A SIMPLICITY CRITERION FOR SYMMETRIC OPERATOR ON A GRAPH

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To Professor Yu. S. Samoilenko on the occasion of his 70th birthday

ABSTRACT. In the present paper we show that the topology of the underlying graph determines the domain and deficiency indices of a certain associated minimal symmetric operator. We obtaine a criterion of simplicity for the minimal operator associated with the graph.

INTRODUCTION

A number of authors in their recent papers, see, e.g., [9, 1, 11, 10, 4, 13, 5, 6], consider a Laplace operator (Laplacian) defined on a metric graph. Such a graph, provided that it is equipped with a differential operator acting on the edges and subject to certain matching conditions at the vertices, is called [10] *a quantum graph.* Quantum graphs play a rôle of natural intuitive models in mathematics, physics, chemistry, and engineering, when one considers the phenomenon of wave propagation (e.g., electromagnetic, acoustic, etc.) in some quasi one–dimensional system.

Matching conditions which determine the domain of the quantum graph comprise of certain conditions for the value of the function and its normal derivatives at all graph vertices. These conditions can be either local (linking the values of the function and its normal derivatives at precisely one vertex) or non-local (when they link together these values pertaining to more than one vertex).

In the paper of Yu. Ershova and A. V. Kiselev [3] the method of boundary triples was successfully applied to the study of an inverse spectral problem on a graph with local matching conditions. This method was suggested and developed by V. I. and M. L. Gorbachuk [7], A. N. Kochubei [8], V. A. Derkach and M. M. Malamud [2]. Within its framework the corresponding self-adjoint differential operator is considered as a proper extension of some symmetric operator with equal deficiency indices. One might argue that this method can be employed quite efficiently in the study of spectra of quantum graphs with non–local matching conditions as well. Under the additional assumption that the symmetric operator in question is simple (i.e., has no reducing subspace such that on it the operator induces a self–adjoint operator), the Weyl-Titchmarsh matrix function allows to investigate *all* of the spectrum of almost solvable extensions of this operator.

In the present paper we show that the topology of the underlying graph determines the domain and deficiency indices of a certain minimal symmetric operator which we call *associated* to the graph and which plays a special rôle in its spectral analysis (Proposition 1). Ershova and Kiselev [3] formulate some conditions sufficient for a minimal operator corresponding to a quantum graph to be simple. Proposition 1 obtained by us provides a criterion of simplicity for the minimal operator associated to the graph.

1. Operator $A_{\min}$ associated to the graph Γ

Assume that the edge set $E = \{e_k\}_{k=1}^n$ of a metric graph $\Gamma = (V, E)$ consists of finite intervals $e_k = [x_{2k-1}, x_{2k}] \subset \mathbb{R}$ of lengths $l_k = x_{2k} - x_{2k-1}$, and that the vertex set V is a certain partition of the set $\{x_j\}_{j=1}^{2n}$. Throughout the text of the paper we use the terminology and notation of [12].

The Hilbert space $L_2(\Gamma)$ corresponding to the graph Γ is the space of measurable square–summable functions defined on all graph edges and equipped with the standard norm

$$\|f\|_{L_2(\Gamma)}^2 = \sum_{e_k \in E} \|f\|_{L_2(e_k)}^2,$$

i.e., it is nothing but an orthogonal sum of $L_2(e_k)$ spaces, $L_2(\Gamma) = \bigoplus_{e \in E} L_2(e_k)$.

The Sobolev space $W^{2,2}(\Gamma) = \bigoplus_{e_k \in E} W^{2,2}(e_k)$ is introduced in an analogous fashion. Here the space $W^{2,2}(e_k)$ consists of functions defined on the edge e_k such that these functions are absolutely continuous together with their first derivatives, and $f''_{e_k}(x) \in L_2(e_k)$.

For functions $f \in W^{2,2}(\Gamma)$ we define the normal derivative $\partial_n f(x_j)$ at the endpoints of each interval e_k,

$$\partial_n f(x_j) = \begin{cases} f'(x_j), & \text{if } x_j \text{ is the left endpoint of the interval,} \\ -f'(x_j), & \text{if } x_j \text{ is the right endpoint of the interval.} \end{cases}$$

Then it appears natural to introduce the following notation for the function $f \in W^{2,2}(\Gamma)$ at any vertex of the graph:

$$f(V_k) = \sum_{x_j \in V_k} f(x_j), \quad \partial_n f(V_k) = \sum_{x_j \in V_k} \partial_n f(x_j).$$

We assign a *marK* (either of the symbols δ or δ', also referred to as *type*) to every interior vertex of the graph Γ, i.e., to any vertex for which the degree is greater than one, $\deg V_k > 1$). The graph obtained as a result of this procedure will be referred to as **marked** and denoted Γ_δ. A marked graph Γ_δ defines the linear manifold

$$D(\Gamma_\delta) := \left\{ f \in W^{2,2}(\Gamma) \; \middle| \; \begin{array}{l} f \text{ is continuous at vertices of } \delta-\text{type,} \\ \partial_n f \text{ is continuous at the vertices of } \delta'-\text{type} \end{array} \right\}.$$

If V_k is a vertex of δ-type then evidently $f(V_k) = \deg V_k f(x_j)$, $x_j \in V_k$. In the same way, for a vertex V_k of δ'-type $\partial_n f(V_k) = \deg V_k \partial_n f(x_j)$, $x_j \in V_k$.

In the Hilbert space $L_2(\Gamma) = \bigoplus_{k=1}^n L_2(e_k)$ we now define the operator $A_{\min}$, which acts on every graph edge as the differential expression

$$(1) \qquad\qquad A_{\min} = -\frac{d^2}{dx^2}$$

on the domain $\mathrm{dom}(A_{\min})$ which is the set of functions $f \in D(\Gamma_\delta)$ subject to conditions

$$(2) \qquad\qquad f(V_k) = 0, \quad \partial_n f(V_k) = 0 \; (\forall k).$$

This operator $A_{\min}$ and the graph Γ_δ will henceforth be called *associated*. For instance, if $A_{\min}$ is associated to the graph Γ_δ in which every vertex is of δ–type, then the domain of the operator consists of functions $f \in D(\Gamma_\delta)$ which satisfy the following conditions:

$$f(x_k) = 0 \; (\forall x_k), \quad \sum_{x_k \in V_i} \partial_n f(x_k) = 0 \quad (\forall V_i).$$

Henceforth we will always assume that the graph Γ_δ and the operator $A_{\min}$ are associated.

Note that $A_{\min}$ is a closed symmetric operator. Adjoint to it, $A_{\max} := A_{\min}^*$, is defined by the same differential expression on the domain $D(\Gamma_\delta)$.

Back to the symmetric operator, we formulate the following

Proposition 1. *Deficiency indices of the operator $A_{\min}$ associated to the graph Γ_δ are equal to the number of graph vertices.*

Proof. Let n be the number of graph edges, m – the number of graph vertices. For $\mathrm{Im}\lambda \neq 0$ the general solution of the equation

$$(3) \qquad (A_{\max} - \lambda I)f = 0$$

on every edge is of the form $f_k(x) = C_k^+ \cos\sqrt{\lambda}x + C_k^- \sin\sqrt{\lambda}x$, i.e., is determined by $2n$ parameters $\{C_k^+,\ C_k^-,\ k = \overline{1,n}.\}$. Moreover, at every vertex and for all $x_i, x_j \in V_k$ the following matching conditions hold:

$$f(x_i) = f(x_j), \qquad \text{if } V_k \text{ is a vertex of } \delta\text{-type},$$
$$\partial_n f(x_i) = \partial_n f(x_j), \quad \text{if } V_k \text{ is a vertex of } \delta'\text{-type}.$$

The total number of these relations pertaining to each vertex is exactly $\deg V_k - 1$. Then for the number of free parameters of the general solution of (3) one obtains

$$2n - \sum_{k=1}^{m} (\deg V_k - 1) = 2n + m - \sum_{k=1}^{m} \deg V_k.$$

Since due to the handshakes lemma $\sum\limits_{k=1}^{m} \deg V_k = 2n$, the dimension of $\ker(A_{\max} - \lambda I)$ is exactly m. $\qquad\square$

2. Criterion of simplicity for the operator $A_{\min}$

The question of whether the operator $A_{\min}$ associated to the graph Γ_δ is simple can be reduced to the question of whether this operator has any point spectrum. Detailed proof of the fact that $A_{\min}$ is simple provided that its point spectrum is empty is contained in [3]. As it will be shown below, the property of simplicity for the operator $A_{\min}$ is completely determined by the structure and marking of the metric graph Γ_δ.

Lemma 1. *If the graph Γ_δ contains a loop, the operator $A_{\min}$ associated to the graph Γ_δ is not simple.*

Proof. In order to simplify the required calculations, accept without loss of generality the following convention. Assume that the left endpoint of each interval $[x_{2k-1}, x_{2k}]$, $k = \overline{1,n}$ is shifted to the point zero, whereas the right endpoint is then at $l_k = x_{2k} - x_{2k-1}$.

Let now graph Γ_δ have a loop $e_k = [0, l_k]$ attached to the vertex V_k.
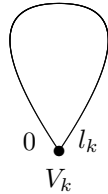
Solutions to the equation $A_{\min}f = \lambda f$ are subject to conditions (2) at the vertex V_k. Moreover, the conditions corresponding to the type of the vertex (δ or δ') have to be satisfied:

$$f(0) = f(l_k) = 0, \quad f'(0) - f'(l_k) = 0, \quad \text{if } V_k \text{ is of } \delta\text{-type},$$
$$f'(0) = f'(l_k) = 0, \quad f(0) + f(l_k) = 0, \quad \text{if } V_k \text{ is of } \delta'\text{-type}.$$

Let $\lambda \neq 0$. Then on e_k the solution assumes the form

$$f_k(x) = C_k^+ \cos\mu x + C_k^- \sin\mu x, \quad \text{where} \quad \mu = \sqrt{\lambda}.$$

Taking into account the above–mentioned conditions, we obtain the following relations for C_k^+, C_k^-:

$$C_k^+ = 0, \quad C_k^- \sin \mu l_k = 0, \quad C_k^-(1 - \cos \mu l_k) = 0, \quad \text{if } V_k \text{ is a } \delta\text{-vertex,}$$
$$C_k^- = 0, \quad C_k^+ \sin \mu l_k = 0, \quad C_k^+(1 + \cos \mu l_k) = 0, \quad \text{if } V_k \text{ is a } \delta'\text{-vertex.}$$

Obviously, in either case there exists a non–trivial solution on the edge e_k for

$$\mu = 2k\pi/l_k, \quad k \in \mathbb{Z}, \qquad \text{if } V_k \text{ is a } \delta\text{-vertex,}$$
$$\mu = (2k+1)\pi/l_k, \quad k \in \mathbb{Z}, \quad \text{if } V_k \text{ is a } \delta'\text{-vertex.}$$

Hence the point spectrum of $A_{\min}$ is non–empty and this operator is not simple. $\qquad \square$

Lemma 2. *The operator $A_{\min}$ associated to the graph Γ_δ is not simple if the graph Γ_δ contains one of the following subgraphs, all vertices of which are of δ'-type:*

(i) *a cycle with an even number of edges;*

(ii) *a subgraph consisting of two cycles with odd numbers of edges each which are connected by a chain or have exactly one common vertex.*

Proof. (i) If the graph Γ_δ contains a cycle with an even number of edges and all of the cycle's vertices are of δ'-type, then on this cycle there exists a non-trivial solution to the equation $A_{\min}f = 0$. Here is an example of the cycle of four edges, on which four constants are shown forming such solution:
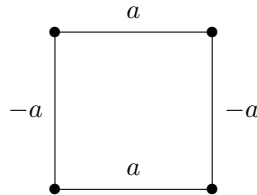
Therefore, the operator $A_{\min}$ is not simple.

(ii) If the graph has a subgraph consisting of two cycles with odd numbers of edges and all vertices of which are of δ'-type, then on these edges there also exists a non-trivial solution to $A_{\min}f = 0$ and hence the operator $A_{\min}$ is not simple. The figure below demonstrates an example of such eigenfunction which is a collection of constants, defined on a graph with two cycles of three edges each. This example admits a natural generalization to the case of arbitrary cycles of odd lengths.

$\qquad \square$
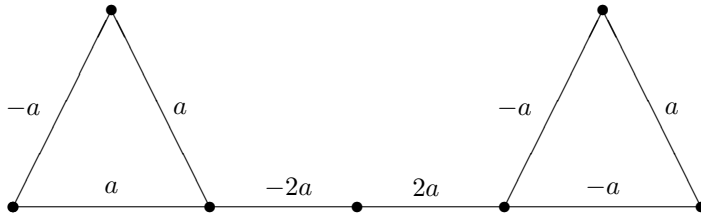
Lemma 3. *The equation $A_{\min}f = \lambda f$, $\lambda \neq 0$, has a non-trivial solution on a cycle of the graph Γ_δ iff all the lengths of this cycles' edges are pairwise rationally dependent.*

Proof. Let the cycle $C = \langle e_1, e_2, \ldots, e_n \rangle$ belong to the graph Γ_δ. Consider the vertex V_k of the cycle C which is incident to the edges e_{k-1} and e_k.
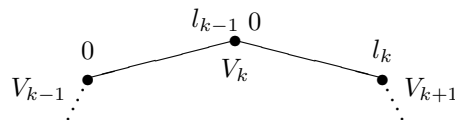
1) Suppose that V_k is the vertex of δ-type. Then eigenvectors are subject to the following conditions:

(4)
$$f_{k-1}(l_{k-1}) = f_k(0) = 0,$$
$$-f'_{k-1}(l_{k-1}) + f'_k(0) = 0.$$

Since $\lambda \neq 0$, solutions on these edges assume the form

$$f_j(x) = C_j^+ \cos \mu x + C_j^- \sin \mu x, \quad \text{where} \quad \mu = \sqrt{\lambda}, \quad j = k-1, k.$$

Due to (4) one obtains the following relations for the coefficients:

$$C_k^+ = 0,$$
$$C_{k-1}^+ \cos \mu l_{k-1} + C_{k-1}^- \sin \mu l_{k-1} = 0,$$
$$C_{k-1}^+ \sin \mu l_{k-1} - C_{k-1}^- \cos \mu l_{k-1} + C_k^- = 0.$$

Therefore,

(5)
$$f_{k-1}(x) = C_k^- \sin \mu(x - l_{k-1}),$$
$$f_k(x) = C_k^- \sin \mu x, \quad \text{where} \quad \mu = \sqrt{\lambda}.$$

Note that from the expressions obtained it follows that on the two neighboring edges of the cycle the solution to the equation $A_{\min} f = \lambda f$, $\lambda \neq 0$, is either non-trivial on both or is necessarily trivial on both.

Two vertices, adjacent to V_k, can also be of either δ or δ'-type. Consider all possible situations one by one.

1.1) If both V_{k-1} and V_{k+1} are of δ-type, the matching conditions at them are of the form $f_{k-1}(0) = 0$, $f_k(l_k) = 0$, that is, taking into account (5),

$$C_k^- \sin \mu l_{k-1} = 0, \quad C_k^- \sin \mu l_k = 0.$$

1.2) If V_{k-1} is a δ-vertex, whereas V_{k+1} is of δ'-type, $f_{k-1}(0) = 0$, $f'_k(l_k) = 0$, which yields the following conditions:

$$C_k^- \sin \mu l_{k-1} = 0, \quad C_k^- \cos \mu l_k = 0.$$

1.3) Let the vertex V_{k-1} be of δ'-type, the vertex V_k – of δ-type. Then the conditions at these vertices lead to the following relations:

$$C_k^- \cos \mu l_{k-1} = 0, \quad C_k^- \sin \mu l_k = 0.$$

1.4) If the vertices V_{k-1} and V_k are of δ'-type, the corresponding matching conditions lead to:

$$C_k^- \cos \mu l_{k-1} = 0, \quad C_k^- \cos \mu l_k = 0.$$

It is quite obvious that in each of these four cases the condition, necessary and sufficient for existence of non-trivial solution, is rational dependence of the lengths l_{k-1} and l_k.

2) Having assumed that V_k is a vertex of δ'-type, one gets the following solutions in place of (5):

$$f_{k-1}(x) = C_k^+ \cos \mu(x - l_{k-1}),$$
$$f_k(x) = C_k^+ \cos \mu x.$$

Suppose that V_{k-1} and V_{k+1} are δ-vertices. Then matching conditions at these vertices yield

$$C_k^+ \cos \mu l_{k-1} = 0, \quad C_k^+ \cos \mu l_k = 0.$$

Therefore in this case as well the solution is non-trivial iff the lengths l_{k-1} and l_k are rationally dependent.

The same result will hold in all other possible cases for the vertices V_{k-1} and V_{k+1}.

Now let some pair of the cycles' edges have rationally independent lengths. This means, that there has to exist a pair of neighboring rationally independent edges. Indeed,

if $e_s, e_{s+1}, \ldots, e_{s+k}$ is the maximal sequence of pairwise rationally dependent edges, then for all $i = \overline{0,k}$ $l_{s+i} = \alpha_i l_s$, $\alpha_i \in \mathbb{Q}$. Hence, k is less then the length of the cycle and the edges e_{s+k} and e_{s+k+1} are rationally independent. On these edges one gets the trivial solution only of the equation $A_{\min} f = \lambda f$, $\lambda \neq 0$, which taking into account the remark made above leads to the trivial solution only on all edges forming the cycle. $\square$

Lemma 4. *If an edge of the graph Γ_δ is incident to a boundary vertex, then on this edge the equation $A_{\min} f = \lambda f$ admits only the trivial solution for any λ.*

Proof. Suppose that the edge e_1 is incident to the boundary vertex V_1. Without loss of generality we can assume that the vertex V_1 is the left endpoint of the edge e_1. Then on this edge one gets the Cauchy problem with zero initial data which has no non-trivial solutions. $\square$

Corollary 1. *If the graph Γ_δ is a tree, the operator $A_{\min}$ is simple.*

The proof follows immediately from Lemma 4.

Lemma 5. *If an edge of the graph Γ_δ is incident to a vertex of δ-type, then on this edge the equation $A_{\min} f = 0$ admits only the trivial solution.*

Proof. Let V_k be a vertex of δ-type and let the edge e_k be incident to this vertex. Without loss of generality one assumes that the vertex V_k is the left endpoint of the edge e_k. Then the general solution on this edge is $f_k(x) = C_k^- x$. If the right endpoint of the edge e_k is a vertex of δ-type, the equality $C_k^- l_k = 0$ must hold; if on the other hand the right endpoint of the edge e_k is a vertex of δ'-type, one has $C_k^- = 0$. In both cases obviously $f_k(x) = 0$. $\square$

These results allow to receive the criterion of simplicity for a symmetric operator associated with a graph.

Theorem 1. *The operator $A_{\min}$ associated with the graph Γ_δ is not simple iff Γ_δ contains at least one of the following subgraphs:*
   1) a loop;
   2) a cycle, all edges of which have pairwise rationally dependent lengths;
   3) a cycle with an even number of edges, all vertices of which are of δ'-type;
   4) a subgraph with all vertices of δ'-type, consisting of two cycles with odd numbers of edges each which are connected by a chain or have exactly one common vertex.

Proof. First observe that the operator $A_{\min}$ is simple iff it is simple on every connected component of the graph. Therefore henceforth without loss of generality consider a connected graph Γ_δ.

If the graph Γ_δ contains the subgraph of the form (1)–(4) of Theorem, then according to Lemmas 1–3 the operator $A_{\min}$ associated with the graph Γ_δ is not simple.

Now assume that the graph Γ_δ doesn't contain the subgraphs of the form (1)–(4) of Theorem. Let us analyze solutions to the equation $A_{\min} f = \lambda f$ on the graph Γ_δ.

a) First consider solutions for $\lambda \neq 0$. Since the lengths of at least two edges of every cycle belonging to the graph Γ_δ are rationally independent, by Lemma 3 one gets only the trivial solution on every cycle. After removal of all cycles, the graph breaks up into a collection of trees. On each of these the solution must be trivial by Corollary 1.

b) Consider the case $\lambda = 0$. By Lemma 5, the equation $A_{\min} f = 0$ admits only the trivial solution on all edges incident to vertices of δ-type. Therefore we can assume that the graph Γ_δ breaks up into connected components having vertices of δ'-type only. Cutting away in accordance with Lemma 4 all edges containing pendant vertices, one ends up with isolated cycles with odd numbers of edges (see the conditions (3) and (4) of Theorem). Observe that the cycles cannot have adjacent edges since otherwise one could

single out a cycle having an even number of edges, which is impossible by assumptions of Theorem.

Consider a solution to the equation $A_{\min} f = 0$ on a cycle with an odd number of edges n. This solution must be of the form $f_k(x) = C_k$ on each of the edges of this graph, and constants C_j at every vertex V_k are subject to the following conditions:

$$C_{k-1} + C_k = 0, \quad k = \overline{2,n}, \quad C_1 + C_n = 0.$$

It is easy to see that this system of linear equations admits no solution but the trivial one. Hence for $\lambda = 0$ there exist no non–trivial solutions of the equation $A_{\min} f = \lambda f$ as well. Therefore the operator $A_{\min}$ associated to the graph Γ_δ has no point spectrum and is thus simple [3]. $\square$

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