0$$

for $\lambda > 0$. It was shown in [11] that $x_\alpha \xrightarrow{\Phi} x \Leftrightarrow x_\alpha \xrightarrow{t(M)} x$ (compare with Proposition 2.1).

An operator $x \in S(M)$ is said to be Φ-*integrable* if there exists a sequence $\{x_n\} \subset M$ such that $x_n \xrightarrow{\Phi} x$ and $\Phi(\mid x_n - x_m \mid) \xrightarrow{t(B)} 0$ at $n, m \to \infty$. It follows from inequalities $\mid \Phi(x_n) - \Phi(x_m) \mid \leq \Phi(\mid x_n - x_m \mid)$ and completeness of the topological $*$-algebra $(S(M), t(M))$ that there exists an element $\hat{\Phi}(x) \in S(B)$ such that $\Phi(x_n) \xrightarrow{t(B)} \hat{\Phi}(x)$. It is shown in [9] that this limit $\hat{\Phi}(x)$ does not depend on the choice of a sequence $\{x_n\} \subset M$, for which $x_n \xrightarrow{\Phi} x$ and $\Phi(\mid x_n - x_m \mid) \xrightarrow{t(B)} 0$. It is clear, any operator x from M is Φ-integrable and $\hat{\Phi}(x) = \Phi(x)$.

Denote by $L^1(M, \Phi)$ the set of all Φ-integrable operators from $S(M)$, and for each $x \in L^1(M, \Phi)$, set $\| x \|_{1,\Phi} = \hat{\Phi}(| x |)$. It is proved in [15] that $L^1(M, \Phi)$ is a linear space in $S(M)$, in addition the following statement holds.

Theorem 2.2. ([8, 9]).

(i) The mapping $\hat{\Phi} : L^1(M, \Phi) \to S(B)$ has the following properties:

1) $\hat{\Phi}$ is a linear positive mapping, in particular, $\hat{\Phi}(x^*) = (\hat{\Phi})^*$;

2) $\hat{\Phi}(xy) = \hat{\Phi}(yx)$ and $| \hat{\Phi}(xy) | \leq \| x \|_M \hat{\Phi}(y)$ for any $x \in M$, $y \in L^1(M, \Phi)$;

3) $x \in L_1(M, \Phi) \Leftrightarrow |x| \in L_1(M, \Phi)$, moreover $\hat{\Phi}(| x^* |) = \hat{\Phi}(| x |)$ and $\hat{\Phi}(| x |) = 0 \Leftrightarrow x = 0$;

4) $\hat{\Phi}(| x + y |) \leq \hat{\Phi}(| x |) + \hat{\Phi}(| y |)$;

5) If $| y | \leq | x |$, $y \in S(M)$, $x \in L^1(M, \Phi)$, then $y \in L^1(M, \Phi)$ and $\|y\|_{1,\Phi} \leq \|x\|_{1,\Phi}$.

(ii) $S(A) \cdot L^1(M, \Phi) \subset L^1(M, \Phi)$, moreover, $\hat{\Phi}(zx) = \psi(z)\hat{\Phi}(x)$ for all $z \in S(A)$, $x \in L^1(M, \Phi)$, where ψ is an extension of the $*$-isomorphism from Theorem 2.1 to an $*$-isomorphism from $S(A)$ onto $S(B)$.

(iii) $(L^1(M, \Phi), \| \cdot \|_{1,\Phi})$ is a Banach-Kantorovich space.

For any $p > 1$, set $L^p(M, \Phi) = \{x \in S(M) : | x |^p \in L^1(M, \Phi)\}$ and $\| x \|_{p,\Phi} = \hat{\Phi}(| x |^p)^{1/p}$ if $x \in L^p(M, \Phi)$.

We need the following properties of the spaces $(L^p(M, \Phi), \| x \|_{p,\Phi})$ from [10].

Theorem 2.3. ([10]). (i) If an element x belongs to $L^p(M, \Phi)$, then the elements x^* and $| x |$ also belong to $L^p(M, \Phi)$ and $\| x^* \|_{p,\Phi} = \| | x | \|_{p,\Phi} = \| x \|_{p,\Phi}$;

(ii) If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in L^p(M, \Phi)$, $y \in L^q(M, \Phi)$ then $xy \in L^1(M, \Phi)$ and $\| xy \|_{1,\Phi} \leq \| x \|_{p,\Phi} \| y \|_{q,\Phi}$;

(iii) $ML^p(M, \Phi)M \subset L^p(M, \Phi)$ and $\| axb \|_{p,\Phi} \leq \| a \|_M \| b \|_M \| x \|_{p,\Phi}$ for all $a, b \in M$, $x \in L^p(M, \Phi)$;

(iv) If $| y | \leq | x |$, $y \in S(M)$, $x \in L^p(M, \Phi)$, then $y \in L^p(M, \Phi)$ and $\| y \|_{p,\Phi} \leq \| x \|_{p,\Phi}$;

(v) $L^p(M, \Phi)$ is a linear subspace in $S(M)$, moreover, $M \subset L^p(M, \Phi)$ and $(L^p(M, \Phi), \| \cdot \|_{p,\Phi})$ is a Banach-Kantorovich space;

(vi) $S(A)L^p(M, \Phi) \subset L^p(M, \Phi)$, moreover, $\| zx \|_{p,\Phi} = \psi(z) \| x \|_{p,\Phi}$ for all $z \in S(A)$, $x \in L^p(M, \Phi)$, where ψ is the $*$-isomorphism from Theorem 2.2 (ii);

(vii) If $\{x_\alpha\} \subset L^p_+(M, \Phi)$, $\{x_\alpha\} \downarrow 0$, then $\| x \|_{p,\Phi} \downarrow 0$.

3. THE ARENS ALGEBRAS $\mathbf{L}^\omega(\mathbf{M}, \mathbf{\Phi})$

Let Φ be a Maharam trace on a von Neumann algebra M with values in $S(B)$. To define the Arens algebras associated with the trace Φ, we need the following version of the Hölder inequality.

Theorem 3.1. If $p, q, r > 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $x \in L^p(M, \Phi)$, $y \in L^q(M, \Phi)$, then $xy \in L^r(M, \Phi)$ and $\| xy \|_{r,\Phi} \leq \| x \|_{p,\Phi} \| y \|_{q,\Phi}$.

Proof. Let Q be the Stone compact corresponding to a complete Boolean algebra $P(B)$ of all project from B. Identify the algebra $S_h(B)$ with the algebra $C_\infty(Q)$ of all continuous functions $f : Q \to [-\infty, +\infty]$ taking values $\pm\infty$ only on nowhere dense sets from Q. As well as in [10], one can show that the element $\Phi(\mathbf{1})$ is reversible in $S(B)$, and a finite trace is defined on M for each $t \in Q$ by the equality $\varphi_t(x) = (\Phi(\mathbf{1})^{-1}\Phi(x))(t)$. According to ([12], 6.2.2), the set $N_t = \{x \in M : \varphi_t(x^*x) = 0\}$ is a two-sided $*$-ideal in M. Consider the factor space M/N_t with the scalar product $([x], [y]) = \varphi_t(y^*x)$ where $[x], [y]$ are the equivalence classes from M/N_t with representatives x and y, respectively.

Denote by $(H_t, (\cdot, \cdot)_t)$ the Hilbert space being the completion of $(M/N_t, (\cdot, \cdot)_t)$. Define the $*$-homomorphism $\pi_t : M \to B(H_t)$, setting $\pi_t(x)([y]) = [xy]$, $x, y \in M$. Denote by $U_t(M)$ the von Neumann algebra in $B(H_t)$ generated by the operators $\pi_t(x)$, $x \in M$.

According to ([11], 6.2), there exists a faithful normal semi-finite trace τ_t on $(U_t(M))_+$ such that

$$\tau_t(\pi_t(x^2)) = ([x],[x]) = \varphi_t(x^*x)$$

for all $x \in M_+$. Hence, $\tau_t(\pi_t(y)) = \varphi_t(y)$ for all $y \in M_+$, moreover, $\tau_t(\mathbf{1}_{B(H_t)}) = \varphi_t(\mathbf{1}) < \infty$, i.e. τ_t is a faithful normal finite trace on $U_t(M)$.

Let $L^p(U_t(M), \tau_t)$ be a noncommutative L^p-space associated with a numerical trace τ_t. According to ([13], Theorem 4.9), we have

$$\tau_t(|\pi_t(xy)|^r)^{\frac{1}{r}} \le \tau_t(|\pi_t(x)|^p)^{\frac{1}{p}} \tau(|\pi_t(y)|^q)^{\frac{1}{q}}$$

for all $x, y \in M$.

Since $\pi_t(\mid x \mid) = \mid \pi_t(x) \mid$ for all $x \in M$, then by virtue of ([11],1.5.3) we have $\pi_t(\mid x \mid^p) = (\pi_t(\mid x \mid))^p$. Hence,

$$\varphi_t(\mid xy \mid^r)^{\frac{1}{r}} \le \varphi_t(\mid x \mid^p)^{\frac{1}{p}} \varphi_t(\mid y \mid^q)^{\frac{1}{q}}$$

or

$$[(\Phi(\mathbf{1}))^{-1}\Phi(\mid xy \mid^r))(t)]^{\frac{1}{r}} \le [((\Phi(\mathbf{1}))^{-1}\Phi(\mid x \mid)^p)(t)]^{\frac{1}{p}}[((\Phi(\mathbf{1}))^{-1}\Phi(\mid y \mid)^q))(t)]^{\frac{1}{q}}$$

for all $t \in Q$ and $x, y \in M$. It means that

$$((\Phi(\mathbf{1}))^{-1}\Phi(\mid xy \mid^r))^{\frac{1}{r}} \le ((\Phi(\mathbf{1}))^{-1}\Phi(\mid x \mid^p))^{\frac{1}{p}}((\Phi(\mathbf{1}))^{-1}\Phi(\mid y \mid^q))^{\frac{1}{q}}.$$

Multiplying the both parts of the last inequality by $\Phi(\mathbf{1})$, we obtain

$$\| xy \|_{r,\Phi} \le \| x \|_{p,\Phi} \| y \|_{q,\Phi}$$

for any $x, y \in M$.

Let now $x \in L^p(M, \Phi)$, $y \in L^q(M, \Phi)$. Let us show that $xy \in L^r(M, \Phi)$ and $\| xy \|_{r,\Phi} \le \| x \|_{p,\Phi} \| y \|_{q,\Phi}$. Set $a_n = E_n(|x|)|x|, b_n = E_n(|y|)|y|$. We have $a_n, b_n \in M_+$ and $a_n \uparrow |x|, b_n \uparrow |y|$, moreover, $a_n \xrightarrow{\Phi} |x|, b_n \xrightarrow{\Phi} |y|$.

Let $x = u \mid x \mid (y = v \mid y \mid$, respectively) be the polar decomposition for x (for y, respectively) with the unitary $u \in M(v \in M)$. It is clear, $ua_n \xrightarrow{\Phi} x, vb_n \xrightarrow{\Phi} y$, and therefore $ua_n vb_n \xrightarrow{\Phi} xy$, in addition, $ua_n vb_n \in M$ for all n. Since the operations $z \mapsto |z|$, $v \mapsto v^r$, $v \ge 0$ are continuous in the topology $t(M)$, we get $|ua_n vb_n|^r \xrightarrow{t(M)} |xy|^r$. The Fatou theorem [11, Theorem 3.2 (iv)] implies that $xy \in L^\omega(M, \Phi)$ and

$$\| xy \|_{r,\Phi}^r = \widehat{\Phi}(\mid xy \mid^r) \le \sup_{n \ge 1} \widehat{\Phi}(\mid ua_n vb_n \mid^r) = \sup_{n \ge 1}\{\| (ua_n)(vb_n) \|_{r,\Phi}^r\}$$

$$\le \sup_{n \ge 1} \| ua_n \|_{p,\Phi}^r \| vb_n \|_{q,\Phi}^r = \sup_{n \ge 1} \| a_n \|_{p,\Phi}^r \| b_n \|_{q,\Phi}^r \le \| x \|_{p,\Phi}^r \| y \|_{q,\Phi}^r,$$

i.e. $\| xy \|_{r,\Phi} \le \| x \|_{p,\Phi} \| y \|_{q,\Phi}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Suppose $L^\omega(M, \Phi) = \bigcap_{p \ge 1} L^p(M, \Phi)$. Theorems 2.3 and 3.1 imply the following

Corollary 3.1. (i) $L^\omega(M, \Phi)$ *is a $*$-subalgebra in $S(M)$, and $M \subset L^\omega(M, \Phi)$;*
(ii) *If $y \in S(M)$, $x \in L^\omega(M, \Phi)$, $|y| \le |x|$, then $y \in L^\omega(M, \Phi)$;*
(iii) *$S(A) \subset S(A)L^\omega(M, \Phi) \subset L^\omega(M, \Phi)$ where A is a $*$-subalgebra in $Z(M)$, $*$-isomorphic to B (see Theorem 3.1), i.e. $L^\omega(M, \Phi)$ is left and right $S(A)$-module.*

An $*$-algebra $L^\omega(M, \Phi)$ is said to be *the Arens algebra* associated with the von Neumann algebra M and the Maharam trace Φ. If $\dim M < \infty$, then $M = S(M)$, and therefore $M = L^\omega(M, \Phi)$. The converse assertion is also true.

Proposition 3.1. *If $M = L^\omega(M, \Phi)$, then $\dim M < \infty$.*

Proof. Suppose that $\dim M = \infty$. Then there exists a countable set of mutually orthogonal nonzero projections $\{q_n\}_{n=1}^{\infty} \subset P(M)$ for which $\sup\{q_n\} = \mathbf{1}$. Since $\Phi(\mathbf{1}) = \sum_{n=1}^{\infty} \Phi(q_n)$, the sequence $\Phi(q_n)$ (o)-converges to zero in $S_h(B)$. Consider the spectral family of projections $\{E_\lambda(\Phi(q_n))\}_{\lambda>0}$ and, using the inequality $\lambda E_\lambda^\perp(\Phi(q_n)) \le \Phi(q_n)$, choose the sequence of numbers $n_1 < \cdots < n_k < \cdots$ such that $e_k = E_{\frac{1}{k^2}}(\Phi(q_{n_k})) \ne 0, k = 1, 2\ldots$ Let $\psi : A \to B$ be the $*$-isomorphism from Theorem 2.1. Set $x = \sum_{k=1}^{\infty}(\ln(k))q_{n_k}\psi^{-1}(e_k)$ (the series converges in the topology $t(M)$). It is clear, $x \in S_+(M)$ and $\Phi(x^p) = \sum_{k=1}^{\infty}(\ln(k)^p)\Phi(q_{n_k})e_k \le \sum_{k=1}^{\infty}\frac{(\ln(k))^p}{k^2}e_k \le (\sum_{k=1}^{\infty}\frac{(\ln(k))^p}{k^2})\mathbf{1_B} \in B$, i.e. $x \in L^p(M,\Phi)$ for all $p \ge 1$. It means that $x \in L^\omega(M,\Phi)$. Since $\Phi(q_{n_k}\psi^{-1}(e_k)) = e_k\Phi(q_{n_k}) \ne 0$, we get $q_{n_k}\psi^{-1}(e_k) \ne 0$ for all $k = 1, 2, \ldots$, i.e. $x \notin M$. $\qquad\square$

Remark 3.1. *If $M = Z(M) = A$, then by Corollary 3.1, (iii), $S(M) = L^\omega(M,\Phi)$.*

Denote by $\mathbf{U}$ the base of zero neighborhoods in $(S(B), t(B))$ consisting of ideal sets in the form (2.1). For any $V \in \mathbf{U}$, $p \ge 1$, set

$$W(V, p) = \{x \in L^p(M, \Phi) : \| x \|_{p,\Phi} \in W\}.$$

According to ([24], chapter I, § 1), there exists a topology $\tau_p(M,\Phi)$ in $L^p(M,\Phi)$, with respect to which $L^p(M,\Phi)$ is a Hausdorff topological vector space, and the system of sets

$$\{x + W(V, p) : V \in \mathbf{U}\}$$

forms the base of neighborhoods of the operator $x \in L^p(M,\Phi)$. Convergence of the net $\{x_\alpha\} \subset L^p(M,\Phi)$ to the operator $x \in L^p(M,\Phi)$ (notation: $x_\alpha \xrightarrow{\tau_p(M,\Phi)} x,$) means that $\|x_\alpha - x\|_{p,\Phi} \xrightarrow{t(B)} 0$.

Since $L^p(M,\Phi)$ is a Banach-Kantorovich space (Theorem 2.3 (v)), we have that $L^p(M,\Phi) - \tau_p(M,\Phi)$ is complete [7], i.e. any $\tau_p(M,\Phi)$-fundamental net from $L^p(M,\Phi)$ converges in $(L^p(M,\Phi), \tau_p(M,\Phi))$.

Now consider the set $W_\omega(V, p) = L^\omega(M,\Phi) \cap W(V, p)$ and denote by $\tau_\omega(M,\Phi)$ the Hausdorff vector topology in $L^\omega(M,\Phi)$, in which the system of sets

$$\{x + W(V, p) : V \in \mathbf{U}, p \ge 1\}$$

forms the base of neighborhoods of the element $x \in L^\omega(M,\Phi)$. For a net $\{x_\alpha\} \in L^\omega(M,\Phi)$, its convergence $x_\alpha \xrightarrow{\tau_\omega(M,\Phi)} x \in L^\omega(M,\Phi)$ means that $\|x_\alpha - x\|_{p,\Phi} \xrightarrow{t(B)} 0$ for all $p \ge 1$.

If $1 \le r < p < \infty$, $q = \frac{rp}{p-r}$, then $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and by Theorem 3.1, we have

$$(3.1) \qquad \|x\|_{r,\Phi} \le \|x\|_{p,\Phi}\|\mathbf{1}\|_{q,\Phi} = \Phi(\mathbf{1})^{\frac{p-r}{rp}}\|x\|_{p,\Phi}$$

for all $x \in L^\omega(M,\Phi)$. It means that the topology $\tau_\omega(M,\Phi)$ has the base of zero neighborhoods consisting of sets in the form of $W_\omega(V, n)$, where $V \in \mathbf{U}$, $n \in \mathbb{N}$, $\mathbb{N}$ is the set of all natural numbers, i.e. the following statement is valid.

Proposition 3.2. *If B is a σ-finite von Neumann algebra, then $\tau_\omega(M,\Phi)$ is a metrizable topology.*

Let A be a von Neumann subalgebra in $Z(M)$ and let ψ be an $*$-isomorphism from $S(A)$ onto $S(B)$ being the extension of the isomorphism from Theorem 2.1. If $x_\alpha, x \in S(A)$, then $x_\alpha, x \in L^\omega(M,\Phi)$ (Corollary 3.1 (iii)), and by virtue of the equality $\|x_\alpha - x\|_{P,\Phi} = \psi(x_\alpha - x)\|\mathbf{1}\|_{P,\Phi}$ (Theorem 2.3 (vi)), the convergence $x_\alpha \xrightarrow{\tau_\omega(M,\Phi)} x$ is equivalent to the convergence $\psi(x_\alpha) \xrightarrow{t(B)} \psi(x)$. Hence,

$$(3.2) \qquad\qquad x_\alpha \xrightarrow{\tau_\omega(M,\Phi)} x \iff x_\alpha \xrightarrow{t(A)} x,$$

i.e. the topology $\tau_\omega(M, \Phi)$ induces on $S(A)$ the topology $t(A)$. Therefore metrizability of the topology $\tau_\omega(M, \Phi)$ implies metrizability of the topology $t(A)$, that is equivalent to σ-finiteness of the von Neumann algebra B. Thus, taking into account Proposition 3.2, we obtain the following criterion for metrizability of $\tau_\omega(M, \Phi)$.

Theorem 3.2. *The following conditions are equivalent for the Arens algebra* $L^\omega(M, \Phi)$ *associated with the von Neumann algebra* M *and an* $S(M)$*-valued trace* Φ :
  (i). *The topology* $\tau_\omega(M, \Phi)$ *is metrizable;*
  (ii). *The von Neumann algebra* B *is* σ*-finite;*
  (iii). *The von Neumann algebra* M *is* σ*-finite.*

Proof. Implications $(i) \Leftrightarrow (ii)$ are already proved. Implication $(iii) \Longrightarrow (ii)$ is evident, since the σ-finiteness of the algebra B is equivalent to σ-finiteness of the von Neumann algebra $A \subset Z(M)$. Implication $(ii) \Longrightarrow (iii)$ is proved as well as in [1], where σ-finiteness of the algebra M is obtained using a center-valued trace $\Phi_M : M \longrightarrow Z(M)$ and σ-finiteness of the center of $Z(M)$. $\square$

Proposition 3.3. $(L^\omega(M, \Phi), \tau_\omega(M, \Phi))$ *is a complete topological* $*$*-algebra.*

Proof. Since $(L^p(M, \Phi), \tau_p(M, \Phi))$ is complete [7], for any $\tau_\omega(M, \Phi)$-fundamental net $\{x_\alpha\} \subset L^\omega(M, \Phi)$ there exists $x_p \in L^\omega(M, \Phi)$ such that $x_\alpha \xrightarrow{\tau_p(M,\Phi)} x_p$. The inequality (3.1) implies that for $1 \leq q < p < \infty$ the topology $\tau_q(M, \Phi)$ is majorized on $L^\omega(M, \Phi)$ by the topology $\tau_p(M, \Phi)$. Therefore the convergence $x_\alpha \xrightarrow{\tau_p(M,\Phi)} x_p$ implies the convergence $x_\alpha \xrightarrow{\tau_q(M,\Phi)} x_p$, what implies the equality $x_p = x_q$ at $1 \leq q < p < \infty$. Hence, $x := x_p$ is an element from $L^\omega(M, \Phi)$, and $x_\alpha \xrightarrow{\tau_\omega(M,\Phi)} x_p$. It means that $(L^\omega(M, \Phi), \tau_\omega(M, \Phi))$ is complete. Further, the equality $\|x\|_{p,\Phi} = \|x^*\|_{p,\Phi}$ (Theorem 2.3 (i)) and inequality $\|xy\|_p \leq \|x\|_{2p}\|y\|_{2p}$ (Theorem 3.1), imply that the involution operation is continuous in $(L^\omega(M, \Phi), \tau_p(M, \Phi))$, and the multiplication operation is continuous in both variables. Thus, $(L^\omega(M, \Phi), \tau_\omega(M, \Phi))$ is a complete topological $*$-algebra. $\square$

Theorem 3.2 and Proposition 3.3 imply the following

Corollary 3.2. *If* B *is a* σ*-finite von Neumann algebra, then* $(L^\omega(M, \Phi), \tau_\omega(M, \Phi))$ *is a complete metrizable topological* $*$*-algebra, in particular,* $(L^\omega(M, \Phi), \tau_\omega(M, \Phi))$ *is a* F*-space.*

Denote by $t_\omega(M)$ the topology on $L^\omega(M, \Phi)$ induced by the topology $t(M)$ from $S(M)$.

Proposition 3.4. (i) $t_\omega(M) \leq \tau_\omega(M, \Phi)$;
(ii) *If* $t_\omega(M) = \tau_\omega(M, \Phi)$, *then* $L^\omega(M, \Phi) = S(M)$.

Proof. (i). Let $\{x_\alpha\} \subset L^\omega(M, \Phi)$, and $x_\alpha \xrightarrow{\tau_\omega(M,\Phi)} 0$, i.e. $\|x_\alpha\|_{p,\Phi} \xrightarrow{t(B)} 0$ for all $p \geq 1$.

Since $\| |x| \| = \| x \|_{p,\Phi}$ (Theorem 2.3 (i)), we obtain $\| |x_\alpha| \|_{p,\Phi} \xrightarrow{t(B)} 0$. Let $\{E_\lambda(|x_\alpha|)\}_{\lambda > 0}$ be the spectral family of projections for $| x_\alpha |$. By virtue of the inequality

$$\Phi(E_\lambda^\perp(| x_\alpha |)) \leq \frac{1}{\lambda}\||x|_\alpha\|_{1,\Phi}, \quad \lambda > 0,$$

we have $x_\alpha \xrightarrow{\Phi} 0$, and therefore $x_\alpha \xrightarrow{t(M)} 0$ [8].

(ii). Suppose that $t_\omega(M) = \tau_\omega(M, \Phi)$. Since $(L^\omega(M, \Phi), \tau_\omega(M, \Phi))$ is complete, $L^\omega(M, \Phi)$ is a closed subalgebra in $(S(M), t(M))$. Let $x \in S_+(M)$, $x_n = E_n(x)x$, $n \in N$. It is clear that $x_n \in M$, and $\lambda E_\lambda^\perp(x - x_n) \leq (x - x_n) \downarrow 0$, that's why $\Phi(E_\lambda^\perp(x - x_n)) \xrightarrow{t(B)} 0$ at $n \to \infty$ for all $\lambda > 0$. The means that $x_n \xrightarrow{\Phi} x$, and therefore $x_n \xrightarrow{t(M)} x$. Hence, $x \in L^\omega(M, \Phi)$. Since each element from $S(M)$ is a linear combination of four elements

from $S_+(M)$, we have $x \in L^\omega(M, \Phi)$ for all $x \in S(M)$, which implies the equality $L^\omega(M, \Phi) = S(M)$. $\square$

Remark 3.2. *If* $\dim M < \infty$, *then* $L^\omega(M, \Phi) = M = S(M)$, *and the both vector topologies* $\tau_\omega(M, t)$ *and* $t(M)$ *coincide on* $L^\omega(M, \Phi)$, *moreover they are equal to the topology generated by the* C^*-*norm* $\| \cdot \|_M$.

4. Disjunct completeness of Arens algebras

Let $L^\omega(M, \Phi)$ be an Arens algebra associated with the von Neumann algebra M and $S(B)$-valued Maharam trace Φ, and let A be a von Neumann subalgebra in $Z(M)$, and ψ be an $*$-isomorphism from $S(A)$ onto $S(B)$ from Theorem 2.2 (ii). Consider arbitrary decomposition $\{e_i\}_{i \in I}$ of the unit $\mathbf{1}_B$ of the complete Boolean algebra $P(B)$ of all projections from the commutative von Neumann algebra B. It is clear that $q_i = \psi^{-1}(e_i)$, $i \in I$, is decomposition of the unit $\mathbf{1}_A = \mathbf{1}_M$ of the Boolean algebra $P(M)$. For each $q_i x \in q_i M$, we have $\Phi(q_i x) = \psi(q_i)\Phi(x) = e_i \Phi(x) \in e_i S(B) = S(e_i B)$. Hence, the restriction Φ_i of the Maharam trace Φ on $e_i M$ is an $S(e_i B)$-valued Maharam trace on the von Neumann algebra $e_i M$ for all $i \in I$. If $p \geq 1$ and $x \in L^p(M, \Phi)$, then, evidently, $q_i x \in L^p(q_i M, \Phi_i)$, and $e_i \|x\|_{P, \Phi} = \|q_i x\|_{P, \Phi_i}$ for all $i \in I$. Since $(L^p(M, \Phi), \| \cdot \|_{P, \Phi})$ is a Banach-Kantorovich space (Theorem 2.3), then $L^p(M, \Phi)$ is disjunct complete ([16], 2.1.5, 2.2.1), i.e. the following assertion is valid.

Proposition 4.1. *If* $x_i \in L^p(q_i M, \Phi_i)$ *for all* $i \in I$, *a there exists the unique element* $x \in L^p(M, \Phi)$, *for which* $q_i x = x_i$ *at all* $i \in I$.

The following property of disjunct completeness for the Arens algebra $L^\omega(M, \Phi)$ follows immediately from Proposition 4.1.

Corollary 4.1. *Let* $\{e_i\}_{i \in I}$ *be arbitrary decomposition of the unit of the Boolean algebra* $P(B)$, $q_i = \psi^{-1}(e_i)$, $x_i \in L^\omega(q_i M, \Phi_i)$, $i \in I$. *Then there exists a unique* $x \in L^\omega(M, \Phi)$ *such that* $q_i x = x_i$ *for all* $i \in I$.

Consider the direct product $\prod_{i \in I} L^\omega(q_i M, \Phi_i)$ of $*$-algebras $L^\omega(q_i M, \Phi_i)$ with coordinate-wise algebraic operations and involution. Define the mappings

$$U : L^\omega(M, \Phi) \longrightarrow \prod_{k \in I} L^\omega(q_i M, \Phi_i),$$

setting $U(x) = \{q_i x\}_{i \in I}$. According to Corollary 4.1, the mapping U is an $*$-isomorphism from $L^\omega(M, \Phi)$ onto the $*$-algebra $\prod_{i \in I} L^\omega(q_i M, \Phi_i)$. Denote by $t(\{q_i\})$ the Tychonoff product of topologies $\tau_\omega(q_i M, \Phi_i)$ in $\prod_{i \in I} L^\omega(q_i M, \Phi_i)$. By virtue of properties of Tychonoff topologies, the pair $\left(\prod_{i \in I} L^\omega(q_i M, \Phi_i), t(\{q_i\}) \right)$ is a complete topological $*$-algebra.

Proposition 4.2. *The mapping*

$$U : (L^\omega(M, \Phi), \tau_\omega(M, \Phi)) \longrightarrow \left(\prod_{i \in I} L^\omega(q_i M, \Phi_i), t(\{q_i\}) \right)$$

is a homeomorphism, in particular, $x_\alpha \xrightarrow{\tau_\omega(M, \Phi)} x, x_\alpha, x \in L^\omega(M, \Phi)$ *iff* $q_i x_\alpha \xrightarrow{\tau_\omega(q_i M, \Phi_i)} q_i x$ *for all* $i \in I$.

The proof of this statement follows from the definition of the Tychonoff topology and equalities $\|q_i x\|_{P, \Phi} = e_i \|x\|_{P, \Phi_i}$ for all $i \in I, p \geq 1, x \in L^\omega(M, \Phi)$ (see Theorem 2.3 (vi)).

The following theorem gives necessary and sufficient conditions for locally convexity of the topology $\tau_\omega(M, \Phi)$.

Theorem 4.1. *Let $L^\omega(M, \Phi)$ be an Arens algebra associated with a von Neumann algebra M and an $S(B)$-valued Maharam trace Φ. The following conditions are equivalent:*
  (i). The topology $\tau_\omega(M, \Phi)$ is locally convex;
  (ii). B is an atomic von Neumann algebra.

Proof. $(i) \Rightarrow (ii)$. Let $\tau_\omega(M, \Phi)$ is a locally convex topology. According to (3.2), the topology $t(A)$ is also locally convex. Hence, by Proposition 2 from ([20], chapter $V, \S\,3$], the von Neumann algebra A is atomic, what implies atomicity of the von Neumann algebra B.

$(ii) \Rightarrow (i)$. If B is an atomic von Neumann algebra, then there exists in $P(B)$ a decomposition $\{e_i\}_{i \in I}$ of the unit such that e_i is an atom in $P(B)$ for each $i \in I$. Therefore $e_i S(B) = \mathbb{C}$ and $S(B) = \mathbb{C}^I$. In this case, for $g_i = \psi^{-1}(e_i)$, the function $\Phi_i(g_i x) = e_i \Phi(x)$ is a faithful normal finite trace on the von Neumann algebra $g_i M$, and thus the topology $\tau_\omega(g_i M, \Phi_i)$ is locally convex [27]. Hence, the Tychonoff topology is also locally convex. It remains to use Proposition 3.2, by virtue of which, the topology $\tau_\omega(M, \Phi)$ is also locally convex. $\qquad\square$

Remark 4.1. *If the topology $\tau_\omega(M, \Phi)$ is normable, then the topology $t(A)$ is also normable (see (3.2)), what implies finite dimensionality of the algebra A (see Proposition 4 from ([27], chapter V, $\S\,3$)). Hence, the algebra B is also finite-dimensional, and therefore there is a finite collection of atoms $\{e_i\}_{i=1}^n$ $P(B)$, for which $\sum\limits_{i=1}^n e_i = \mathbf{1}_B$. In this case the $*$-algebra is $B - *$-isomorphic to $\mathbb{C}^n$, and $\tau(x) = \sum\limits_{i=1}^n e_i \Phi(x)$ is a faithful normal finite numeric trace on M, for which $L^\omega(M, \Phi) = L^\omega(M, \tau)$.*

5. COMPARISON OF THE (o)-TOPOLOGY AND THE TOPOLOGY $\tau_\omega(\mathbf{M}, \mathbf{\Phi})$

Let $L^\omega(M, \Phi)$ be the Arens algebra associated with a von Neumann algebra M and an $S(B)$-valued Maharam trace Φ. Since the involution in $L^\omega(M, \Phi)$ is continuous with respect to the topology $\tau_\omega(M, \Phi)$ (Proposition 3.3), the set $L_h^\omega(M, \Phi)$ is closed in $(L^\omega(M, \Phi), \tau_\omega(M, \Phi))$. It was proved in [26] that the set $S_+(M)$ is closed in $(S(M), t(M))$. Therefore, according to Proposition 3.4 (i), the set $L_+^\omega(M, \Phi)$ is closed in $(L^\omega(M, \Phi), \tau_\omega(M, \Phi))$. Denote by $\tau_{\omega h}(M, \Phi)$ the topology in $L_h^\omega(M, \Phi)$ induced by the topology $\tau_\omega(M, h)$ from $L^\omega(M, \Phi)$, and by $\tau_o(M, \Phi)$ – the (o)-topology in $L_h^\omega(M, \Phi)$, i.e. the strongest topology in an ordered linear space $L_h^\omega(M, \Phi)$, for which (o)-convergence of nets implies their topological convergence.

Theorem 5.1. $(i).$ $\tau_{\omega,h}(M, \Phi) \le \tau_o(M, \Phi)$;
  $(ii).$ $\tau_{\omega h}(M, h) = \tau_o(M, \Phi)$ *if and only if a von Neumann algebra B is σ-finite.*

Proof. (i). If $\{x_\alpha\}_{\alpha \in A} \subset L_h^\omega(M, \Phi)$, and $\{x_\alpha\}$ (o)-converges in $L_h^\omega(M, \Phi)$ to an element $x \in L_h^\omega(M, \Phi)$, then by definition there exist nets $\{y_\alpha\}_{\alpha \in A}$, $\{z_\alpha\}_{\alpha \in A}$ from $L_h^\omega(M, \Phi)$ such that $y_\alpha \le x_\alpha \le z_\alpha$ for all $\alpha \in A$ $y_\alpha \uparrow x, z_\alpha \uparrow x$. Since $y_\alpha - x \le x_\alpha - x \le z_\alpha - x$, we have
$$0 \le (x_\alpha - x) + (x - y_\alpha) \le (z_\alpha - x) + (x - y_\alpha).$$
Using convergences $(z_\alpha - x) \downarrow 0$, $(x - y_\alpha) \downarrow 0$, and Theorem 2.3 (iv), (vii), we obtain
$$\|x_\alpha - x\|_{p,\Phi} \le \|(x_\alpha - x) + (x - y_\alpha)\|_{p,\Phi} + \|x - y_\alpha\|_{p,\Phi} \le$$
$$\le \|z_\alpha - x\|_{p,\Phi} + 2\|x - y_\alpha\|_{p,\Phi} \xrightarrow{t(B)} 0$$
for all $p \ge 1$. Therefore $x_\alpha \xrightarrow{\tau_\omega(M,\Phi)} x$, which implies of the inequality
$$\tau_{\omega,h}(M, \Phi) \le \tau_o(M, \Phi).$$

(ii). If the von Neumann algebra B is σ-finite, then the topology $\tau_\omega(M, \Phi)$ is metrizable (Theorem 3.2). Repeating the proof of Theorem 2 from [2], we obtain $\tau_o(M, \Phi) \le \tau_{\omega h}(M, \Phi)$. Therefore, according to (i), the equality $\tau_o(M, \Phi) = \tau_{\omega h}(M, \Phi)$ is valid.

Now suppose that $\tau_o(M, \Phi) = \tau_{\omega h}(M, \Phi)$. Let A and ψ by the same as in Theorem 2.2. (ii). Since $S(A) \subset L^\omega(M, \Phi)$, and the topology $\tau_{\omega, h}(M, \Phi)$ induces on $S(A)$ the topology $t(A)$ (see (3.2)), repeating the proof of Theorem 2 from [5], we obtain that the (o)-topology in $S_h(A)$ coincides with the topology $t_h(A)$, where $t_h(A)$ is the topology $S_h(M)$ induced by the topology $t(A)$ from $S(A)$. Hence, by Theorem 2 from [5] the von Neumann algebra A is σ-finite. Since ψ is $*$-isomorphism from A onto B, the algebra B is also σ-finite. □

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Tashkent Islamic University, 11 Abdulla Qodiriy str., Tashkent, 100011, Uzbekistan
E-mail address: alimovakrom63@yandex.ru