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CONTINUITY OF OPERATOR-VALUED FUNCTIONS IN THE $*$-ALGEBRA OF LOCALLY MEASURABLE OPERATORS

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Dedicated to Yuryi Stevanofich Samoilenko on the occasion of his 70th birthday

Abstract. In the present paper we establish sufficient conditions for a complex-valued function f defined on $\mathbb{R}$ which guarantee continuity of an operator-function $T \mapsto f(T)$ w.r.t. the topology of local measure convergence in the $*$-algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated to a von Neumann algebra $\mathcal{M}$.

1. Introduction

The development of integration theory for a faithful normal semifinite trace τ defined on a von Neumann algebra $\mathcal{M}$ has led to a need for consideration of the $*$-algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators affiliated with $\mathcal{M}$, see, e.g., [11]. This algebra is a solid $*$-subalgebra of the $*$-algebra $S(\mathcal{M})$ of all measurable operators affiliated with $\mathcal{M}$. The $*$-algebra $S(\mathcal{M})$ was introduced by I. Segal [13] in order to describe a "noncommutative version" of the $*$-algebra of measurable complex-valued functions. If $\mathcal{M}$ is a commutative von Neumann algebra, then $\mathcal{M}$ can be identified with the $*$-algebra $L_\infty(\Omega,\Sigma,\mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space (Ω,Σ,μ) with a measure μ having the direct sum property. In this case, the $*$-algebra $S(\mathcal{M})$ is identified with the $*$-algebra $L_0(\Omega,\Sigma,\mu)$ of all measurable complex-valued functions defined on (Ω,Σ,μ) [13].

The $*$-algebras $S(\mathcal{M},\tau)$ and $S(\mathcal{M})$ are substantive examples of EW^*-algebras E of closed linear operators, affiliated with the von Neumann algebra $\mathcal{M}$, which act on the same Hilbert space $\mathcal{H}$ as $\mathcal{M}$ and have the bounded part $E_b = E\cap\mathcal{B}(\mathcal{H})$ coinciding with $\mathcal{M}$ [7], where $\mathcal{B}(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$. A natural desire of obtaining a maximal EW^*-algebra E with $E_b = \mathcal{M}$ has led to a construction of the $*$-algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with the von Neumann algebra $\mathcal{M}$ (see, for example, [17]). It was shown in [3] that any EW^*-algebra E satisfying $E_b = \mathcal{M}$ is a solid $*$-subalgebra of $LS(\mathcal{M})$.

In the case when there exists a faithful normal finite trace τ on $\mathcal{M}$, all three $*$-algebras $LS(\mathcal{M})$, $S(\mathcal{M})$, and $S(\mathcal{M},\tau)$ coincide [10, § 2.6], and a natural topology that endows these $*$-algebras with the structure of a topological $*$-algebra is the measure topology t_τ induced by the trace τ [11]. If τ is a semifinite but not a finite trace, then one can consider the τ-local measure topology $t_{\tau\, l}$ and the weak τ-local measure topology $t_{w\tau\, l}$ [2]. However, in the case where $\mathcal{M}$ is not of finite type, the multiplication is not jointly continuous in the two variables with respect to these topologies. In this respect, it makes sense to use, for the $*$-algebra $LS(\mathcal{M})$, the local measure topology $t(\mathcal{M})$, which was defined in [17] for any von Neumann algebras and which endows $LS(\mathcal{M})$ with the structure of a complete topological $*$-algebra [10, § 3.5].

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It is known that for all $T \in LS_h(\mathcal{M}) = \{S \in LS(\mathcal{M}) : S^* = S\}$ and for any Borel complex function f defined on the set of real numbers $\mathbb{R}$ and bounded on compact subsets of $\mathbb{R}$, a locally measurable operator $f(T) \in LS(\mathcal{M})$ is correctly defined [10, § 2.3]. Therefore it is natural to consider an operator-function $T \mapsto f(T)$ from $LS_h(\mathcal{M})$ to $LS(\mathcal{M})$ and study the question of its continuity w.r.t. topology $t(\mathcal{M})$ of convergence locally in measure. In the case of strong operator topology, when $LS(\mathcal{M}) = \mathcal{M} = \mathcal{B}(\mathcal{H})$, this problem was investigated by I. Kaplansky [9], R. V. Kadison [8] and E. V. Davies [5]. In the case of the EW^*-algebra $S(\mathcal{M}, \tau)$ and measure topology t_τ this problem was investigated by O. E. Tikhonov [16].

In this paper we prove that, for $\{T_\alpha\} \subset LS_h(\mathcal{M})$, $T \in LS_h(M)$ and Borel function f continuous on the spectrum $\sigma(T)$ of the operator T, the convergence $T_\alpha \xrightarrow{t(\mathcal{M})} T$, implies the convergence $f(T_\alpha) \xrightarrow{t(\mathcal{M})} f(T)$.

We use the von Neumann algebra terminology, notations and results from [14, 15], and those that concern the theory of measurable and locally measurable operators from [10, 17].

2. Preliminaries

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$ of complex numbers, let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$, let I be the identity operator on $\mathcal{H}$, $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, let $\mathcal{P}(\mathcal{M}) = \{P \in \mathcal{M} : P^2 = P = P^*\}$ be the lattice of all projections in $\mathcal{M}$, and let $\mathcal{P}_{\text{fin}}(\mathcal{M})$ be the sublattice of its finite projections. The center of a von Neumann algebra $\mathcal{M}$ will be denoted by $\mathcal{Z}(\mathcal{M})$.

A closed linear operator T affiliated with the von Neumann algebra $\mathcal{M}$ with everywhere dense domain $\mathcal{D}(T) \subset \mathcal{H}$ is called *measurable* if there exists a sequence $\{P_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$ such that $P_n \uparrow I$, $P_n(\mathcal{H}) \subset \mathcal{D}(T)$, and $P_n^{\perp} = I - P_n \in \mathcal{P}_{\text{fin}}(\mathcal{M})$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. The set $S(\mathcal{M})$ of all measurable operators is a $*$-algebra with identity I over the field $\mathbb{C}$ with respect to the strong sum $\overline{T+S}$, strong product $\overline{TS}$ and the adjoint operation T^* [13]. It is clear that $\mathcal{M}$ is a $*$-subalgebra of $S(\mathcal{M})$.

A closed linear operator T affiliated with $\mathcal{M}$ with everywhere dense domain $\mathcal{D}(T) \subset \mathcal{H}$ is called *locally measurable* with respect to $\mathcal{M}$ if there exists a sequence $\{Z_n\}_{n=1}^{\infty}$ of central projections in $\mathcal{M}$ such that $Z_n \uparrow I$ and $TZ_n \in S(\mathcal{M})$ for all $n \in \mathbb{N}$.

The set $LS(\mathcal{M})$ of all locally measurable operators with respect to $\mathcal{M}$ is a $*$-algebra with identity I over the field $\mathbb{C}$ with respect to the same algebraic operations as in $S(\mathcal{M})$ [12], [17], in addition $S(\mathcal{M})$ is a $*$-subalgebra of $LS(\mathcal{M})$. If $\mathcal{M}$ is finite, or if $\mathcal{M}$ is a factor, the algebras $S(\mathcal{M})$ and $LS(\mathcal{M})$ coincide.

For every subset $E \subset LS(\mathcal{M})$, the set of all selfadjoint (resp., positive) operators in E is denoted by E_h (resp., E_+). The partial order in $LS_h(\mathcal{M})$ defined by its cone $LS_+(\mathcal{M})$ is denoted by $\leq$.

Let T be a closed operator with dense domain $\mathcal{D}(T)$ in $\mathcal{H}$, let $T = U|T|$ be the polar decomposition of the operator T, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the partial isometry in $\mathcal{B}(\mathcal{H})$ such that $r(T) = U^*U$ is the right support of T. It is known that $T \in LS(\mathcal{M})$ (respectively, $T \in S(\mathcal{M})$) if and only if $|T| \in LS(\mathcal{M})$ (respectively, $|T| \in S(\mathcal{M})$) and $U \in \mathcal{M}$ [10, § 2.3].

Denote by $\mathcal{B}(\mathbb{R})$ the C^*-algebra of all Borel complex-valued functions defined on $\mathbb{R}$ and bounded on compact subsets of $\mathbb{R}$. It is know that $f(T) \in LS(\mathcal{M})$ for all $T \in LS_h(\mathcal{M})$ and $f \in \mathcal{B}(\mathbb{R})$ [10, § 2.3]. In particular, for the real-value bounded Borel function φ_λ at $\mathbb{R}$, such that $\varphi_\lambda(t) = 1$ for $t \leq \lambda$ and $\varphi_\lambda(t) = 0$ for $t > \lambda$, where λ is a fixed number from $\mathbb{R}$, the inclusion $E_\lambda(T) := \varphi_\lambda(T) \in \mathcal{M}$ holds for all $T \in LS_h(\mathcal{M})$.

Denote by $\|\cdot\|_{\mathcal{M}}$ the C^*-norm in the von Neumann algebra $\mathcal{M}$. We need the following property of the partial order in the algebra $LS(\mathcal{M})$.

Proposition 1. ([1, Proposition 6.1]) *Let $\mathcal{M}$ be a von Neumann algebra, $T, S \in LS_+(\mathcal{M})$ and $S \leq T$. Then $S^{1/2} = AT^{1/2}$ for some $A \in \mathcal{M}$, $\|A\|_{\mathcal{M}} \leq 1$, in particular, $S = ATA^*$.*

Let us now recall the definition of the local measure topology. Firstly let $\mathcal{M}$ be a commutative von Neumann algebra. Then $\mathcal{M}$ is $*$-isomorphic to the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space (Ω, Σ, μ) with the measure μ satisfying the direct sum property (we identify functions that are equal almost everywhere). The direct sum property of a measure μ means that the Boolean algebra of all projections of the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$ is order complete, and for any nonzero $P \in \mathcal{P}(\mathcal{M})$ there exists a nonzero projection $Q \leq P$ such that $\mu(Q) < \infty$.

Consider the $*$-algebra $LS(\mathcal{M}) = S(\mathcal{M}) = L_0(\Omega, \Sigma, \mu)$ of all measurable almost everywhere finite complex-valued functions defined on (Ω, Σ, μ) (functions that are equal almost everywhere are identified). On $L_0(\Omega, \Sigma, \mu)$, define the local measure topology $t(\mathcal{M})$, that is, the linear Hausdorff topology, whose base of neighborhoods of zero is given by

$$\begin{gathered} W(B, \varepsilon, \delta) = \{f \in L_0(\Omega, \Sigma, \mu)\colon \text{ there exists a set } E \in \Sigma \text{ such that} \\ E \subseteq B,\ \mu(B \setminus E) \leq \delta,\ f\chi_E \in L_\infty(\Omega, \Sigma, \mu),\ \|f\chi_E\|_{L_\infty(\Omega,\Sigma,\mu)} \leq \varepsilon\}, \end{gathered}$$

where $\varepsilon,\ \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$, and

$$\chi(\omega) = \begin{cases} 1, & \omega \in E, \\ 0, & \omega \notin E. \end{cases}$$

Convergence of a net $\{f_\alpha\}$ to f in the topology $t(\mathcal{M})$, denoted by $f_\alpha \xrightarrow{t(\mathcal{M})} f$, means that $f_\alpha \chi_B \longrightarrow f\chi_B$ in measure μ for any $B \in \Sigma$ with $\mu(B) < \infty$. It is clear that the topology $t(\mathcal{M})$ does not change if the measure μ is replaced with an equivalent measure.

Let now $\mathcal{M}$ be an arbitrary von Neumann algebra and let φ be a $*$-isomorphism from $\mathcal{Z}(\mathcal{M})$ onto the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$, where μ is a measure satisfying the direct sum property. Denote by $L_+(\Omega, \Sigma, m)$ the set of all measurable real-valued functions defined on (Ω, Σ, μ) and taking values in the extended half-line $[0, \infty]$ (functions that are equal almost everywhere are identified). It was shown in [13] that there exists a mapping

$$d\colon \mathcal{P}(\mathcal{M}) \to L_+(\Omega, \Sigma, \mu)$$

that possesses the following properties:

(i) $d(P) = 0$ if and only if $P = 0$;
(ii) $d(P) \in L_0(\Omega, \Sigma, \mu) \Longleftrightarrow P \in \mathcal{P}_{\text{fin}}(\mathcal{M})$;
(iii) $d(P \vee Q) = d(P) + d(Q)$ if $PQ = 0$;
(iv) $d(U^*U) = d(UU^*)$ for any partial isometry $U \in \mathcal{M}$;
(v) $d(ZP) = \varphi(Z)d(P)$ for any $Z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ and $P \in \mathcal{P}(\mathcal{M})$;
(vi) if $\{P_\alpha\}_{\alpha \in A}, P \in \mathcal{P}(\mathcal{M})$ and $P_\alpha \uparrow P$, then $d(P) = \sup\limits_{\alpha \in A} d(P_\alpha)$.

A mapping $d\colon \mathcal{P}(\mathcal{M}) \to L_+(\Omega, \Sigma, \mu)$ that satisfies properties (i)-(vi) is called a *dimension function* on $\mathcal{P}(\mathcal{M})$.

For arbitrary numbers $\varepsilon, \delta > 0$ and a set $B \in \Sigma$, $\mu(B) < \infty$, we set

$$\begin{gathered} V(B, \varepsilon, \delta) = \{T \in LS(\mathcal{M})\colon \text{ there exist } P \in \mathcal{P}(\mathcal{M}),\ Z \in \mathcal{P}(\mathcal{Z}(\mathcal{M})), \\ \text{such that } TP \in \mathcal{M}, \|TP\|_{\mathcal{M}} \leq \varepsilon,\ \varphi(Z^\perp) \in W(B, \varepsilon, \delta),\ d(ZP^\perp) \leq \varepsilon\varphi(Z)\}. \end{gathered}$$

It was shown in [17] that the system of sets

$$\{\{T + V(B, \varepsilon, \delta)\}\colon\ T \in LS(\mathcal{M}),\ \varepsilon,\ \delta > 0,\ B \in \Sigma,\ \mu(B) < \infty\} \tag{1}$$

defines the Hausdorff vector topology $t(\mathcal{M})$ on $LS(\mathcal{M})$ such that sets (1) form a neighborhood base of the operator $T \in LS(\mathcal{M})$. The topology $t(\mathcal{M})$ is called the *local measure topology* (or the topology of *convergence locally in measure*). It is known that

$(LS(\mathcal{M}), t(\mathcal{M}))$ is a complete topological $*$-algebra, and the topology $t(\mathcal{M})$ does not depend on a choice of dimension function $\mathcal{D}$ and on a choice of $*$-isomorphism φ [10, § 3.5], [17].

We need the following criterion for convergence of nets with respect to this topology.

Proposition 2. ([10, § 3.5])

(i) *A net* $\{P_\alpha\}_{\alpha\in A} \subset \mathcal{P}(\mathcal{M})$ *converges to zero with respect to the topology* $t(\mathcal{M})$ *if and only if there exists a net* $\{Z_\alpha\}_{\alpha\in A} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ *such that* $Z_\alpha P_\alpha \in \mathcal{P}_{\text{fin}}(\mathcal{M})$ *for all* $\alpha \in A$, $\varphi(Z_\alpha^\perp) \stackrel{t(L_0(\Omega))}{\longrightarrow} 0$, *and* $d(Z_\alpha P_\alpha) \stackrel{t(L_0(\Omega))}{\longrightarrow} 0$, *where* $t(L_0(\Omega))$ *is the local measure topology on* $L_0(\Omega, \Sigma, \mu)$, *and* φ *is a* $*$*-isomorphism of* $\mathcal{Z}(\mathcal{M})$ *onto* $L_\infty(\Omega, \Sigma, \mu)$.

(ii) *A net* $\{T_\alpha\}_{\alpha\in A} \subset LS(\mathcal{M})$ *converges to zero with respect to the topology* $t(\mathcal{M})$ *if and only if* $E_\lambda^\perp(|T_\alpha|) \stackrel{t(\mathcal{M})}{\longrightarrow} 0$ *for every* $\lambda > 0$, *where* $\{E_\lambda^\perp(|T_\alpha|)\}$ *is the spectral family for the operator* $|T_\alpha|$.

It follows from Proposition 2 that the topology $t(\mathcal{M})$ induces the topology $t(\mathcal{Z}(\mathcal{M}))$ on $LS(\mathcal{Z}(\mathcal{M}))$; hence, $S(\mathcal{Z}(\mathcal{M}))$ is a closed $*$-subalgebra of $(LS(\mathcal{M}), t(\mathcal{M}))$.

It is clear that

$$X \cdot V(B, \varepsilon, \delta) \subset V(B, \varepsilon, \delta)$$

for any $X \in \mathcal{M}$ with $\|X\|_{\mathcal{M}} \le 1$. Since $V^*(B, \varepsilon, \delta) \subset V(B, 2\varepsilon, \delta)$ [10, § 3.5], we have

$$V(B, \varepsilon, \delta) \cdot Y \subset V(B, 4\varepsilon, \delta)$$

for all $Y \in \mathcal{M}$ satisfying $\|Y\|_{\mathcal{M}} \le 1$. Hence,

$$X \cdot V(B, \varepsilon, \delta) \cdot Y \subset V(B, 4\varepsilon, \delta) \tag{2}$$

for any $\varepsilon, \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$, $X, Y \in \mathcal{M}$ with $\|X\|_{\mathcal{M}} \le 1$, $\|Y\|_{\mathcal{M}} \le 1$.

Since the involution is continuous in the topology $t(\mathcal{M})$, the set $LS_h(\mathcal{M})$ is closed in $(LS(\mathcal{M}), t(\mathcal{M}))$. The cone $LS_+(\mathcal{M})$ of positive elements is also closed in $(LS(\mathcal{M}), t(\mathcal{M}))$ [17].

It follows from the definition of the topology $t(\mathcal{M})$ that the convergence locally in measure of a net $\{T_\alpha\}_{\alpha\in J}$ to T means that for any $\varepsilon, \delta > 0$ and $B \in \Sigma$, $\mu(B) < \infty$, there exists $\alpha_0 = \alpha(B, \varepsilon, \delta)$ such that, for each $\alpha \ge \alpha_0$, there exists a projection $P(\alpha) \in P(\mathcal{M})$, $Z(\alpha) \in P(\mathcal{Z}(\mathcal{M}))$, $Z(\alpha)P^\perp(\alpha) \in \mathcal{P}_{\text{fin}}(\mathcal{M})$ satisfying

$$\|(T_\alpha - T)P(\alpha)\|_{\mathcal{M}} \le \varepsilon \tag{3}$$

and

$$\varphi(Z^\perp(\alpha)) \in W(B, \varepsilon, \delta), \quad d(Z(\alpha)P^\perp(\alpha)) \le \varepsilon\varphi(Z(\alpha)).$$

If inequality (3) is replaced with the inequality

$$\|P(\alpha)(T_\alpha - T)P(\alpha)\|_{\mathcal{M}} \le \varepsilon, \tag{3'}$$

then it is said that the net $\{T_\alpha\}_{\alpha\in J}$ *converges to* T *two-sided locally in measure.*

It is easy to see that the two-sided convergence locally in measure is equivalent to the convergence in the vector topology in $LS(\mathcal{M})$, with the base of neighborhoods of zero formed by the sets

$$\begin{aligned} U(B, \varepsilon, \delta) = \{T \in LS(\mathcal{M}):\ &\text{there exists } P \in \mathcal{P}(\mathcal{M}), Z \in \mathcal{P}(\mathcal{Z}(\mathcal{M})),\\ &ZP^\perp \in \mathcal{P}_{\text{fin}}(\mathcal{M}),\ \text{such that } PTP \in \mathcal{M},\ \|PTP\|_{\mathcal{M}} \le \varepsilon,\\ &\varphi(Z^\perp) \in W(B, \varepsilon, \delta),\ \ d(ZP^\perp) \le \varepsilon\varphi(Z)\}, \end{aligned}$$

where $\varepsilon, \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$. In fact, this vector topology coincides with the topology $t(\mathcal{M})$, which is directly implied by the following inclusions:

$$V(B, \varepsilon, \delta) \subseteq U(B, \varepsilon, \delta) \subseteq V(A, 2\varepsilon, \delta)$$

for any $\varepsilon, \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$ [10, § 3.5].

We need the following criterion for the local measure topology $t(\mathcal{M})$ on $LS(\mathcal{M})$ to be metrizable. Remind that the center $\mathcal{Z}(\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$ is σ-finite if any family of nonzero mutually orthogonal projections in $\mathcal{P}(\mathcal{Z}(\mathcal{M}))$ is at most countable.

Proposition 3. ([10, Theorem 3.5.2]) *Local measure topology $t(\mathcal{M})$ on $LS(\mathcal{M})$ is metrizable if and only if there center $\mathcal{Z}(\mathcal{M})$ is σ-finite.*

Remark 1. As we have already noted, the center $\mathcal{Z}(\mathcal{M})$ of an arbitrary von Neumann algebra $\mathcal{M}$ is $*$-isomorphic to the commutative von Neumann algebra $L_\infty(\Omega, \Sigma, \mu)$, with measure μ satisfying the direct sum property. It means that there exists a family of nonzero mutually orthogonal central projections $\{Z_j\}_{j\in J} \subset P(\mathcal{Z}(\mathcal{M}))$ such that $\sup\limits_{j\in J} Z_j = I$ and the von Neumann algebra $Z_j\mathcal{Z}(\mathcal{M})$ is σ-finite for all $j \in J$.

Using Proposition 2, Remark 1 and [4, Proposition 8] we obtain the following corollary.

Corollary 1. *Let $\{Z_j\}_{j\in J} \subset P(\mathcal{Z}(\mathcal{M}))$ be a family of nonzero mutually orthogonal central projections such that $\sup\limits_{j\in J} Z_j = I$. For $\{T_\alpha\}_{\alpha\in A} \subset LS(\mathcal{M})$ and $T \in LS(\mathcal{M})$ the following conditions are equivalent:*

(i) $T_\alpha \xrightarrow{t(\mathcal{M})} T$;

(ii) $Z_jT_\alpha \xrightarrow{t(\mathcal{M})} Z_jT$ *for all* $j \in J$;

(iii) $Z_jT_\alpha \xrightarrow{t(Z_j\mathcal{M})} Z_jT$ *for all* $j \in J$.

3. Continuity of operator-valued functions

This section contains the main results of the paper concerning continuity of an operator-function $T \mapsto f(T)$ from $LS_h(\mathcal{M})$ to $LS(\mathcal{M})$ w.r.t. the local measure topology.

Theorem 1. *Let $f \in \mathcal{B}(\mathbb{R})$ be a continuous function on the spectrum $\sigma(T)$ of operator $T \in LS_h(\mathcal{M})$. If the net of operators $\{T_\alpha\} \subset LS_h(\mathcal{M})$ converges to T in the topology $t(\mathcal{M})$, then $f(T_\alpha) \xrightarrow{t(\mathcal{M})} f(T)$.*

To prove Theorem 1, we need several lemmas.

Note, from the definition of the operator $f(T)$ [10, §2.3] and Proposition 2.3.17 [10] it follows that $Zf(T) = f(ZT)$ for all $f \in \mathcal{B}(\mathbb{R})$, $Z \in \mathcal{P}(Z(\mathcal{M}))$ and $T \in LS_h(\mathcal{M})$. Remark 1 and Corollary 1 implies that for the proof of Theorem 1 it is sufficient to consider the case of σ-finite center $\mathcal{Z}(\mathcal{M})$ of the von Neumann algebra $\mathcal{M}$ only. And therefore, by Proposition 3, we need only to verify the implication

$$(T_n \xrightarrow{t(\mathcal{M})} T) \Rightarrow (f(T_n) \xrightarrow{t(\mathcal{M})} f(T)).$$

Lemma 1. *Let $T_k, T \in LS_h(\mathcal{M})$, and let $f, f_n \in \mathcal{B}(\mathbb{R}), k, n \in \mathbb{N}$ be such that*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} |f_n(t) - f(t)| = 0$$

and $f_n(T_k) \xrightarrow{t(\mathcal{M})} f_n(T)$ for $k \to \infty$ and for all $n \in \mathbb{N}$. Then $f(T_k) \xrightarrow{t(\mathcal{M})} f(T)$ for $k \to \infty$.

Proof. Since the topology of convergence locally in measure $t(\mathcal{M})$ and the topology of two-sided convergence locally in measure coincide, it is sufficient to show that for all $\varepsilon > 0, \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$ there exists $K = K(B, \varepsilon, \delta) \in \mathbb{N}$, such that for any $k \geq K$ there exist projections $P_k \in \mathcal{P}(\mathcal{M})$ and $Z_k \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, such that

$$P_k(f(T_k) - f(T))P_k \in \mathcal{M}, \quad \|P_k(f(T_k) - f(T))P_k\|_{\mathcal{M}} \leq \varepsilon,$$
$$\varphi(Z_k^\perp) \in W(B, \varepsilon, \delta), \quad d(Z_kP_k^\perp) \leq \varepsilon\varphi(Z_k).$$

Since $\lim\limits_{n\to\infty}\sup\limits_{t\in\mathbb{R}}|f_n(t)-f(t)|=0$, there exists a number n_0, such that

$$\sup_{t\in\mathbb{R}}|f_{n_0}(t)-f(t)|<\frac{\varepsilon}{3}.$$

This means that $(f_{n_0}(T)-f(T))\in\mathcal{M}$, $(f_{n_0}(T_k)-f(T_k))\in\mathcal{M}$, and

$$\|f_{n_0}(T)-f(T)\|_{\mathcal{M}}<\frac{\varepsilon}{3}\quad\text{and}\quad\|f_{n_0}(T_k)-f(T_k)\|_{\mathcal{M}}<\frac{\varepsilon}{3}\quad\text{for all}\quad k\in\mathbb{N}.$$

Since $f_{n_0}(T_k)\xrightarrow{t(\mathcal{M})}f_{n_0}(T)$, there exists $K=K(B,\varepsilon,\delta)\in\mathbb{N}$, such that for any $k\geq K$ there exist projections $P_k\in\mathcal{P}(\mathcal{M})$ and $Z_k\in\mathcal{P}(\mathcal{Z}(\mathcal{M}))$, such that

$$P_k(f_{n_0}(T_k)-f_{n_0}(T))P_k\in\mathcal{M},\quad\|P_k(f_{n_0}(T_k)-f_{n_0}(T))P_k\|_{\mathcal{M}}\leq\frac{\varepsilon}{3},$$

$$\varphi(Z_k^{\perp})\in W(B,\frac{\varepsilon}{3},\delta),\quad d(Z_kP_k^{\perp})\leq\frac{\varepsilon}{3}\varphi(Z_k).$$

Since

$$f(T_k)-f(T)=[f(T_k)-f_{n_0}(T_k)]+[f_{n_0}(T_k)-f_{n_0}(T)]+[f_{n_0}(T)-f(T)],$$

there exist partial isometries U,V and W of the von Neumann algebra $\mathcal{M}$ ([10, Theorem 2.4.5]), such that

$$\begin{aligned}P_k|f(T_k)-f(T)|P_k&\leq UP_k|f(T_k)-f_{n_0}(T_k)|P_kU^*\\&+VP_k|f_{n_0}(T_k)-f_{n_0}(T)|P_kV^*+WP_k|f_{n_0}(T)-f(T)|P_kW^*.\end{aligned}$$

Thus, the inclusion $P_k(f(T_k)-f(T))P_k\in\mathcal{M}$ and conditions

$$\|P_k(f(T_k)-f(T))P_k\|_{\mathcal{M}}\leq\varepsilon,\quad\varphi(Z_k^{\perp})\in W(B,\frac{\varepsilon}{3},\delta)\subset W(B,\varepsilon,\delta),$$

$$d(Z_kP_k^{\perp})\leq\frac{\varepsilon}{3}\varphi(Z_k)\leq\varepsilon\varphi(Z_k)$$

hold for any $k\geq K$. This means, that

$$f(T_k)\xrightarrow{t(\mathcal{M})}f(T),\quad k\to\infty.$$

□

Lemma 2. *If $\lambda\in\mathbb{C}\setminus\mathbb{R}$, $T_n,T\in LS_h(\mathcal{M}),n\in\mathbb{N}$ and $T_n\xrightarrow{t(\mathcal{M})}T$, then*

$$(T_n-\lambda I)^{-1}\xrightarrow{t(\mathcal{M})}(T-\lambda I)^{-1}.$$

Proof. The operators T_n and T are self-adjoint, and therefore $\sigma(T)\subset\mathbb{R}$ and $\sigma(T_n)\subset\mathbb{R}$, and for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$ there exist operators $(T_n-\lambda I)^{-1}$ and $(T-\lambda I)^{-1}$. These operators are operator-value functions for the continuous function $f(t)=(t-\lambda)^{-1},t\in\mathbb{R}$ of operators T_n and T respectively. In addition, $(T_n-\lambda I)^{-1},(T-\lambda I)^{-1}\in LS(\mathcal{M})$ [10, Propositions 2.3.17 (iii)]. Since

$$|f(t)|=\frac{1}{|t-\mathrm{Re}\lambda-i\mathrm{Im}\lambda|}=\frac{1}{\sqrt{(t-\mathrm{Re}\lambda)^2+(\mathrm{Im}\lambda)^2}}\leq\frac{1}{|\mathrm{Im}\lambda|},$$

we have that $(T_n-\lambda I)^{-1},(T-\lambda I)^{-1}\in\mathcal{M}$ and

$$\|(T_n-\lambda I)^{-1}\|_{\mathcal{M}}\leq\frac{1}{|\mathrm{Im}\lambda|},\quad\|(T-\lambda I)^{-1}\|_{\mathcal{M}}\leq\frac{1}{|\mathrm{Im}\lambda|}.$$

The equalities

$$\begin{aligned}(T_n-\lambda I)^{-1}-(T-\lambda I)^{-1}&=(T-\lambda I)^{-1}[(T-\lambda I)-(T_n-\lambda I)](T_n-\lambda I)^{-1}\\&=(T-\lambda I)^{-1}[T-T_n](T_n-\lambda I)^{-1}\end{aligned}$$

and the convergence $T_n\xrightarrow{t(\mathcal{M})}T$ imply that

$$(T_n-\lambda I)^{-1}\xrightarrow{t(\mathcal{M})}(T-\lambda I)^{-1},\quad n\to\infty.\quad\square$$

Lemma 3. *Let f be a continuous function on $\mathbb{R}$ such that $f(z) \to 0$ for $|z| \to +\infty$. If $T_n, T \in LS_h(\mathcal{M}), n \in \mathbb{N}$ and $T_n \stackrel{t(\mathcal{M})}{\longrightarrow} T$, then*

$$f(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} f(T).$$

Proof. If the function $f(z) = \dfrac{p(z)}{q(z)}$ is rational, with $q(z)$ having no real root, then $f(z)$ is the sum of a polynomial $r(z)$ and finite number of summands of the form

$$\varphi(z) = \frac{b}{(z-\lambda)^k}, \quad b \in \mathbb{C}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Since $T_n \stackrel{t(\mathcal{M})}{\longrightarrow} T$ and $(LS(\mathcal{M}), t(\mathcal{M}))$ is a topological $*$-algebra [10, § 3.5], we have $r(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} r(T)$. Continuity of multiplication with respect to topology $t(\mathcal{M})$ and Lemma 2 we obtain $\varphi(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} \varphi(T)$. Therefore

$$f(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} f(T).$$

If $f \in \mathcal{B}(\mathbb{R})$ is an arbitrary continuous function, such that $f(z) \to 0$ for $|z| \to +\infty$, then there exists a sequence of rational functions $f_n(z) = \dfrac{p_n(z)}{q_n(z)}$, such that $q_n(z) \neq 0$ for $z \in \mathbb{R}$ and f_n converges uniformly to f on $\mathbb{R}$. Then, by Lemma 1, we obtain

$$f(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} f(T).$$

□

Lemma 4. *If $T_n, T \in LS_h(\mathcal{M}), n \in \mathbb{N}$ and $T_n \stackrel{t(\mathcal{M})}{\longrightarrow} T$, then for $\lambda \to +\infty$*

$$E_\lambda^\perp(|T_n|) \stackrel{t(\mathcal{M})}{\longrightarrow} 0$$

uniformly with respect to n, where $\{E_\lambda(|T_n|)\}_{\lambda \in \mathbb{R}}$ is the spectral family for the operator $|T_n|$.

Proof. We denote $S = T^2, \quad S_n = T_n^2$ and fix an arbitrary neighborhood of zero $V(B, \varepsilon, \delta)$ of the topology $t(\mathcal{M})$. Since $S \in LS(\mathcal{M})$, by Proposition 2 and Proposition 2.3.4 from [10] we have $E_\lambda^\perp(S) \stackrel{t(\mathcal{M})}{\longrightarrow} 0$ for $\lambda \to +\infty$. Then there exists $\lambda_\varepsilon > 0$ such that $E^\perp_{\frac{\lambda_\varepsilon}{2}}(S) \in V\left(B, \frac{\varepsilon}{2}, \frac{\delta}{2}\right)$.

Since $T_n \stackrel{t(\mathcal{M})}{\longrightarrow} T$, it follows that $S_n \stackrel{t(\mathcal{M})}{\longrightarrow} S$. Then, by Proposition 2, we have that

$$E_\lambda^\perp(|S_n - S|) \stackrel{t(\mathcal{M})}{\longrightarrow} 0$$

for $n \to \infty$ for all $\lambda > 0$. Therefore there exists a number n_ε such that $E^\perp_{\frac{\lambda_\varepsilon}{2}}(|S_n - S|) \in V\left(B, \frac{\varepsilon}{2}, \frac{\delta}{2}\right)$ for all $n \geq n_\varepsilon$.

Suppose $Q_\lambda = E_{\frac{\lambda}{2}}(|S_n - S|) \wedge E_{\frac{\lambda}{2}}(S)$. From the inequalities

$$-\frac{\lambda}{2} Q_\lambda \leq Q_\lambda (S_n - S) Q_\lambda \leq \frac{\lambda}{2} Q_\lambda, \quad 0 \leq Q_\lambda S Q_\lambda \leq \frac{\lambda}{2} Q_\lambda$$

we obtain

$$-\lambda Q_\lambda \leq Q_\lambda (S_n - S) Q_\lambda + Q_\lambda S Q_\lambda = Q_\lambda S_n Q_\lambda = Q_\lambda (S_n - S) Q_\lambda + Q_\lambda S Q_\lambda \leq \lambda Q_\lambda.$$

Consequently, $Q_\lambda S_n Q_\lambda \in \mathcal{M}$ and $\|Q_\lambda S_n Q_\lambda\|_\mathcal{M} \leq \lambda$. It means that $T_n Q_\lambda \in \mathcal{M}$ and $\|T_n Q_\lambda\|_\mathcal{M} \leq \lambda$. Therefore, by Lemma 2.2.4 [10], we have

$$E_\lambda^\perp(|T_n|) \precsim Q_\lambda^\perp = E_{\frac{\lambda}{2}}(|S_n - S|) \vee E_{\frac{\lambda}{2}}(S) \leq E_{\frac{\lambda}{2}}(|S_n - S|) + E_{\frac{\lambda}{2}}(S) \in V(B, \varepsilon, \delta)$$

for all $\lambda \geq \lambda_\varepsilon$ and $n \geq n_\varepsilon$. It remain to use the property $d(E_\lambda^\perp(|T_n|)) \leq d(Q_\lambda^\perp)$ of dimension function d, which implies the inclusion

$$E_\lambda^\perp(|T_n|) \in V(B, \varepsilon, \delta)$$

for $\lambda \geq \lambda_\varepsilon$ and $n \geq n_\varepsilon$. Since $E_\lambda^\perp(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} 0$ for $\lambda \to +\infty$ for all fixed $n \in \mathbb{N}$, there exists $\lambda'_\varepsilon \geq \lambda_\varepsilon$ such that $E_\lambda^\perp(|T_n|) \in V(B, \varepsilon, \delta)$ for all $\lambda \geq \lambda'_\varepsilon$ and for all fixed $n \in \mathbb{N}$. □

Proof of Theorem 1. Let $T_n, T \in LS_h(\mathcal{M}), n \in \mathbb{N}$ and let $T_n \stackrel{t(\mathcal{M})}{\longrightarrow} T$. Firstly, suppose that f is a real continuous function on $\mathbb{R}$, then

$$f(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} f(T).$$

Fix an arbitrary neighborhood of zero $V(B, \varepsilon, \delta)$ of the topology $t(\mathcal{M})$, $0 < \varepsilon < 1$. By Lemma 4, there exists $\lambda_V > 0$ such that $E_\lambda^\perp(|T_n|) \in V\left(B, \frac{\varepsilon}{3}, \frac{\delta}{3}\right)$ for all $\lambda \geq \lambda_V$ and $n \in \mathbb{N}$. Furthermore, by the convergence $E_\lambda^\perp(|T|) \stackrel{t(\mathcal{M})}{\longrightarrow} 0$ for $\lambda \to +\infty$, we can choose the number λ_V so that $E_\lambda^\perp(|T|) \in V\left(B, \frac{\varepsilon}{3}, \frac{\delta}{3}\right)$ for all $\lambda \geq \lambda_V$.

Let $g(t)$ be a real continuous function on $\mathbb{R}$ such that $g(t) = f(t)$ for $t \in [-\lambda_V, \lambda_V]$ and $g(t) \to 0$ for $|t| \to +\infty$. Let $\varphi(t) = f(t) - g(t)$. Since

$$f(T_n) - f(T) = g(T_n) - g(T) + \varphi(T_n) - \varphi(T),$$

there exist partial isometries U, V and W of the von Neumann algebra $\mathcal{M}$ ([10, Theorem 2.4.5]), such that

$$|f(T_n) - f(T)| \leq U|g(T_n) - g(T)|U^* + V|\varphi(T_n)|V^* + W|\varphi(T)|W^*. \tag{4}$$

By Lemma 3, we have $g(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} g(T)$. Consequently, by inclusion (2), there exists a number n_V such that

$$U|g(T_n) - g(T)|U^* \in V\left(B, \frac{\varepsilon}{3}, \frac{\delta}{3}\right)$$

for all $n \geq n_V$.

Since $\varphi(t) = 0$ for $t \in [-\lambda_V, \lambda_V]$ we have

$$\varphi(|T|) = \varphi(|T|(E_{\lambda_V}(|T|) + E_{\lambda_V}^\perp(|T|)) = \varphi(|T|E_{\lambda_V}^\perp(|T|)) = \varphi(|T|)E_{\lambda_V}^\perp(|T|)).$$

The definition of the neighborhoods $V(B, \varepsilon, \delta)$, the inclusion

$$Q \in V(B, \varepsilon, \delta) \cap \mathcal{P}(\mathcal{M})$$

for $0 < \varepsilon < 1$ implies the following inclusion $TQ \in V(B, \varepsilon, \delta)$ for all $T \in LS(\mathcal{M})$. Hence, from inclusion $E_{\lambda_V}^\perp(|T|) \in V\left(B, \frac{\varepsilon}{3}, \frac{\delta}{3}\right)$ we infer the inclusion $\varphi(|T|) \in V\left(B, \frac{\varepsilon}{3}, \frac{\delta}{3}\right)$. Similarly, $\varphi(|T_n|) \in V\left(B, \frac{\varepsilon}{3}, \frac{\delta}{3}\right)$ for all $n \in \mathbb{N}$. Now, by inequality (4), Proposition 2 and inclusion (2), we obtain that $|f(T_n) - f(T)| \in V(B, \varepsilon, \delta)$ for all $n \geq n_V$. This means that $f(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} f(T)$.

Now, let f be an arbitrary continuous function on $\sigma(T)$ from $\mathcal{B}(\mathbb{R})$. Let us show that in this case the convergence

$$f(T_n) \stackrel{t(\mathcal{M})}{\longrightarrow} f(T)$$

also holds. Since algebraic operations are continuous on $(LS(\mathcal{M}), t(\mathcal{M}))$ and

$$tf(S) = (\mathrm{Re} f)(S) \dot{+} (i \mathrm{Im} f)(S)$$

for all $S \in LS_h(\mathcal{M})$, without loss of generality we may assume that the function f is real-valued from $\mathcal{B}(\mathbb{R})$.

Suppose first that $|f(t)| \leq 1$ for all $t \in \mathbb{R}$. Since the spectrum $\sigma(T)$ of the operator T is closed in $\mathbb{R}$, by Tietze-Uryson Theorem on extension (see., for example, [6], Theorem

4.5.1), there exists a continuous function $g : \mathbb{R} \to [-1, +1]$ such that $g(t) = f(t)$ for all $t \in \sigma(T)$. Let

$$\sigma_n = \left\{ t \in \mathbb{R} : |f(t) - g(t)| \geq \frac{1}{2^n} \right\}.$$

It is clear, that $\sigma(T) \cap \sigma_n = \emptyset$ for all $n \in \mathbb{N}$. Since the function g is continuous on $\mathbb{R}$, and the function f is continuous on $\sigma(T)$, then $\sigma(T) \cap \overline{\sigma_n} = \emptyset$. For $t \in \mathbb{R}$ and $A \subset \mathbb{R}$ denote by $\rho(t, A) = \inf\limits_{a \in A} |t - a|$ the distance from the point t to the set A. Consider the following function:

$$h(t) = \sum_{n=0}^{\infty} \frac{2^{-n} \rho(t, \sigma(T))}{\rho(t, \sigma(T)) + \rho(t, \overline{\sigma_n})}.$$

Since the distance $\rho(t, A)$ is a continuous function on $\mathbb{R}$, the function $h(t)$ is also continuous on $\mathbb{R}$. Moreover,

$$0 \leq h(t) \leq \sum_{n=0}^{\infty} 2^{-n} = 2, \ h(t) = 0 \quad \text{for all} \quad t \in \sigma(T)$$

and

$$g - h \leq f \leq g + h.$$

Since the functions $g(t)$ and $h(t)$ are continuous on $\mathbb{R}$, by the proven above, we have

$$h(T_n) \xrightarrow{t(\mathcal{M})} h(T) = 0$$

and

$$g(T_n) \xrightarrow{t(\mathcal{M})} g(T) = f(T).$$

Using the inequality $0 \leq f - g + h \leq 2h$, we obtain

$$0 \leq (f - g + h)(T_n) \leq 2h(T_n) \xrightarrow{t(\mathcal{M})} 0.$$

Therefore, $(f - g + h)(T_n) \xrightarrow{t(\mathcal{M})} 0$ and

$$f(T_n) = (f - g + h)(T_n) + g(T_n) - h(T_n) \xrightarrow{t(\mathcal{M})} f(T).$$

Thus, Theorem 1 is proven in the case, when $|f(t)| \leq 1$, $t \in \mathbb{R}$.

Let now, the condition $|f(t)| \leq 1$, $t \in \mathbb{R}$ is not realized. Since

$$\sup_{t \in [n, n+1]} |f(t)| < \infty \quad \text{for all} \quad n \in \mathbb{N},$$

we can choose a piecewise-linear continuous function $\varphi(t)$ on $\mathbb{R}$ so that

$$\varphi(t) \geq |f(t)| + 1 \quad \text{for all} \quad t \in \mathbb{R}.$$

By the proven above, for the function $\frac{f(t)}{\varphi(t)}$ we obtain the convergence

$$\left(\frac{f}{\varphi} \right) (T_n) \xrightarrow{t(\mathcal{M})} \left(\frac{f}{\varphi} \right) (T).$$

On the other hand, continuity of the function φ implies that

$$\varphi(T_n) \xrightarrow{t(\mathcal{M})} \varphi(T).$$

Since $(LS(\mathcal{M}), t(\mathcal{M}))$ is a topological $*$-algebra [10, § 3.5], we have

$$f(T_n) = \left(\varphi \cdot \frac{f}{\varphi} \right) (T_n) = \varphi(T_n) \left(\frac{f}{\varphi} \right) (T_n) \xrightarrow{t(\mathcal{M})} \varphi(T) \cdot \left(\frac{f}{\varphi} \right) (T) = f(T).$$

Thus Theorem 1 is proven. □

Theorem 1 immediately implies two following useful Corollaries.

Corollary 2. *If $\{T_\alpha\}$ is a net of operators from $LS(\mathcal{M})$, $T \in LS(\mathcal{M})$ and $T_\alpha \xrightarrow{t(\mathcal{M})} T$, then $|T_\alpha|^p \xrightarrow{t(\mathcal{M})} |T|^p$ for all $p > 0$.*

Proof. Since $(LS(\mathcal{M}), t(\mathcal{M}))$ is a complete topological $*$-algebra, it follows that

$$|T_\alpha|^2 = T_\alpha^* T_\alpha \xrightarrow{t(\mathcal{M})} T^*T = |T|^2.$$

Using Theorem 1 for the continuous function $f(t) = |t|^{p/2}$ we obtain that $|T_\alpha|^p \xrightarrow{t(\mathcal{M})} |T|^p$ for all $p > 0$. □

Denote by $\{E_\lambda(T)\}_{\lambda\in\mathbb{R}}$ the a spectral family of projections for the operator $T \in LS_h(\mathcal{M})$. Since $E_\lambda(T) = \varphi_\lambda(T)$, where $\varphi_\lambda(t) = 1$ for $t \leq \lambda$ and $\varphi_\lambda(t) = 0$ for $t > \lambda$, Theorem 1 gives the following

Corollary 3. *If λ does not belong to the spectrum of the operator $T \in LS_h(\mathcal{M})$, $\{T_\alpha\}$ be a net of operators from $LS_h(\mathcal{M})$ such that $T_\alpha \xrightarrow{t(\mathcal{M})} T$, then $E_\lambda(T_\alpha) \xrightarrow{t(\mathcal{M})} E_\lambda(T)$.*

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