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SOME REMARKS ON HILBERT REPRESENTATIONS OF POSETS

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To 70-th birthday of our Teacher Yu. S. Samoilenko

Abstract. For a certain class of finite posets, we prove that all their irreducible orthoscalar representations are finite-dimensional and describe those, for which there exist essential (non-degenerate) irreducible orthoscalar representations.

1. Introduction

Let S be a finite partially ordered set (poset). A representation of S in a linear space V is a collection of subspaces V_g, $g \in S$, for which $V_g \subset V_h$ if $g < h$, and one considers representations V_g in V and V'_g in V' to be equivalent if there exists an invertible operator $T\colon V \to V'$ such that $V_g = TV_g$, $g \in S$. Representations of posets have been extensively studied by A. V. Roiter and his colleagues (see [1, 2] and others), in particular, classes of posets of finite type, tame type and wild type were described.

In the case of a Hilbert space H, a representation of S is a collection of closed subspaces H_g, $g \in S$, for which $H_g \subset H_h$ if $g < h$, and they are studied up to a unitary equivalence: representations H_g in H and H'_g in H' are equivalent if there exists a *unitary* operator $U\colon H \to H'$ such that $H_g = UH_g$, $g \in S$. It appeared that posets of (Hilbert) tame type ($*$-tame type) have a very simple structure [3], — they are chains or semichains. In Section 2 we provide a short description of them (Section 2.1). Also, in Section 2.2 we introduce and describe a class of unitarily one-parameter poset (for them, the continuous series of irreducible representations naturally depend on a single parameter) and calculate the spectrum of a linear combination of the corresponding projections (Section 2.3).

On the other hand, it was discovered in several recent papers ([4, 5, 6] and others) that in the case of primitive posets, an additional condition of *orthoscalarity* (see Section 3.1 for the definition of orthoscalar representations) leads to results very similar to the ones in a linear representations theory. Moreover, it was shown in [7, 8] that this similarity can be extended to some cases of non-primitive posets by using a unitarization technique; there exists a correspondence between classes of linear representations in V and orthoscalar representations in H.

Therefore, it is still a problem to develop results on orthoscalar representations of posets in the non-primitive case. In this paper, we study orthoscalar representations of the class of posets which can be decomposed into a union of two unitarily one-parameter posets. We start with the simplest example of the primitive $(1, 1, 1, 1)$ poset (Section 3.2), orthoscalar representations of which are four-tuples of projections in H whose linear combination is a scalar operator. Here we summarize some results of [9, 10, 11], and give explicit formulas for representations similar to the ones established in [12]. The main

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result is that any irreducible orthoscalar representation of any poset which is a union of two one-parameter posets, is finite-dimensional (Section 3.3).

In Section 5 we consider examples of such posets and their orthoscalar representations.

2. Posets of $*$-tame type

2.1. **Description of $*$-tame posets.** Recall that given a finite poset S, its (Hilbert) representation is a collection of subspaces $S \ni g \mapsto H_g$ of closed subspaces of some Hilbert space H, such that $H_g \subset H_h$ for $g < h$, $g, h \in S$. Obviously, each subspace H_g is uniquely determined by an orthogonal projection P_g onto H_g, $g \in S$, therefore, representations of a poset S are described by representations of a $*$-algebra generated by projections P_g, $g \in S$, such that $P_g P_h = P_g$, $g < h$, and vice versa. Notions of indecomposable, irreducible representations and unitary equivalence of representations are standard for representations of $*$-algebras and thus can be applied to Hilbert representations of posets as well.

For Hilbert representations of posets, it is well-known that (1) and $(1,1)$ are posets of tame type, and the posets $(1,2)$ and $(1,1,1)$ are of $*$-wild type.

Proposition 1. *A poset S is of tame type if and only if its width is 1 or 2 and S does not contain the $(1,2)$ poset.*

Proof. If S contains the $(1,2)$ poset, it is evidently of $*$-wild type. If the width of S is 3 or more, it contains the $(1,1,1)$ poset and is again of $*$-wild type [9].

It S is of width 1, the corresponding algebra is commutative, therefore S is of tame type.

Let S be of width 2 and does not contain $(1,2)$. It is easy to see that in this case $S = S_1 \cup \dots \cup S_m$ such that each S_i is either (1) or $(1,1)$, $i = 1, \dots, m$, and $S_j > S_k$, $j < k$. The latter means that and for any $f \in S_j$, $g \in S_k$ we have $g < f$. But in this case any representation of S is a tensor product of representations of S_i, $i = 1, \dots, m$. $\square$

It is easy to see that any irreducible representation of a poset of $*$-tame type is one- or two-dimensional (see, e.g., [3]), moreover, two-dimensional irreducible representations exist if and only if the poset is of width 2. As noticed above, any poset S of tame type can be represented as

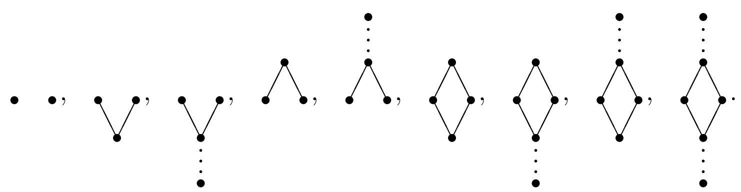
$$S = S_1 \cup \dots \cup S_m, \tag{1}$$

such that each S_i is either (1) or $(1,1)$, $i = 1, \dots, m$, and $S_j > S_k$, $j < k$.

2.2. **One-parameter posets.** We introduce the class of one-parameter posets. For these posets, the set of irreducible representations consists of a finite number of one-dimensional and a one-parameter continuous family of two-dimensional representations.

Definition 1. *We say that S is a (unitarily) one-parameter poset if it is of tame type and in its decomposition* (1) *exactly one set S_i is of width 2.*

In other words, unitarily one-parameter posets are those having the following Hasse diagrams:



Remark 1. In the theory of linear (non-unitary) representations of poset, the class of one-parameter posets is defined in other terms and differs from the one introduced above. Below, speaking of one-parameter posets we mean unitarily one-parameter posets unless specified explicitly.

The description of irreducible representations given in [3] in the case of one-parameter posets can be specified as follows.

Let S be a one-parameter poset, and let k be the unique index for which S_k in (1) is of width 2. Then S_j, $j \neq k$ consists of a single element g_j, while S_k consists of two elements $g_{k,1}$, $g_{k,2}$.

Proposition 2. *Any irreducible representation of $S \ni g \mapsto P_g$ has dimension one or two. There exists a finite number of one-dimensional irreducible representations, $P_g = p_g \in \{0,1\}$, where*

$$p_1 \leq \cdots \leq p_{k-1} \leq \frac{p_{k,1}+p_{k,2}}{2} \leq p_{k+1} \leq \cdots \leq p_m,$$

and a one-parameter family of two-dimensional irreducible non-equivalent representations,

$$P_1 = \cdots = P_{k-1} = 0,$$
$$P_{k,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{k,2} = \begin{pmatrix} \tau & \sqrt{\tau(1-\tau)} \\ \sqrt{\tau(1-\tau)} & 1-\tau \end{pmatrix}, \quad 0 < \tau < 1,$$
$$P_{k+1} = \cdots = P_m = I.$$

2.3. **Linear combinations of projections.** The latter proposition enables one to obtain in a standard way a spectral decomposition of an arbitrary (reducible) representation into a direct sum or integral of irreducible ones. We use such a decomposition to describe the spectrum of the operator $\sum_{g_s \in S} \alpha_s P_s$, which will be used below.

Proposition 3. *Let $S \ni g \mapsto P_g$ be a Hilbert representation of a one-parameter poset S. Denote*

$$\sum_{g_s \in S} \alpha_s P_s.$$

Then

$$\sigma(A) \subset \Delta = \Delta_d \cup \frac{\Sigma \pm (|\alpha_{k,1} - \alpha_{k,2}|, \alpha_{k,1} + \alpha_{k,2})}{2},$$

where

$$\begin{aligned} (2) \quad \Delta_d = \{&0, \alpha_m, \alpha_{m-1} + \alpha_m, \ldots, \alpha_{k+1} + \cdots + \alpha_m, \\ &\alpha_{k,1} + \alpha_{k+1} + \cdots + \alpha_m, \alpha_{k,2} + \alpha_{k+1} + \cdots + \alpha_m, \alpha_{k,1} + \alpha_{k,2} + \alpha_{k+1} + \cdots + \alpha_m, \\ &\alpha_{k-1} + \alpha_{k,1} + \alpha_{k,2} + \alpha_{k+1} + \cdots + \alpha_m, \ldots \\ &\alpha_1 + \cdots + \alpha_{k-1} + \alpha_{k,1} + \alpha_{k,2} + \alpha_{k+1} + \cdots + \alpha_m\}, \end{aligned}$$

$$(3) \quad \Sigma = \alpha_{k,1} + \alpha_{k,2} + 2\sum_{j>k} \alpha_j.$$

Moreover, the parts of the spectrum in the continuous area corresponding to the plus and minus signs have the same type and multiplicity. In particular, the number $(\Sigma + \lambda)/2$, $|\alpha_{k,1} - \alpha_{k,2}| < \lambda < \alpha_{k,1} + \alpha_{k,2}$, is an eigenvalue if and only if $(\Sigma - \lambda)/2$ is an eigenvalue of the same multiplicity.

Proof. In the case of a two-dimensional irreducible representation, by routine calculations we obtain that the spectrum $\sigma(\sum_{g_s\in S}\alpha_s P_s)$ consists of two points,

$$\lambda = \frac{(\alpha_{k,1}+\alpha_{k,2}) \pm \sqrt{\alpha_{k,1}^2+\alpha_{k,2}^2+2\alpha_{k,1}\alpha_{k,2}(2\tau-1)}}{2} + \sum_{j>k}\alpha_j.$$

Since the parameter τ can take arbitrary value in $(0,1)$, in the general case these representations give two segments symmetric with respect to the point $\frac{(\alpha_{k,1}+\alpha_{k,2})}{2}+\sum_{j>k}\alpha_j$,

$$\frac{1}{2}\Big((\alpha_{k,1}+\alpha_{k,2}) \pm (|\alpha_{k,1}-\alpha_{k,2}|, \alpha_{k,1}+\alpha_{k,2})\Big) + \sum_{j>k}\alpha_j.$$

The rest of the possible points of $\sigma(\sum_{g_s\in S}\alpha_s P_s)$ arise from one-dimensional representations. □

Given $\lambda\in\sigma(A)$, $\lambda\in(\Sigma\pm(|\alpha_{k,1}-\alpha_{k,2}|,\alpha_{k,1}+\alpha_{k,2}))/2$, we can restore the corresponding projections. Indeed, let

$$\tau = \frac{(\sum_{j>k}\alpha_j+\alpha_{k,1}-\lambda)(\sum_{j>k}\alpha_j+\alpha_{k,2}-\lambda)}{\alpha_{k,1}\alpha_{k,2}}.$$

If $\lambda\in\sigma(A)$ is an eigenvalue corresponding to the continuous part of Δ, then in the corresponding eigen-basis of A, we have

$$P_{k,1} = \frac{1}{2}\begin{pmatrix} 1+\epsilon_1 & -\sqrt{1-\epsilon_1^2} \\ -\sqrt{1-\epsilon_1^2} & 1-\epsilon_1 \end{pmatrix}, \quad P_{k,2} = \frac{1}{2}\begin{pmatrix} 1+\epsilon_2 & \sqrt{1-\epsilon_2^2} \\ \sqrt{1-\epsilon_2^2} & 1-\epsilon_2 \end{pmatrix},$$

where after routine calculations,

$$\epsilon_1 = \frac{2\mu^2-(2\mu-\alpha_{k,1})(\alpha_{k,1}+\alpha_{k,2})}{\alpha_{k,1}(2\mu-\alpha_{k,1}-\alpha_{k,2})}, \quad \epsilon_2 = \frac{2\mu^2-(2\mu-\alpha_{k,2})(\alpha_{k,1}+\alpha_{k,2})}{\alpha_{k,2}(2\mu-\alpha_{k,1}-\alpha_{k,2})}.$$

Here $\mu=\lambda-\sum_{j>k}\alpha_j$.

3. Orthoscalar representations of finite posets

3.1. **Definition of orthoscalarity.** Let S be a finite poset, $S\ni g\mapsto P_g$ be a collection of orthoprojections which form its representation, i.e., $P_gP_h=P_g$, $g<h$.

We use the term *character* for a positive function on S, $S\ni g\mapsto\alpha_g>0$.

Definition 2. *We say that a representation $S\ni g\mapsto P_g$, $P_gP_h=P_g$, $g<h$, is orthoscalar with a character $\alpha=(\alpha_g)_{g\in S}$, if*

$$\sum_{g\in S}\alpha_g P_g = I.$$

Orthoscalar representations of primitive posets (in terms of orthoscalar representations of graphs or quivers) were studied in [4, 5, 6] and other papers, some results for the non-primitive case are obtained in [7, 8] and others.

Notice the following simple properties of orthoscalar representations.

1. If $\sum_{g\in S}\alpha_g<1$, there are no representations.
2. If $\sum_{g\in S}\alpha_g=1$, then all $P_g=I$, $g\in S$.
3. If $\alpha_g>1$ for some $g\in S$, then $P_g=0$.
4. If $\alpha_g=1$ for some $g\in S$, then in any irreducible representation either $P_g=0$ or $P_g=I$.

To exclude these degenerated cases, in what follows we assume that $0<\alpha_g<1$, $g\in S$, and $\sum_{g\in S}\alpha_g>1$.

In this paper we study the class of finite posets S, such that $S=S_1\cup S_2$, where S_1 and S_2 are unitarily one-parameter posets of tame type. We admit that some elements of

S_1 can be comparable with some elements of S_2, however, we do not use such relations, and they should be taken into account to narrow the result obtained without them.

3.2. **Orthoscalar four-tuples of projections.** The simplest case of poset of such a kind is the $(1, 1, 1, 1)$ poset that is a primitive poset consisting of four elements, any two of which are non-comparable. An orthoscalar representation of this poset with character $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a four-tuples of projections, $P_1, \ldots, P_4$, in some Hilbert space H, for which

$$\alpha_1 P_1 + \alpha_2 P_4 + \alpha_3 P_3 + \alpha_4 P_4 = I. \tag{4}$$

Such four-tuples have been studied in [9, 12, 10, 11] and others. In particular, the following theorem has been proved (see also [13]).

Theorem 1. *Any orthoscalar irreducible four-tuple of projections is finite-dimensional.*

Here we give an independent proof of this fact, which involves constructions which we will apply in a more general case.

3.2.1. *Case of* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$. *Continuous series.*

Proposition 4. *Let* $P_1, \ldots, P_4$ *be an irreducible family of projections in* H *satisfying* (4)*, for which* $\ker P_j \cap \ker P_k = \{0\}$, $\operatorname{ran} P_j \cap \operatorname{ran} P_k = \{0\}$, $\ker P_j \cap \operatorname{ran} P_k = \{0\}$, $j \neq k$. *Then* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$ *and* $\dim H = 2$.

Proof. Introduce operators $A_1 = \alpha_1 P_1 + \alpha_2 P_2$, $A_2 = \alpha_3 P_3 + \alpha_4 P_4$. The orthoscalarity condition means that $A_1 + A_2 = I$. The conditions that the kernels and the ranges of the projections are zero imply, due to the structure theorem for a pair of projections, that the space H can be decomposed as $H = \mathcal{H} \oplus \mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}$ so that

$$A_1 = \frac{\alpha_1 + \alpha_2}{2} I + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes C_1, \quad A_2 = \frac{2 - \alpha_1 - \alpha_2}{2} I + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes C_1, \tag{5}$$

where $\frac{|\alpha_1 - \alpha_2|}{2} I < C_1 < \frac{\alpha_1 + \alpha_2}{2} I$, is a self-adjoint operator in $\mathcal{H}$. Applying the same structure theorem to P_3, P_4, we conclude that A_2 can be represented (probably for another decomposition $H = \mathbb{C}^2 \otimes \mathcal{H}'$) as

$$A_2 = \frac{\alpha_3 + \alpha_4}{2} I + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes C_2, \quad \frac{|\alpha_3 - \alpha_4|}{2} I < C_2 < \frac{\alpha_3 + \alpha_4}{2} I.$$

Comparing this to (5), we have $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$.

It is easy to see that the operator $\left(A_1 - \frac{\alpha_1 + \alpha_2}{2} I\right)^2 = I \otimes C_1^2$ commutes with P_1 and P_2. In the same way, the operator $(A_2 - \frac{\alpha_3 + \alpha_4}{2} I)^2 = I \otimes C_1^2$ commutes with P_3, P_4. Therefore, $I \otimes C_1^2$ commutes with $P_1, \ldots, P_4$, and therefore, is a scalar operator in an irreducible representation. Thus, $C_1 = cI$ for some

$$c \in \Big(\frac{|\alpha_1 - \alpha_2|}{2}, \frac{\alpha_1 + \alpha_2}{2}\Big) \cap \Big(\frac{|\alpha_3 - \alpha_4|}{2}, \frac{\alpha_3 + \alpha_4}{2}\Big),$$

and the irreducibility implies $\mathcal{H} = \mathbb{C}$. □

Remark 2. One can obtain explicit formulas for the corresponding two-dimensional representations. Write the projections P_1, P_2 in the form

$$P_1 = \frac{1}{2} \begin{pmatrix} 1 + \lambda_1 & \sqrt{1 - \lambda_1^2} \\ \sqrt{1 - \lambda_1^2} & 1 - \lambda_1 \end{pmatrix}, \quad P_2 = \frac{1}{2} \begin{pmatrix} 1 + \lambda_2 & -\sqrt{1 - \lambda_2^2} \\ -\sqrt{1 - \lambda_2^2} & 1 - \lambda_2 \end{pmatrix}, \tag{6}$$

with

$$\lambda_1 = \frac{\alpha_1^2 - \alpha_2^2 + 4c^2}{4c\alpha_1}, \quad \lambda_2 = \frac{\alpha_2^2 - \alpha_1^2 + 4c^2}{4c\alpha_2},$$

so that $A_1 = \alpha_1 P_1 + \alpha_2 P_2 = c\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The projections P_3 and P_4 can be represented as

$$P_3 = \frac{1}{2}\begin{pmatrix} 1+\lambda_3 & \gamma\sqrt{1-\lambda_3^2} \\ \bar{\gamma}\sqrt{1-\lambda_3^2} & 1-\lambda_3 \end{pmatrix}, \quad P_4 = \frac{1}{2}\begin{pmatrix} 1+\lambda_4 & -\gamma\sqrt{1-\lambda_4^2} \\ -\bar{\gamma}\sqrt{1-\lambda_4^2} & 1-\lambda_4 \end{pmatrix}, \tag{7}$$

with

$$\lambda_3 = \frac{\alpha_3^2 - \alpha_4^2 + 4c^2}{4c\alpha_3}, \quad \lambda_4 = \frac{\alpha_4^2 - \alpha_3^2 + 4c^2}{4c\alpha_4}, \quad |\gamma| = 1.$$

Therefore, the set of two-dimensional irreducible representations, for which $\ker P_j \cap \ker P_k = \{0\}$, $\operatorname{ran} P_j \cap \operatorname{ran} P_k = \{0\}$, $\ker P_j \cap \operatorname{ran} P_k = \{0\}$, $j \neq k$, is described by two continuous parameters, c and γ.

3.2.2. *Case* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$. *Discrete series.* Now consider the case where at least one of the subspaces $\ker P_j \cap \ker P_k$, $\operatorname{ran} P_j \cap \operatorname{ran} P_k$, $\ker P_j \cap \operatorname{ran} P_k$, $\operatorname{ran} P_j \cap \ker P_k$, $j \neq k$, is nonzero. We obviously can assume $j = 1$, $k = 2$. Introduce sets

$$\begin{aligned} \Delta_{1,d} &= \{0, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, & \Delta_{1,c} &= (0, \alpha_1) \cup (\alpha_2, \alpha_1 + \alpha_2), \\ \Delta_{2,d} &= \{0, \alpha_3, \alpha_4, \alpha_3 + \alpha_4\}, & \Delta_{2,c} &= (0, \alpha_3) \cup (\alpha_4, \alpha_3 + \alpha_4), \\ \Delta_1 &= [0, \alpha_1] \cup [\alpha_2, \alpha_1 + \alpha_2], & \Delta_2 &= [0, \alpha_3] \cup [\alpha_4, \alpha_3 + \alpha_4], \end{aligned}$$

so that $\Delta_1 = \Delta_{1,d} \cup \Delta_{1,c}$, $\Delta_2 = \Delta_{2,d} \cup \Delta_{2,c}$.

Then there exists a number $\lambda_0 \in \Delta_{1,d}$ which is an eigenvalue of the operator A_1. Let f_0 be the corresponding unit eigenvector. Since $A_1 + A_2 = I$, f_0 is also an eigenvector of A_2, $A_2 f_0 = \mu_0 f_0$, where $\mu_0 = 1 - \lambda_0$. The following two cases can arise.

(i) $\mu_0 \in \Delta_{2,d}$. Then the space spanned by f_0 is invariant w.r.t. $P_1, \ldots, P_4$, and due to the irreducibility, is the whole H, $\dim H = 1$, and

$$P_1 = \delta_1, \quad P_2 = \delta_2, \quad P_3 = \delta_3, \quad P_4 = \delta_4, \quad \delta_1, \delta_2, \delta_3, \delta_4 \in \{0, 1\}. \tag{8}$$

Notice that in this case there exists such permutation σ of indexes, that

$$\alpha_{\sigma(1)} + \alpha_{\sigma(2)} = \alpha_{\sigma(3)} + \alpha_{\sigma(4)} = 1.$$

(ii) $\mu_0 \in \Delta_{2,c}$. Then $\mu_1 = \alpha_3 + \alpha_4 - \mu_0$ is also an eigenvalue of A_2 with some unit eigenvector f_1.

In the latter case, since $A_1 + A_2 = I$, the vector f_1 is an eigenvector of A_1 as well, $A_1 f = (I - A_2) f = \lambda_1 f_1$, $\lambda_1 = 1 - \mu_1$. Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$, one can see that $\lambda_1 \in \Delta_{1,d}$. Indeed, otherwise $\lambda_1 \in \Delta_{1,c}$ and $\alpha_1 + \alpha_2 - \lambda_1 = \lambda_0 \in \Delta_{1.c}$ which contradicts the initial setting $\lambda_0 \in \Delta_{1,d}$. Therefore, H is spanned by (f_0, f_1). The projections are

$$\begin{aligned} P_1 &= \begin{pmatrix} \delta_1 & 0 \\ 0 & 1-\delta_1 \end{pmatrix}, & P_3 &= \frac{1}{2}\begin{pmatrix} 1+\tau_1 & \sqrt{1-\tau_1^2} \\ \sqrt{1-\tau_1^2} & 1-\tau_1 \end{pmatrix}, \\ P_2 &= \begin{pmatrix} \delta_2 & 0 \\ 0 & 1-\delta_2 \end{pmatrix}, & P_4 &= \frac{1}{2}\begin{pmatrix} 1+\tau_2 & -\sqrt{1-\tau_2^2} \\ -\sqrt{1-\tau_2^2} & 1-\tau_2 \end{pmatrix}, \end{aligned} \tag{9}$$

where $\delta_1, \delta_2 \in \{0, 1\}$ are defined from $\alpha_1 \delta_1 + \alpha_2 \delta_2 = \lambda_0 \in \Delta_{1,d}$, and

$$\tau_1 = \frac{2\mu_0^2 - (2\mu_0 - \alpha_3)(\alpha_3 + \alpha_4)}{\alpha_3(2\mu_0 - \alpha_3 - \alpha_4)}, \quad \tau_2 = \frac{2\mu_0^2 - (2\mu_0 - \alpha_4)(\alpha_3 + \alpha_4)}{\alpha_4(2\mu_0 - \alpha_3 - \alpha_4)},$$

$\mu_0 = 1 - \lambda_0 \in \Delta_{2,c}$.

3.2.3. *Case* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \neq 2$. In the case where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \neq 2$, for λ_1 there can be two possibilities.

(i) $\lambda_1 \in \Delta_{1,d}$. Then the space spanned by (f_0, f_1) is invariant w.r.t. $P_1, \ldots, P_4$, and due to the irreducibility, is the whole H, $\dim H = 2$, the projections are given by (9).

(ii) $\lambda_1 \in \Delta_{1,c}$. In this case, $\lambda_2 = \alpha_1 + \alpha_2 - \lambda_1$ is also an eigenvalue of A_1 with some unit eigenvector f_2.

In the latter case, since $A_1 + A_2 = I$, the vector f_2 is an eigenvector of A_2 and we proceed as above.

Consider two sequences, $\lambda_0, \lambda_1, \dots$, and $\mu_0, \mu_1, \dots$, constructed from λ_0 by the following rules. For $j = 2k$, $k \geq 0$ let $\mu_j = 1 - \lambda_j$, $\mu_{j+1} = \alpha_3 + \alpha_4 - \mu_j$, $\lambda_{j+1} = 1 - \mu_{j+1}$, $\lambda_{j+2} = \alpha_1 + \alpha_2 - \lambda_{j+1}$. We have

$$\begin{aligned}
\lambda_{2k} &= \lambda_0 + k(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2),\\
\lambda_{2k+1} &= \alpha_1 + \alpha_2 - \lambda_0 - (k+1)(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2),\\
\mu_{2k} &= 1 - \lambda_0 - k(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2),\\
\mu_{2k+1} &= \lambda_0 + \alpha_3 + \alpha_4 - 1 + k(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2), \quad k \geq 0.
\end{aligned}$$

Then the arguments above imply that $\sigma(A_1) \subset (\lambda_k)_{k=0}^{\infty}$, $\sigma(A_2) \subset (\mu_k)_{k=0}^{\infty}$. Since $\sigma(A_1) \subset \Delta_1$, $\sigma(A_2) \subset \Delta_2$, then for $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2 \neq 0$ only finite number of λ_k may belong to $\sigma(A_1)$, and the same number of μ_k may belong to $\sigma(A_2)$.

Therefore for $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2 \neq 0$, $\sigma(A_1) = (\lambda_k)_{k=0}^{m}$, $\sigma(A_2) = (\mu_k)_{k=0}^{m}$, where $m \geq 0$ is determined by the following conditions:

$$\begin{aligned}
&m = 2l\colon && \lambda_k \in \Delta_{1,c}, \quad 1 \leq k \leq m, \\
& && \mu_k \in \Delta_{2,c}, \quad 0 \leq k \leq m-1, \quad \mu_m \in \Delta_{2,d};
\end{aligned}\tag{10}$$

$$\begin{aligned}
&m = 2l+1\colon && \lambda_k \in \Delta_{1,c}, \quad 1 \leq k \leq m-1, \quad \lambda_m \in \Delta_{1,d},\\
& && \mu_k \in \Delta_{2,c}, \quad 0 \leq k \leq m.
\end{aligned}\tag{11}$$

The dimension of the space H is equal to $m+1$.

3.2.4. *Description of representations.* As we already shown, in the case where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2 = 0$, or $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2 \neq 0$, $\dim H \leq 2$, the projections are given by the formulas (6), (7), (8) or (9). Now assume $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2 \neq 0$, $\dim H > 2$. In order to give explicit formulas for the projections, let us introduce projections in $\mathbb{C}^2$

$$P_\tau = \frac{1}{2}\begin{pmatrix} 1+\tau & \sqrt{1-\tau^2} \\ \sqrt{1-\tau^2} & 1-\tau \end{pmatrix}, \quad Q_\tau = \frac{1}{2}\begin{pmatrix} 1+\tau & -\sqrt{1-\tau^2} \\ -\sqrt{1-\tau^2} & 1-\tau \end{pmatrix}, \quad \tau \in (0,1). \tag{12}$$

In the case $m = 2l$, $l \geq 1$, the space $H = \mathbb{C}^{m+1}$ spanned by the joint eigenvectors f_0, $\dots$, f_m, of A_1 and A_2, can be written as $\mathbb{C} \oplus \underbrace{\mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2}_{l \text{ times}}$, or as $\underbrace{\mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2}_{l \text{ times}} \oplus \mathbb{C}$, so that the projections take the form

(13)

$$\begin{aligned}
&P_1 = \delta_1 \oplus P_{p_1} \oplus \dots \oplus P_{p_l}, \quad P_2 = \delta_2 \oplus Q_{q_1} \oplus \dots \oplus Q_{q_l}, \quad H = \mathbb{C} \oplus \underbrace{\mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2}_{l \text{ times}},\\
&P_3 = P_{r_0} \oplus \dots \oplus P_{r_{l-1}} \oplus \delta_3, \quad P_4 = Q_{s_0} \oplus \dots \oplus Q_{s_{l-1}} \oplus \delta_4, \quad H = \underbrace{\mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2}_{l \text{ times}} \oplus \mathbb{C},
\end{aligned}$$

where $\delta_1, \delta_2, \delta_3, \delta_4 \in \{0, 1\}$ are defined from the conditions

$$\alpha_1\delta_1 + \alpha_2\delta_2 = \lambda_0 \in \Delta_{1,d}, \quad \alpha_3\delta_3 + \alpha_4\delta_4 = \mu_m \in \Delta_{2,d},$$

and

$$\begin{aligned}
&p_j = \frac{2\lambda_{2j-1}^2 - (2\lambda_{2j-1} - \alpha_1)(\alpha_1 + \alpha_2)}{\alpha_1(2\lambda_{2j-1} - \alpha_1 - \alpha_2)}, \quad q_j = \frac{2\lambda_{2j-1}^2 - (2\lambda_{2j-1} - \alpha_2)(\alpha_1 + \alpha_2)}{\alpha_2(2\lambda_{2j-1} - \alpha_1 - \alpha_2)},\\
&\qquad 1 \leq j \leq l,\\
&r_j = \frac{2\mu_{2j}^2 - (2\mu_{2j} - \alpha_3)(\alpha_3 + \alpha_4)}{\alpha_3(2\mu_{2j} - \alpha_3 - \alpha_4)}, \quad s_j = \frac{2\mu_{2j}^2 - (2\mu_{2j} - \alpha_4)(\alpha_3 + \alpha_4)}{\alpha_4(2\mu_{2j} - \alpha_3 - \alpha_4)},\\
&\qquad 0 \leq j \leq l-1.
\end{aligned}$$

The conditions (10) are equivalent to

$$\begin{aligned}\alpha_1+\alpha_2+\alpha_3+\alpha_4-2&=(1-\lambda_0-\mu_{2l})/l,\\ \lambda_0+\frac{k}{l}(1-\lambda_0-\mu_m)&\in\Delta_{1,c},\\ \mu_m+\frac{k}{l}(1-\lambda_0-\mu_m)&\in\Delta_{2.c},\quad 1\le k\le l,\end{aligned}$$

where $\lambda_0\in\Delta_{1,d}$, $\mu_m\in\Delta_{2,d}$ (total of 16 possibilities).

In the case $m=2l-1$, $l\ge 1$, the space H spanned by $f_0,\dots,f_m$ can be written as $\mathbb{C}\oplus\underbrace{\mathbb{C}^2\oplus\dots\oplus\mathbb{C}^2}_{l-1\text{ times}}\oplus\mathbb{C}$, or as $\underbrace{\mathbb{C}^2\oplus\dots\oplus\mathbb{C}^2}_{l\text{ times}}$, so that the projections take the form

$$\begin{aligned}&P_1=\delta_1\oplus P_{p_1}\oplus\dots\oplus P_{p_{l-1}}\oplus\delta_2,\\ &P_2=\delta_3\oplus Q_{q_1}\oplus\dots\oplus Q_{q_{l-1}}\oplus\delta_4, && H=\mathbb{C}\oplus\underbrace{\mathbb{C}^2\oplus\dots\oplus\mathbb{C}^2}_{l-1\text{ times}}\oplus\mathbb{C},\\ &P_3=P_{r_0}\oplus\dots\oplus P_{r_{l-1}},\\ &P_4=Q_{s_0}\oplus\dots\oplus Q_{s_{l-1}}, && H=\underbrace{\mathbb{C}^2\oplus\dots\oplus\mathbb{C}^2}_{l\text{ times}},\end{aligned}\tag{14}$$

where $\delta_1,\delta_2,\delta_3,\delta_4\in\{0,1\}$ are defined from the conditions

$$\alpha_1\delta_1+\alpha_2\delta_2=\lambda_0\in\Delta_{1,d},\quad \alpha_1\delta_3+\alpha_2\delta_4=\lambda_m\in\Delta_{1,d},$$

and

$$\begin{aligned}p_j&=\frac{2\lambda_{2j-1}^2-(2\lambda_{2j-1}-\alpha_1)(\alpha_1+\alpha_2)}{\alpha_1(2\lambda_{2j-1}-\alpha_1-\alpha_2)},\\ q_j&=\frac{2\lambda_{2j-1}^2-(2\lambda_{2j-1}-\alpha_2)(\alpha_1+\alpha_2)}{\alpha_2(2\lambda_{2j-1}-\alpha_1-\alpha_2)},\quad 1\le j\le l-1,\\ r_j&=\frac{2\mu_{2j}^2-(2\mu_{2j}-\alpha_3)(\alpha_3+\alpha_4)}{\alpha_3(2\mu_{2j}-\alpha_3-\alpha_4)},\\ s_j&=\frac{2\mu_{2j}^2-(2\mu_{2j}-\alpha_4)(\alpha_3+\alpha_4)}{\alpha_4(2\mu_{2j}-\alpha_3-\alpha_4)},\quad 0\le j\le l-1.\end{aligned}$$

The conditions (11) are equivalent to

$$\begin{aligned}\alpha_1+\alpha_2+\alpha_3+\alpha_4-2&=\frac{1}{l}(\alpha_1+\alpha_2-\lambda_1-\lambda_m),\\ \lambda_0+\frac{k}{l}(\alpha_1+\alpha_2-\lambda_0-\lambda_m)&\in\Delta_{1,c},\quad 1\le k\le l-1,\\ 1-\lambda_0-\frac{k}{l}(1-\lambda_0-\mu_{2l})&\in\Delta_{2.c},\quad 0\le k\le l-1,\end{aligned}$$

where $\lambda_0,\lambda_m\in\Delta_{1,d}$ (total of 16 possibilities).

3.3. **Main theorem.** The main result of this paper is the following.

Theorem 2. *Any irreducible orthoscalar representation of a finite poset S such that S can be decomposed into a union of two unitarily one-parameter sets as described above, is finite-dimensional.*

Proof. Let $S\ni g\mapsto P_g$ be an irreducible orthoscalar representation of S with character $\alpha=(\alpha_g)_{g\in S}$.

Introduce operators

$$A_1 = \sum_{g \in S_1} \alpha_g P_g, \quad A_2 = \sum_{g \in S_2} \alpha_g P_g \tag{15}$$

and let Δ_1, Δ_2 be the corresponding sets described by Proposition 3, so that $\sigma(A_1) \subset \Delta_1$, $\sigma(A_2) \subset \Delta_2$. Then the orthoscalarity condition is equivalent to $A_1 + A_2 = I$. We also write Σ_1 and Σ_2 for numbers (3) corresponding to S_1 and S_2 respectively.

First, we show that A_1 has an eigenvalue. Let $\lambda_0 \in \sigma(A_1)$. Then, since $A_1 + A_2 = I$, $\mu_0 = 1 - \lambda_0 \in \sigma(A_2) \subset \Delta_2$.

For μ_0 there can be two possibilities. If μ_0 lies in the discrete part of Δ_2, then μ_0 is an eigenvalue of A_2 and therefore, λ_0 is an eigenvalue of A_1.

If μ_0 lies in the continuous part of Δ_2, then $\mu_1 = \Sigma_2 - 1 \in \sigma(A_2)$, and since $A_1 + A_2 = I$, $\lambda_1 = 1 - \mu_1 \in \sigma(A_1) \subset \Delta_1$. If λ_1 belongs to the discrete part of Δ_1, then it is an eigenvalue of A_1, otherwise $\lambda_2 = \Sigma_1 - \lambda_1 \in \sigma(A_1)$ etc.

Thus, we have the following sequence of numbers:

$$\begin{aligned} &\lambda_0 \to \mu_0 = 1 - \lambda_0 \to \mu_1 = \Sigma_2 - \mu_0 \\ &\qquad \to \lambda_1 = 1 - \mu_1 \to \lambda_2 = \Sigma_1 - \lambda_1 \to \mu_2 = 1 - \lambda_2 \to \dots, \end{aligned} \tag{16}$$

and we terminate this sequence as soon as λ_k hits into the discrete part of Δ_1 or μ_k hits into the discrete part of Δ_2 which would mean that all the numbers λ_k are eigenvalues of A_1, and μ_k are eigenvalues of A_2. Introduce $\Lambda = \Sigma_1 + \Sigma_2 - 2$, then simple calculations yield

$$\begin{aligned} \lambda_{2k} &= \lambda_0 + k\Lambda, \\ \lambda_{2k+1} &= \Sigma_1 - \lambda_0 - (k+1)\Lambda, \\ \mu_{2k} &= 1 - \lambda_0 - k\Lambda, \\ \mu_{2k+1} &= \Sigma_2 - 1 + \lambda_0 + k\Lambda, \quad k = 0, 1, \dots . \end{aligned} \tag{17}$$

If $\Lambda \neq 0$, these sequences are unbounded, therefore, assuming $\lambda_k \in \sigma(A_1)$, $\mu_k \in \sigma(A_2)$ we conclude that the sequence (16) terminates, therefore, it consists of eigenvalues of A_1 and A_2. If $\Lambda = 0$, then $(2A_1 - \Sigma_1)^2 = (2A_2 - \Sigma_2)^2$ commutes with all P_g, $g \in S$, and due to the irreducibility is a scalar operator. Then $\sigma(A_1) = 1 - \sigma(A_2)$ consists of two points, which are eigenvalues.

This way, we have shown that in the case where $\Lambda \neq 0$ one can assume that A_1 has at least one eigenvalue in the discrete part of Δ_1. Taking this eigenvalue as λ_0 in (16) and repeating the argument above, we conclude that the spectrum of A_1 consists of a finite number of eigenvalues, $\lambda_0, \dots, \lambda_n$. Moreover, similarly to the case of quadruples of projections considered in Section 3.2 one can construct a series of corresponding eigenvectors $f_0, \dots, f_n$, span of which is an invariant subspace for all P_g, $g \in S$ and thus is the whole space H.

If $\Sigma_1 + \Sigma_2 - 2 = 0$, we have that either $\mu_0 = 1$ belongs to the discrete part of Δ_2 and irreducible representation is one-dimensional, or $\mu_0 = 1$ belongs to the continuous part of Δ_2, then $\lambda_1 = 2 - \Sigma_2 = 2 - \Sigma_1 - \Sigma_2 + \Sigma_1 = \Sigma_1$ belongs to the discrete part of Δ_1 and irreducible representation is two-dimensional. □

Remark 3. The proof in fact establishes a method to describe all irreducible representations of S, their dimensions and explicit formulas for the projections.

Remark 4. For the case where $\cap_{g \in S_1} \ker P_g \neq \{0\}$, the value of Λ enables one to obtain a rough estimate for the dimension of irreducible representations: for dimension $k \geq 2$, one can see that $\Lambda > 0$, $1 - (k-1)\Lambda \geq 0$, which implies $k \leq \Lambda^{-1} + 1$.

4. Essential posets

Let S be a poset, and let $S \ni g \mapsto P_g$ be its orthoscalar representation,

$$\sum_{g \in S} \alpha_g P_g = I.$$

If $P_h = 0$ for some $h \in S$, then the corresponding term can be excluded from the sum above, and the family P_g, $g \neq h$ forms an orthoscalar representation of the $S \setminus h$ poset. The same way, if $P_h = I$ for some $h \in S$, then

$$\sum_{g \in S, g \neq h} \frac{\alpha_g}{1 - \alpha_h} P_g = I,$$

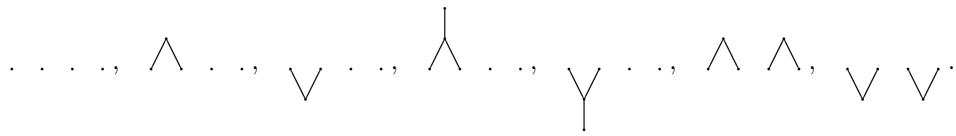
and the family P_g, $g \neq h$ forms an orthoscalar representation of the $S \setminus h$ poset.

Also, if $h < k$ and $P_h = P_k$, then the family P_g, $g \neq h$, forms an orthoscalar representation of the $S \setminus h$ poset with α_k replaced by $\alpha'_k = \alpha_h + \alpha_k$.

In all these cases, the representation of S is essentially determined by a representation of a smaller poset $S \setminus h$.

Definition 3. *We say that an orthoscalar representation $S \ni g \mapsto P_g$ of S is essential, if $P_g \neq 0$, $P_g \neq I$ for all $g \in S$, and $P_g \neq P_h$ for all $g, h \in S$, $g < h$. We say that a poset S is essential if it possesses an irreducible essential orthoscalar representation.*

Theorem 3. *Let S be an essential poset which is a union of two unitarily one-parameter posets. Then S is one of the following posets:*



Proof. We keep the notations used in the previous Section. First consider the case where $\Lambda = 0$. Then any irreducible representation is one or two-dimensional. If S possesses two elements $g < h$, then either $P_h = I$, or $P_g = 0$, or $P_g = P_h$, i.e. irreducible orthoscalar representation is not essential. Therefore, for $\Lambda = 0$, the only poset with essential irreducible representations is $S = (1, 1, 1, 1)$, the poset considered in Section 3.2.

From now on, we assume $\Lambda \neq 0$. In this case we can assume that $\sigma(A_1)$ contains an eigenvalue λ_0 in the discrete part of Δ_1. Then the argument used in the proof of Theorem 2 implies that there can be the following two possibilities.

(i). Dimension $\dim H = n + 1$ is even, λ_n lies in the discrete part of Δ_1, other eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ lie in the continuous part of Δ_1, all μ_k, $k = 0, \ldots, n$, lie in the continuous part of Δ_2.

(ii). Dimension $\dim H = n + 1$ is odd, eigenvalues $\lambda_1, \ldots, \lambda_n$ lie in the continuous part of Δ_1, eigenvalue μ_n lies in the discrete part of Δ_2, all other μ_k, $k = 0, \ldots, n - 1$ lie in the continuous part of Δ_2.

Consider the case (i). Let h_1, h_2 be a (unique) pair of incomparable elements of S_2. Since all μ_k, $k = 0, \ldots, n$, lie in the continuous part of Δ_2, we see from the structure theorem for a pair of projections P_{h_1}, P_{h_2}, that $P_h = 0$ for any $h < h_1$, $h < h_2$, and $P_h = I$ for any $h > h_1$, $h > h_2$. Therefore, an essential irreducible orthoscalar representation of S exists in even dimension only if S_2 consists of two incomparable points, $S_2 = (1, 1)$.

For the set S_1, we have the following. Let g_1, g_2 be a (unique) pair of incomparable elements in S_1. By the structure theorem for a pair of projections P_{g_1}, P_{g_2} we decompose

$$H = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2 \oplus \mathbb{C}^1$$

into invariant with respect to P_g, $g \in S_1$, irreducible subspaces. Then

$$\begin{aligned} P_g &= \delta_1 \oplus I_2 \oplus \cdots \oplus I_2 \oplus \delta_2, \quad g > g_1, \quad g > g_2, \\ P_h &= \delta_3 \oplus 0_2 \oplus \cdots \oplus 0_2 \oplus \delta_4, \quad h < g_1, \quad h < g_2, \end{aligned} \tag{18}$$

where $\delta_1, \delta_2, \delta_3, \delta_4 \in \{0, 1\}$. Moreover, to exclude the cases $P_g = I$ and $P_h = 0$ we assume that $\delta_1 + \delta_2 < 2$, $\delta_3 + \delta_4 > 0$. In each of the two invariant one-dimensional blocks, the sum $P_{g_1} + P_{g_2}$ can take values 0, 1, or 2, and the following cases can arise.

1). In the both blocks the sum is 0. Then for P_g in (18) we can have $\delta_1 + \delta_2 = 0$, or $\delta_1 + \delta_2 = 1$, thus there can be at most two different nontrivial projections P_{g_3}, P_{g_4}, $g_3 > g_1$, $g_3 > g_2$, $g_4 > g_3$. For P_h we have $\delta_3 = \delta_4 = 0$, thus there are no nonzero P_h.

2). In one block the sum is 0, and in the other one it is 1. Then for P_g in (18) we have $\delta_1 + \delta_2 = 1$, and there can be at most one nontrivial projection P_{g_3}, $g_3 > g_1$, $g_3 > g_2$. For P_h we again have $\delta_3 = \delta_4 = 0$, thus there are no nonzero P_h.

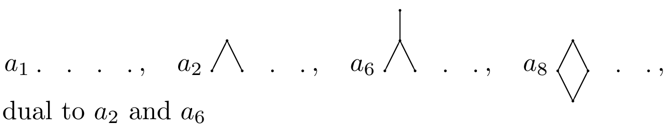
3). In the both blocks the sum is 1. Then for P_g in (18) we have $\delta_1 + \delta_2 = 2$, and for P_h we have $\delta_3 = \delta_4 = 0$, thus there are no $P_g \neq I$, $P_h \neq 0$.

4). In the first block the sum is 0, in the second one the sum is 2, or in the first block the sum is 2, in the second one the sum is 0. For P_g in (18) we have $\delta_1 + \delta_2 = 1$, and for P_h we have $\delta_3 + \delta_4 = 1$. There can be at most one nontrivial projection P_h, $h < g_1$, $h < g_2$, and at most one nontrivial projection P_g, $g > g_1$, $g > g_2$.

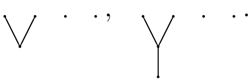
5) In one block the sum is 2, in the other one the sum is 1. Then similarly to the case 2 there can be at most one nontrivial projection P_h, $h < g_1$, $h < g_2$. For any $g > g_1$, $g > g_2$ we have $P_g = 0$.

6) In the both blocks the sum is 2. Then similarly to the case 1 there can be at most two different nontrivial projections P_{h_1}, P_{h_2}, $h_1 < g_1$, $h_1 < g_2$, $h_2 < h_1$. For any $g > g_1$, $g > g_2$ we have $P_g = 0$.

Therefore, an essential irreducible orthoscalar representation of even dimension can exist only for the following posets (we use the notation from [14]):



and the posets dual to a_2 and a_6

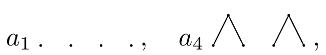


We show that the a_8 poset arising in the case 4 above is not in fact essential, i.e., any its irreducible orthoscalar representation is not essential. Indeed, in the case 4 above assume that in the first one-dimensional block $P_{g_1} = P_{g_2} = 0$, and in the second one $P_{g_1} = P_{g_2} = 1$, then $\delta_1 = \delta_3 = 0$, $\delta_2 = \delta_4 = 1$, and in essential representation $\sigma(A_1)$ contains 0 and $\alpha_g + \alpha_{g_1} + \alpha_{g_2} + \alpha_h$. Then the sequence (16) is

$$\begin{aligned} \lambda_0 = 0 \to \mu_0 = 1 \to \mu_1 = \Sigma_2 - 1 \to \lambda_1 = 2 - \Sigma_2 \to \lambda_2 = \Lambda \to \dots \\ \to \mu_{2n+1} = \Sigma_2 - 1 + n\Lambda \to \lambda_{2n+1} = \alpha_g + \alpha_{g_1} + \alpha_{g_2} + \alpha_h. \end{aligned}$$

Since we have already shown that an essential orthoscalar representation of the a_8 poset must have dimension more than 2, we conclude that $\lambda_2 \in \sigma(A_1)$ and therefore $\Lambda > 0$. On the other side, since $\lambda_{2k+1} = \Sigma_1 - (k+1)\Lambda$, we have $\lambda_1 > \lambda_3 > \cdots > \lambda_{2n+1}$, therefore $\lambda_{2n+1} \leq \lambda_1 = \Sigma_1 - \Lambda < \alpha_g + \alpha_{g_1} + \alpha_{g_2}$.

In the case (ii) of odd dimension, similar arguments lead to the following posets:



and the poset dual to a_4

$$\bigvee \ \bigvee .$$ □

In the following section we will show that all posets listed in Theorem 3 admit essential irreducible orthoscalar representations, and therefore they are essential ones.

5. Examples

As was shown above, the essential posets that are unions of two unitarily one-parameter posets may have 4, 5 or 6 elements. Let $S = S_1 \cup S_2$ be one of them with $S_1 \cap S_2 = \emptyset$. Obviously S and $S_2 \cup S_1$ are isomorphic. Therefore we consider only the posets in which S_1 has two comparable elements. The operators A_1 and A_2 will be defined by the formula (15), where P_g is the orthoprojection corresponding to the element $g \in S$ in an essential orthoscalar irreducible representation of S. Note that in all posets a_2, a_4 and a_6 below the element g_5 (or g_6) is the maximal element of S_1. Whence the ranges of A_1 and P_5 (or P_6) coincide. Thus if A_1 is invertible, then $P_5 = I$ (or $P_6 = I$) and hence the representation is not essential. Therefore A_1 is singular for every essential representation of a_2, a_4, a_6. Let $\Lambda = \Sigma_1 + \Sigma_2 - 2$.

1) *Representations of* a_2. Let $a_2 = S_1 \cup S_2$, $S_1 = \{g_1, g_2, g_5 \mid g_5 > g_1, g_5 > g_2\}$, $S_2 = \{g_3, g_4\}$ and orthoprojections $P_1, \dots, P_5$ form an essential orthoscalar representation of a_2 with character $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. The operator A_1 is singular so there exists f_0 such that $A_1 f_0 = \lambda_0 f_0 = 0$. According to Theorem 2, the sequence λ_i obtained by the formulas (17) consists of eigenvalues of A_1 and the sequence μ_j consists of eigenvalues of A_2 for $i = 0, 1, \dots, m'$ and $j = 0, 1, \dots, m'$ with some positive m'. Let f_0, $f_1, \dots, f_{m'}$ be the corresponding eigenvectors. This consequence can be obtain from f_0 using special linear combinations of P_i. Let $D_1(x) = P_1 + \phi(x) P_2$, $D_2(x) = P_3 + \psi(x) P_4$, where

$$\phi(x) = \frac{\alpha_2(\lambda - \alpha_2 - \alpha_5)}{\alpha_1(\alpha_1 + \alpha_5 - \lambda)}, \quad \psi(x) = \frac{\alpha_4(\lambda - \alpha_4)}{\alpha_3(\alpha_3 - \lambda)}.$$

Then $f_1 = D_2(\mu_0)$, $f_{2j} = D_1(\lambda_{2j-1})$, $f_{2j+1} = D_2(\mu_{2j})$. Assume that

$$\lambda_{2i} \in (\alpha_5, \alpha_1 + \alpha_5) \cup (\alpha_2 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_5) \tag{19}$$

and

$$\mu_{2j-1} \in (0, \alpha_3) \cup (\alpha_4, \alpha_3 + \alpha_4), \tag{20}$$

where $i, j = 1, \dots, m$. There exist only two cases in which the representation can be reconstructed: $\lambda_{2m+1} \in \{\alpha_5, \alpha_1 + \alpha_5, \alpha_2 + \alpha_5\}$ with $m' = 2m + 1$ or $\mu_{2m} \in \{0, \alpha_3, \alpha_4\}$ with $m' = 2m$. We consider both cases in details.

(**i**) $\lambda_{2m+1} \in \{\alpha_5, \alpha_1 + \alpha_5, \alpha_2 + \alpha_5\}$. The subspace

$$\operatorname{span}(f_0, f_1, f_2, \dots, f_{2m+1}) \tag{21}$$

is invariant under the act of $P_1, \dots, P_5$ for $\alpha_1 \neq \alpha_2$. Operators P_1, $P_2, \dots, P_5$ are restored up to unitary equivalence: $P_5 = 0 \oplus I_{2m}$, $P_1, \dots, P_4$ have the form (14) with $\delta_1 = \delta_3 = 0$, and $\delta_2 = \delta_4 = 0$ if $\lambda_{2m+1} = \alpha_5$ or

$$\delta_2 = 1 - \delta_4 = \begin{cases} 1, & \text{if } \lambda_{2m+1} = \alpha_5 + \alpha_1, \\ 0, & \text{if } \lambda_{2m+1} = \alpha_5 + \alpha_2. \end{cases}$$

The parameters p_j, q_j, r_j, s_j are calculated by λ_i and μ_i after the substitution $\lambda_i = \lambda_i - \alpha_5$.

Let now $\alpha_1 = \alpha_2$, then $\lambda_{2m+1} = \alpha_1 + \alpha_5 = \alpha_2 + \alpha_5$. Note, that

$$A_1 f_{2m+1} = (\alpha_1 + \alpha_5) f_{2m+1} = (\alpha_1 + \alpha_5)(P_1 + P_2) f_{2m+1}.$$

So $f'_{2m+1} = P_1 f_{2m+1}$ is an eigenvector of A_1 too. Therefore we get two non-equivalent representations of a_2 with the same formulas on P_i as above except the relation

$$\delta_2 = 1 - \delta_4 = \begin{cases} 1, & \text{if } f'_{2m+1} = f_{2m+1}, \\ 0, & \text{if } f'_{2m+1} = 0. \end{cases}$$

The vector f'_{2m+1} must be a multiple of f_{2m+1} since otherwise f'_{2m+1} does not belong to the subspace (21) and hence we have a new eigenvector of A_1 and the elements of the sequence λ_i are eigenvalues of A_1 with $i > 2m+1$. It is easy to see then that $\lambda_{2m+i} = \lambda_{2m+3-i}$ and $\mu_{2m+i} = \mu_{2m+3-i}$. So if $\lambda_{2m_2+1} = \alpha_5$ or $\mu_{2m_2} \in \{0, \alpha_3, \alpha_4\}$, then the relation (19) or (20) does not hold. If at last $\lambda_{2m_2+1} = 0$, then the subspace

$$\operatorname{span}(f'_{2m+1}, D_2(\mu_{2m+2})f'_{2m+1}, D_1(\lambda_{2m+3})D_2(\mu_{2m+2})f'_{2m+1}, \dots,$$
$$D_2(\mu_{2m_2})D_1(\lambda_{2m_2-1})\dots D_2(\mu_{2m+2})f'_{2m+1})$$

is invariant under the act of $P_1, \dots, P_5$ and so the representation is reducible.

(**ii**) $\mu_{2m} \in \{0, \alpha_3, \alpha_4\}$. The subspace

$$\operatorname{span}(f_0, f_1, f_2, \dots, f_{2m}) \tag{22}$$

is invariant under the act of $P_1, \dots, P_5$ for $\alpha_3 \neq \alpha_4$. Operators $P_1, P_2, \dots, P_5$ are restored up to unitary equivalence and has the form (13) with $\delta_1 = \delta_2 = 0$, and $\delta_3 = \delta_4 = 0$ if $\mu_{2m} = 0$ or

$$\delta_3 = 1 - \delta_4 = \begin{cases} 1, & \text{if } \mu_{2m} = \alpha_3, \\ 0, & \text{if } \mu_{2m} = \alpha_4, \end{cases}$$

and $P_5 = 0 \oplus I_{2m}$. Note that the parameters p_j, q_j, r_j, s_j are calculated here also after the substitution $\lambda_i = \lambda_i - \alpha_5$.

As above in (**i**), we get two different irreducible representations for $\mu_{2m} = \alpha_3 = \alpha_4$. The formulas for P_i are the same except the relation

$$\delta_3 = 1 - \delta_4 = \begin{cases} 1, & \text{if } f'_{2m} = f_{2m}, \\ 0, & \text{if } f'_{2m} = 0, \end{cases}$$

where $f'_{2m} = P_3 f_{2m}$. The vector f'_{2m} must be a multiple of f_{2m} since otherwise we obtain an invariant subspace with smaller dimension or the violation in (19) or (20). The proof of the fact is similar and we leave it to the reader.

Thus we proved that with fixed coefficients α_i there exist at most two non-equivalent essential representations of a_2. The most simple way to construct the examples is to put $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2 + \epsilon$, $\alpha_5 = 1/(2m+5) - 2\epsilon + 8\epsilon/(2m+5)$. Then $\lambda_{2m+1} = \alpha_5$, (19) and (20) hold for small irrational ϵ and hence we obtain one essential representation of a_2 of dimension $2m$. If we put $\alpha_5 = 1/(4m) - 2\epsilon - 3\epsilon/(2m)$, then $\lambda_{2m+1} = \alpha_1$ and we have two non-equivalent representations of a_2 of dimension $2m$. It easy to see that if $\alpha_5 = 1/(4m) - 2\epsilon - \epsilon/(2m)$, then $\mu_{2m} = 1/2 + \epsilon = \alpha_3$, that is we have two essential representations of a_2 of the dimension $2m+1$ in this case. For last case we put $\alpha_5 = 1/(2m) - 2\epsilon$, then $\mu_{2m} = 0$ and for small irrational ϵ, the poset a_2 has the only one up to unitary equivalence essential representation in the dimension $2m+1$.

2) *Representations of* a_4. Let $a_4 = S_1 \cup S_2$, $S_1 = \{g_1, g_2, g_5 \mid g_5 > g_1, g_5 > g_2\}$, $S_2 = \{g_3, g_4, g_6 \mid g_6 > g_3, g_6 > g_4\}$ and orthoprojections $P_1, \dots, P_6$ form an essential orthoscalar representation of a_4 with character $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$. The operator A_1 is singular so we can construct the sequences λ_i and μ_j of eigenvalues of A_1 and A_2 as we did for the representations of a_2. Note that A_2 is also singular since otherwise P_6 will be the identity matrix. Therefore there exist m such that (19) and (20) hold for every $i, j = 0, \dots, m$ and $\mu_{2m} = 0$. Whence there exists only one up to unitary equivalence representation. The operators P_i are restored by the formulas (13) where $\delta_i = 0$, the parameters p_j, q_j, r_j, s_j are calculated by λ_i and μ_i after two substitutions $\lambda_i = \lambda_i - \alpha_5$ and $\mu_i = \mu_i - \alpha_6$, $P_5 = 0 \oplus I_{2m}$ and $P_6 = I_{2m} \oplus 0$.

To find an appropriate character we set

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2 + \epsilon, \quad \alpha_5 = \epsilon/2. \tag{23}$$

Then the relation $\mu_{2m} = 0$ yields $\alpha_6 = 1/(2m) - 5\epsilon/2$. The inclusions (19) and (20) hold for small ϵ.

3) *Representations of* a_6. Let $a_6 = S_1 \cup S_2$, $S_1 = \{g_1, g_2, g_5, g_6 \mid g_6 > g_5 > g_1, g_5 > g_2\}$, $S_2 = \{g_3, g_4\}$ and orthoprojections $P_1, \dots P_6$ form an essential orthoscalar representation of a_6 with character $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$. The operator A_1 is singular, hence we have again the sequences λ_i and μ_j. If $\lambda_{2m+1} \neq \alpha_6$ for every m, then $P_6 = P_5$. Really, the operator A_1 is a sum of four nonnegative operators and if $\alpha_6 \notin \sigma(A_1)$, then every nonzero number of $\sigma(A_1)$ is greater or equal to $\alpha_5 + \alpha_6$. So the ranges of A_1, P_5 and P_6 coincide. Whence $P_5 = P_6$.

Thus $\lambda_{2m+1} = \alpha_6$ for some $m > 0$ and we have only one up to unitary equivalence essential representation of a_6. The operator P_i, $i = 1, \dots, 4$ have the form (14) with $\delta_i = 0$ and the parameters p_j, q_j, r_j, s_j calculated by λ_i and μ_i after the substitution $\lambda_i = \lambda_i - \alpha_5 - \alpha_6$. The operator $P_5 = 0 \oplus I_{2m} \oplus 0$ and $P_6 = 0 \oplus I_{2m+1}$.

To find the character for which the orthoscalar representation exists we set α_i as in (23) and $\alpha_6 = 1/(2m+1) - (5m+2)\epsilon/(2m+1)$. Then $\lambda_{2m+1} = \alpha_6$ and for small ϵ the inclusions (19) and (20) hold.

4) *Representations of dual to* a_i. All essential representations of the posets dual to a_2, a_4 and a_6 can be calculated from the described representations using duality and the formulas $Q_g = I - P_g$.

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References

1. L. A. Nazarova, A. V. Roiter, *Representations of partially ordered sets*, Investigations on Representation Theory, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) **28** (1972), 5–31. (Russian); English transl. J. Sov. Math. **23** (1975), 585–607.
2. M. M. Kleiner, *Partially ordered sets of finite type*, Investigations on Representation Theory, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) **28** (1972), 32–41. (Russian); English transl. J. Sov. Math. **23** (1975), 607–615.
3. V. M. Bondarenko, V. Futorny, T. Klimchuk, V. V. Sergeichuk, K. Yusenko, *Systems of subspaces of a unitary space*, Linear Algebra Appl. **438** (2013), no. 5, 2561–2573.
4. S. A. Kruglyak, V. I. Rabanovich, Yu. S. Samoilenko, *On sums of projections*, Funktsional. Anal. i Prilozhen. **36** (2002), no. 3, 20–35. (Russian); English transl. Funct. Anal. Appl. **36** (2002), no. 3, 182–195.
5. S. A. Kruglyak, A. V. Roiter, *Locally scalar representations of graphs in the category of Hilbert spaces*, Funktsional. Anal. i Prilozhen. **39** (2005), no. 2, 13–30. (Russian); English transl. Funct. Anal. Appl. **39** (2005), no. 2, 91–105.
6. S. Albeverio, V. Ostrovskyi, Yu. Samoilenko, *On functions on graphs and representations of a certain class of $*$-algebras*, J. Algebra **308** (2007), 567–582.
7. T. Weist, K. Yusenko, *Unitarizable Representations of Quivers*, Algebr. Represent. Theory **16** (2013), no. 5, 1349–1383.
8. Yu. Samoilenko, K. Yusenko, *Kleiner's theorem for unitary representations of posets*, Linear Algebra Appl. **437** (2012), no. 2, 581–588.
9. V. Ostrovskyi, Yu. Samoilenko, *Introduction to the Theory of Representations of Finitely Presented $*$-Algebras.* I. *Representations by Bounded Operators*, Rev. Math. & Math. Phys. **11** Harwood Academic Publishers, Amsterdam, 1999.
10. A. A. Kirichenko, *On linear combinations of orthoprojectors*, Uch. Zap. Tavrich. Nat. Univ. **54** (2002), no. 2, 31–39. (Russian)
11. K. A. Yusenko, *On quadruples of projectors connected by a linear relation*, Ukrain. Mat. Zh. **58** (2006), no. 9, 1289–1295. (Ukrainian); English transl. Ukrainian Math. J. **58** (2006), no. 9, 1462–1470.
12. V. L. Ostrovskyi and Yu. S. Samoilenko, *On spectral theorems for families of linearly connected selfadjoint operators with prescribed spectra associated with extended Dynkin graphs*, Ukrain.

Mat. Zh. **58** (2006), no. 11, 1556–1570. (Ukrainian); English transl. Ukrainian Math. J. **58** (2006), no. 11, 1768-1785.

13. V. Ostrovskyi, *On ∗-representations of a certain class of algebras related to a graph*, Methods Funct. Anal. Topology **11** (2005), no. 3, 250–256.
14. V. V. Otrashevskaya, *On representations of one-parameter partially ordered sets*, Matrix problems, Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1977, pp. 144–149. (Russian).
15. S. A. Kruglyak, L. A. Nazarova, A. V. Roiter, *Orthoscalar representations of quivers in the category of Hilbert spaces*, J. Math. Sci. **145** (2007), no. 1, 4793–4804.

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