

ON INVERSE SPECTRAL PROBLEMS FOR SELF-ADJOINT DIRAC OPERATORS WITH GENERAL BOUNDARY CONDITIONS

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Dedicated to my teacher Yaroslav Mykytyuk on the occasion of his 60th birthday

ABSTRACT. We consider the self-adjoint Dirac operators on a finite interval with summable matrix-valued potentials and general boundary conditions. For such operators, we study the inverse problem of reconstructing the potential and the boundary conditions of the operator from its eigenvalues and suitably defined norming matrices.

1. INTRODUCTION

In this paper, we consider the self-adjoint Dirac operators on $(-1, 1)$ generated by the differential expressions

$$\mathfrak{t}_q := \frac{1}{i} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix}$$

and general boundary conditions of the form

$$Ay(-1) + By(1) = 0.$$

Here, q is an $r \times r$ matrix-valued function with entries belonging to $L_p(-1, 1)$, $p \in [1, \infty)$, called the *potential* of the operator, I is the $r \times r$ identity matrix, A and B are $2r \times 2r$ matrices with complex entries such that the operator is self-adjoint. For such operators, we introduce the notion of the spectral data – eigenvalues and suitably defined norming matrices. We then study the inverse problem of reconstructing the potential and the boundary conditions of the operator from its spectral data.

Inverse spectral problems for Dirac and Sturm–Liouville operators with matrix-valued potentials arise in many areas of modern physics and mathematics. For instance, the inverse problems for quantum graphs (see, e.g., [10]) in some cases can be reduced to the ones for the operators with matrix-valued potentials. Among the recent investigations in the area of inverse problems for Dirac-type systems we mention, e.g., the ones by Albeverio, Hryniv and Mykytyuk [1], Gesztesy et al. [4, 6, 7], Malamud et al. [11, 12, 13], Sakhnovich [20, 21]. The inverse problem of reconstructing the skew-self-adjoint Dirac system with a rectangular matrix-valued potential from the Weyl function was recently solved in [5]. Problems similar to the ones considered in this paper were recently treated for Sturm–Liouville operators with matrix-valued potentials in [2, 3, 17]. A considerable contribution to the spectral theory of differential operators with matrix-valued potentials was made by Roĭe-Beketov et al. (see, e.g., [19]). The operators with general boundary conditions also remain an object of interest these days. For instance, the completeness problem of root functions of general boundary value problems for the first order Dirac-type systems was recently solved in [14]. We refer the reader to the extensive reference lists in [1–7, 10–12, 16, 18–20] for further results on the subject.

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Recently, the inverse problem of reconstructing the self-adjoint Dirac operators with some separated boundary conditions from eigenvalues and norming matrices was solved in [16] (for the operators with square-integrable matrix-valued potentials) and [18] (for the more general case of the operators with summable matrix-valued potentials). In the present paper, we extend the results of [18] in order to solve the inverse spectral problem for the operators with general (especially, non-separated) boundary conditions.

The approach consists in reducing the problem to the one for the operators with separated boundary conditions acting in the space $L_2((0, 1), \mathbb{C}^{4r})$. We then develop the Krein accelerant method [16, 17, 18] in order to solve the inverse spectral problem for the operators with general separated boundary conditions. We show that the accelerants of such operators do not depend on boundary conditions and uniquely determine the potentials of the operators. The boundary conditions can be then reconstructed from the asymptotics of the spectral data.

The paper is organized as follows. In the following section, we give the precise setting of the problem. In Sect. 3, we introduce the approach and formulate the main results. In Sect. 4, we prove the main results of this paper.

Notations. Throughout this paper, we write $\mathcal{M}_r$ for the set of all $r \times r$ matrices with complex entries and identify $\mathcal{M}_r$ with the Banach algebra of linear operators in $\mathbb{C}^r$ endowed with the standard norm. We write $I = I_r$ for the $r \times r$ identity matrix and $\mathcal{U}_r$ for the set of all unitary matrices $U \in \mathcal{M}_r$. For an arbitrary $p \in [1, \infty)$, we write $\mathcal{Q}_p := L_p((-1, 1), \mathcal{M}_r)$ for the set of all $r \times r$ matrix-valued functions with entries belonging to $L_p(-1, 1)$ and endow $\mathcal{Q}_p$ with the norm

$$\|q\|_{\mathcal{Q}_p} := \left(\int_{-1}^1 \|q(s)\|^p ds \right)^{1/p}, \quad q \in \mathcal{Q}_p.$$

Similarly, we set $\mathfrak{Q}_p := L_p((0, 1), \mathcal{M}_{2r})$. The superscript $\top$ designates the transposition of vectors and matrices, e.g., $(c_1, c_2)^\top$ is the column vector $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

2. SETTING OF THE PROBLEM

Given an arbitrary $q \in \mathcal{Q}_p$, we set

$$(2.1) \quad J := \frac{1}{i} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad Q := \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix}$$

and consider the differential expression

$$\mathfrak{t}_q := J \frac{d}{dx} + Q$$

on the domain $D(\mathfrak{t}_q) = W_2^1((-1, 1), \mathbb{C}^{2r})$, where W_2^1 is the Sobolev space (see Appendix A). In the Hilbert space $\mathcal{H} := L_2((-1, 1), \mathbb{C}^{2r})$, we introduce the *maximal operator* T_q by the formula $T_q y = \mathfrak{t}_q(y)$,

$$D(T_q) = \{y \in D(\mathfrak{t}_q) \mid \mathfrak{t}_q(y) \in \mathcal{H}\}.$$

The adjoint operator $T_q^0 := T_q^*$ is the restriction of T_q onto the domain

$$D(T_q^0) = \{y \in D(T_q) \mid y(-1) = y(1) = 0\}.$$

By definition, the operator T_q^0 will be called the *minimal* one. The objects of our study are self-adjoint extensions of the minimal operator T_q^0 .

It is known (see, e.g., [8]) that every self-adjoint extension of the minimal operator T_q^0 is the restriction of the maximal operator T_q onto the domain

$$(2.2) \quad D(T) = \{y \in D(T_q) \mid Ay(-1) + By(1) = 0\},$$

where $A, B \in \mathcal{M}_{2r}$ are such that

$$(2.3) \quad \text{rank}(A \ B) = 2r, \quad AJA^* = BJB^*.$$

Evidently, the self-adjoint extensions of T_q^0 cannot be parametrized by the matrices A and B uniquely since different pairs (A, B) may lead to the same self-adjoint extension. However, using the standard technique involving the concept of boundary triplets one can prove the following lemma providing a *unique* characterization of all self-adjoint extensions of the minimal operator T_q^0 :

Lemma 2.1. *A linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is a self-adjoint extension of the minimal operator T_q^0 if and only if there exists a unitary matrix $U \in \mathcal{U}_{2r}$ such that T is the restriction of the maximal operator T_q onto the domain (2.2) with*

$$(2.4) \quad A = A_U := P_2 + UP_1, \quad B = B_U := P_1 + UP_2,$$

where

$$(2.5) \quad P_1 := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

According to Lemma 2.1, we can parametrize all self-adjoint extensions of the minimal operator T_q^0 by unitary matrices $U \in \mathcal{U}_{2r}$. For an arbitrary $U \in \mathcal{U}_{2r}$, we denote by $T_{q,U}$ the restriction of the maximal operator T_q onto the domain (2.2) with $A = A_U$ and $B = B_U$ given by formula (2.4). For the operators $T_{q,U}$, we introduce the notion of the *spectral data* – eigenvalues and suitably defined norming matrices.

More precisely, the spectrum of the operator $T_{q,U}$ is purely discrete and consists of countably many isolated real eigenvalues of finite multiplicity accumulating only at $+\infty$ and $-\infty$. Throughout this section, we denote by $\lambda_j := \lambda_j(q, U)$, $j \in \mathbb{Z}$, the pairwise distinct eigenvalues of the operator $T_{q,U}$ labeled in increasing order so that $\lambda_0 < 0 \leq \lambda_1$.

In order to introduce the norming matrices of the operator $T_{q,U}$, it is convenient to use the constructive definition which is similar to the one suggested in [2]. For every $\lambda \in \mathbb{C}$, we denote by $Y_q(\cdot, \lambda) \in W_2^1((-1, 1), \mathcal{M}_{2r})$ a $2r \times 2r$ matrix-valued solution of the Cauchy problem

$$(2.6) \quad J \frac{d}{dx} Y + QY = \lambda Y, \quad Y(0, \lambda) = I_{2r},$$

where I_{2r} is the $2r \times 2r$ identity matrix. For every $j \in \mathbb{Z}$, we set

$$M_j := \frac{1}{2} \int_{-1}^1 Y_q(s, \lambda_j)^* Y_q(s, \lambda_j) ds.$$

It follows that for every $j \in \mathbb{Z}$, $M_j = M_j^* > 0$. We then denote by $P_j : \mathbb{C}^{2r} \rightarrow \mathbb{C}^{2r}$ the orthogonal projector onto $\mathcal{E}_j := \ker[A_U Y_q(-1, \lambda_j) + B_U Y_q(1, \lambda_j)]$ and define the positive self-adjoint operator $B_j : \mathcal{E}_j \rightarrow \mathcal{E}_j$ by setting

$$(2.7) \quad B_j := (P_j M_j P_j)|_{\mathcal{E}_j}.$$

Definition 2.1. *For every $j \in \mathbb{Z}$, we set*

$$A_j(q, U) := B_j^{-1} P_j$$

and call $A_j(q, U)$ the norming matrix of the operator $T_{q,U}$ corresponding to the eigenvalue $\lambda_j(q, U)$. The sequence

$$\mathfrak{a}_{q,U} := ((\lambda_j(q, U), A_j(q, U)))_{j \in \mathbb{Z}}$$

will be called the spectral data of the operator $T_{q,U}$.

Remark 2.1. It follows from the definition of the norming matrices that $A_j M_j A_j = A_j$ for every $j \in \mathbb{Z}$, $A_j := A_j(q, U)$. This will play an important role in the proof of Lemma 3.1 below.

For the operators $T_{q,U}$, we study the inverse problem of reconstructing the potential q and the unitary matrix U from the spectral data. We shall give a complete description of the class

$$(2.8) \quad \mathfrak{A}_p := \{\mathfrak{a}_{q,U} \mid q \in \mathcal{Q}_p, U \in \mathcal{U}_{2r}\}$$

of the spectral data, show that the spectral data of the operator $T_{q,U}$ determine the potential q and the unitary matrix U uniquely and suggest how to find q and U from the spectral data.

3. THE APPROACH AND THE MAIN RESULTS

Our approach consists in reducing the problem for the operators $T_{q,U}$ to the one for the operators with separated boundary conditions that we now introduce.

Let $V \in \mathfrak{Q}_p$ (see Notations) be an arbitrary $2r \times 2r$ matrix-valued function with entries belonging to $L_p(0, 1)$, $p \in [1, \infty)$. Set

$$(3.1) \quad J := \frac{1}{i} \begin{pmatrix} I_{2r} & 0 \\ 0 & -I_{2r} \end{pmatrix}, \quad V := \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$$

and consider the differential expression

$$\mathfrak{s}_V := J \frac{d}{dx} + V$$

on the domain

$$D(\mathfrak{s}_V) = \left\{ f := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mid f_1, f_2 \in W_2^1((0, 1), \mathbb{C}^{2r}) \right\}.$$

In the Hilbert space $\mathbb{H} := L_2((0, 1), \mathbb{C}^{2d})$, $d := 2r$, we introduce the auxiliary operator $S_{V,U}$, where $U \in \mathcal{U}_{2r}$, by the formula $S_{V,U}f = \mathfrak{s}_V(f)$,

$$D(S_{V,U}) = \{f \in D(\mathfrak{s}_V) \mid \mathfrak{s}_V(f) \in \mathbb{H}, f_1(0) = f_2(0), f_1(1) = Uf_2(1)\}.$$

As in the case of the operators $T_{q,U}$, the spectrum of the operator $S_{V,U}$ is purely discrete and consists of countably many isolated real eigenvalues of finite multiplicity accumulating only at $+\infty$ and $-\infty$. In what follows, we denote by $\zeta_j := \zeta_j(V, U)$, $j \in \mathbb{Z}$, the pairwise distinct eigenvalues of the operator $S_{V,U}$ labeled in increasing order so that $\zeta_0 < 0 \leq \zeta_1$.

For the operator $S_{V,U}$, the notion of the *Weyl–Titchmarsh function* can be defined as in [4] (see (4.4) below for a precise definition); the Weyl–Titchmarsh function of the operator $S_{V,U}$ is a $2r \times 2r$ matrix-valued meromorphic Herglotz function and $\{\zeta_j\}_{j \in \mathbb{Z}}$ is the set of its poles. This allows us to introduce the spectral data of the operator $S_{V,U}$ as in [16, 18]:

Definition 3.1. Let $M_{V,U}$ be the Weyl–Titchmarsh function of the operator $S_{V,U}$. For every $j \in \mathbb{Z}$, we set

$$C_j(V, U) := - \operatorname{res}_{\zeta=\zeta_j(V, U)} M_{V,U}(\zeta)$$

and call $C_j(V, U)$ the norming matrix of the operator $S_{V,U}$ corresponding to the eigenvalue $\zeta_j(V, U)$. The sequence

$$\mathfrak{b}_{V,U} := ((\zeta_j(V, U), C_j(V, U)))_{j \in \mathbb{Z}}$$

will be called the spectral data of the operator $S_{V,U}$. The $2r \times 2r$ matrix-valued measure

$$\mu_{V,U} := \sum_{j \in \mathbb{Z}} C_j(V, U) \delta_{\zeta_j(V, U)},$$

where δ_ζ is the Dirac delta measure centered at the point ζ , will be referred to as its spectral measure.

As in [16, 18], it follows that for every $j \in \mathbb{Z}$, $C_j(V, U) \geq 0$ and the rank of $C_j(V, U)$ equals the multiplicity of the eigenvalue $\zeta_j(V, U)$.

We now state a connection between the operators $T_{q,U}$ and $S_{V,U}$:

Lemma 3.1. *For an arbitrary $q \in \mathcal{Q}_p$ and $U \in \mathcal{U}_{2r}$, the operator $T_{q,U}$ is unitarily equivalent to the operator $S_{V,U}$, where*

$$(3.2) \quad V(x) = \begin{pmatrix} 0 & q(x) \\ q(-x)^* & 0 \end{pmatrix}, \quad x \in (0, 1).$$

Moreover, the spectral data of the operator $T_{q,U}$ coincide with the spectral data of the operator $S_{V,U}$ with V given by formula (3.2).

It thus follows from Lemma 3.1 that every sequence $\mathbf{a} \in \mathfrak{A}_p$ (see (2.8)) is the spectral data of the operator $S_{V,U}$ with the potential V of the form (3.2). We now extend the results of [18] in order to solve the inverse spectral problem for the operators $S_{V,U}$. For such operators, we shall give a complete description of the class

$$\mathfrak{B}_p := \{\mathfrak{b}_{V,U} \mid V \in \mathfrak{Q}_p, U \in \mathcal{U}_{2r}\}$$

of the spectral data, show that the spectral data of the operator $S_{V,U}$ determine the potential V and the unitary matrix U uniquely and suggest how to find V and U from the spectral data.

3.1. The inverse problem for the operators $S_{V,U}$. In what follows, let

$$\mathbf{a} := ((\lambda_j, A_j))_{j \in \mathbb{Z}}$$

stand for an arbitrary sequence, where $(\lambda_j)_{j \in \mathbb{Z}}$ is a strictly increasing sequence of real numbers such that $\lambda_0 < 0 \leq \lambda_1$ and A_j , $j \in \mathbb{Z}$, are non-zero non-negative matrices in $\mathcal{M}_{2r}$. We first give the necessary and sufficient conditions for a sequence $\mathbf{a}$ to belong to the class $\mathfrak{B}_p$. In order to formulate these conditions, we need to introduce some preliminaries.

We start by describing the asymptotics of $(\lambda_j)_{j \in \mathbb{Z}}$ and $(A_j)_{j \in \mathbb{Z}}$. The description will be much clearer after the following remark:

Remark 3.1. Let $U \in \mathcal{U}_{2r}$ and $\gamma_1 < \gamma_2 < \dots < \gamma_s$ be real numbers from the interval $[0, \pi)$ such that $e^{2i\gamma_k}$, $k = 1, \dots, s$, are all distinct eigenvalues of U . Then all distinct eigenvalues of the free operator $S_{0,U}$ take the form

$$(3.3) \quad \zeta_{ns+k}^0 = \gamma_k + \pi n, \quad k \in \{1, \dots, s\}, \quad n \in \mathbb{Z}.$$

The norming matrix of $S_{0,U}$ corresponding to the eigenvalue ζ_{ns+k}^0 appears to be the orthogonal projector onto $\ker(U - e^{2i\gamma_k} I_{2r})$.

Definition 3.2. We say that a sequence $\mathbf{a}$ satisfies the condition (C_1) if:

- (i) there exist real numbers $\gamma_1 < \gamma_2 < \dots < \gamma_s$ from the interval $[0, \pi)$ such that with the numbers ζ_m^0 of (3.3), $m \in \mathbb{Z}$, it holds

$$(3.4) \quad \sum_{\lambda_j \in \Delta_m} |\lambda_j - \zeta_m^0| = o(1), \quad |m| \rightarrow \infty,$$

and

$$(3.5) \quad \sup_{m \in \mathbb{Z}} \sum_{\lambda_j \in \Delta_m} 1 < \infty,$$

where

$$\Delta_m := \left[\frac{\zeta_{m-1}^0 + \zeta_m^0}{2}, \frac{\zeta_m^0 + \zeta_{m+1}^0}{2} \right);$$

(ii) there exist pairwise orthogonal projectors $P_1^0, \dots, P_s^0 \in \mathcal{M}_{2r}$ such that

$$\sum_{k=1}^s P_k^0 = I_{2r}$$

and for every $k \in \{1, \dots, s\}$,

$$(3.6) \quad \left\| P_k^0 - \sum_{\lambda_j \in \Delta_{ns+k}} A_j \right\| = o(1), \quad |n| \rightarrow \infty.$$

For every sequence $\mathbf{a}$ satisfying the condition (C_1) , we define the unitary matrix $U_{\mathbf{a}} \in \mathcal{U}_{2r}$ by the formula

$$(3.7) \quad U_{\mathbf{a}} := \sum_{k=1}^s e^{2i\gamma_k} P_k^0.$$

Next, we denote by $\mu^{\mathbf{a}}$ the $2r \times 2r$ matrix-valued measure given by the formula

$$(3.8) \quad \mu^{\mathbf{a}} := \sum_{j \in \mathbb{Z}} A_j \delta_{\lambda_j}$$

and associate with $\mu := \mu^{\mathbf{a}}$ the $\mathbb{C}^{2r}$ -valued distribution defined via

$$(\mu, f) := \int_{\mathbb{R}} f \, d\mu, \quad f \in \mathcal{S}^{2r},$$

where $\mathcal{S}^{2r}$ is the Schwartz space of rapidly decreasing $\mathbb{C}^{2r}$ -valued functions (see Appendix A). As in [18], we introduce a Fourier-type transform of $\mu^{\mathbf{a}}$:

Definition 3.3. For an arbitrary measure $\mu := \mu^{\mathbf{a}}$, we denote by $\widehat{\mu}$ the $\mathbb{C}^{2r}$ -valued distribution given by the formula

$$(\widehat{\mu}, f) := (\mu, \widehat{f}), \quad f \in \mathcal{S}^{2r},$$

where $\widehat{f}(\lambda) := \int_{-\infty}^{\infty} e^{2i\lambda s} f(s) \, ds$, $\lambda \in \mathbb{R}$.

For an arbitrary sequence $\mathbf{a}$ satisfying the condition (C_1) , set $\mu := \mu^{\mathbf{a}}$ and let H_{μ} be the restriction of the distribution $\widehat{\mu} - \widehat{\mu}_0$ to the interval $(-1, 1)$, i.e.,

$$(3.9) \quad (H_{\mu}, f) := (\widehat{\mu} - \widehat{\mu}_0, f), \quad f \in \mathcal{S}^{2r}, \quad \text{supp } f \subset (-1, 1),$$

where $\mu_0 := \mu_{0,U}$ is the spectral measure of the free operator $S_{0,U}$, $U := U_{\mathbf{a}}$. Then the following lemma gives the necessary and sufficient conditions for a sequence $\mathbf{a}$ to belong to the class $\mathfrak{B}_p$:

Lemma 3.2. A sequence $\mathbf{a}$ belongs to the class $\mathfrak{B}_p$, $p \in [1, \infty)$, if and only if it satisfies the condition (C_1) and

(C_2) there exists $n_0 \in \mathbb{N}$ such that for all natural $n > n_0$,

$$\sum_{m=-ns+1}^{ns} \sum_{\lambda_j \in \Delta_m} \text{rank } A_j = 4nr;$$

(C_3) the system of functions $\{e^{i\lambda_j t} v \mid j \in \mathbb{Z}, v \in \text{Ran } A_j\}$ is complete in the space $L_2((-1, 1), \mathbb{C}^{2r})$;

(C_4) the distribution H_{μ} , where $\mu := \mu^{\mathbf{a}}$, belongs to $L_p((-1, 1), \mathcal{M}_{2r})$.

By definition, every sequence $\mathbf{a} \in \mathfrak{B}_p$ is the spectral data of some operator $S_{V,U}$. It turns out that the operator $S_{V,U}$ is determined by its spectral data uniquely:

Lemma 3.3. For every $p \in [1, \infty)$, the mapping $\mathfrak{Q}_p \times \mathcal{U}_{2r} \ni (V, U) \mapsto \mathbf{b}_{V,U} \in \mathfrak{B}_p$ is bijective.

We then solve the inverse problem of finding the operator $S_{V,U}$ from its spectral data. As in [16, 17, 18], we base our procedure on Krein's accelerant method:

Lemma 3.4. *Take an arbitrary $\mathfrak{a} \in \mathfrak{B}_p$ and denote by $S_{V,U}$ the operator for which $\mathfrak{a}$ is the spectral data. Then $U = U_{\mathfrak{a}}$ is defined by formula (3.7) and $V = \Theta(H)$; here $H := H_{\mu}$ is a Krein accelerant (see Appendix B) given by (3.9), with $\mu := \mu^{\mathfrak{a}}$ of formula (3.8), and $\Theta : \mathfrak{H}_p \rightarrow \mathfrak{Q}_p$ is the Krein mapping of (B.2).*

The function $H := H_{\mu}$, where $\mu := \mu^{\mathfrak{a}}$ and $\mathfrak{a} \in \mathfrak{B}_p$ is the spectral data of the operator $S_{V,U}$, will be called the *accelerant* of the operator $S_{V,U}$.

3.2. The inverse problem for the operators $T_{q,U}$. We now use the results of the previous subsection to solve the inverse spectral problem for the operators $T_{q,U}$.

Recall that by virtue of Lemma 3.4, for every sequence $\mathfrak{a}$ satisfying the conditions $(C_1) - (C_4)$ from Lemma 3.2 the distribution $H := H_{\mu}$, where $\mu := \mu^{\mathfrak{a}}$, appears to be an accelerant and belongs to the class $\mathfrak{H}_p$. Then the following theorem gives a complete description of the class $\mathfrak{A}_p$ of the spectral data of the operators $T_{q,U}$:

Theorem 3.1. *A sequence $\mathfrak{a}$ belongs to the class $\mathfrak{A}_p$, $p \in [1, \infty)$, if and only if it satisfies the conditions $(C_1) - (C_4)$ from Lemma 3.2 and*

(C_5) the function $V = \Theta(H)$, where $H := H_{\mu}$, $\mu := \mu^{\mathfrak{a}}$ and $\Theta : \mathfrak{H}_p \rightarrow \mathfrak{Q}_p$ is the Krein mapping given by formula (B.2), satisfies the anti-commutative relation

$$V(x)J = -JV(x)$$

a.e. on $(0, 1)$.

By definition, every sequence $\mathfrak{a} \in \mathfrak{A}_p$ is the spectral data of some operator $T_{q,U}$. It turns out that the operator $T_{q,U}$ is determined by its spectral data uniquely:

Theorem 3.2. *For every $p \in [1, \infty)$, the mapping $\mathcal{Q}_p \times \mathcal{U}_{2r} \ni (q, U) \mapsto \mathfrak{a}_{q,U} \in \mathfrak{A}_p$ is bijective.*

We then solve the inverse problem of finding the operator $T_{q,U}$ from its spectral data:

Theorem 3.3. *Take an arbitrary $\mathfrak{a} \in \mathfrak{A}_p$ and denote by $T_{q,U}$ the operator for which $\mathfrak{a}$ is the spectral data. Then $U = U_{\mathfrak{a}}$ is defined by formula (3.7) and*

$$(3.10) \quad q(x) = \begin{cases} V_{12}(x), & x \in (0, 1), \\ V_{21}(-x)^*, & x \in (-1, 0), \end{cases}$$

where $V = (V_{ij})_{i,j=1}^2 := \Theta(H)$, $H := H_{\mu}$ is a Krein accelerant (see Appendix B) given by (3.9), with $\mu := \mu^{\mathfrak{a}}$ of formula (3.8), and $\Theta : \mathfrak{H}_p \rightarrow \mathfrak{Q}_p$ is the Krein mapping of (B.2).

The procedure of finding the operator $T_{q,U}$ from its spectral data can be visualized by means of the following diagram:

$$\begin{array}{c} \mathfrak{A}_p \ni \mathfrak{a} \xrightarrow[s_1]{(3.7)} U = U_{\mathfrak{a}}; \\ \mathfrak{A}_p \ni \mathfrak{a} \xrightarrow[s_2]{(3.8)} \mu := \mu^{\mathfrak{a}} \xrightarrow[s_3]{(3.9)} H := H_{\mu} \xrightarrow[s_4]{(B.2)} V := \Theta(H) \xrightarrow[s_5]{(3.10)} q. \end{array}$$

Here, s_j denotes the step number j . The steps s_1 , s_2 , s_3 and s_5 are trivial. The basic and non-trivial step is s_4 which requires solving the Krein equation (B.1).

Remark 3.2. By virtue of the condition (C_5) , the description of the class $\mathfrak{A}_p$ is not formulated in terms of eigenvalues and norming matrices directly. Unfortunately, this condition cannot be easily formulated even in terms of the accelerant $H := H_{\mu}$. However, a certain complication as compared to the case of the separated boundary conditions is naturally expected. For instance, recall the classical results [15] on the inverse problem

of reconstructing Sturm–Liouville operators from two spectra: therein, a description of the two spectra in the case of the periodic/antiperiodic boundary conditions appears to be much more complicated than the one for the operators with separated ones.

Remark 3.3. We define the norming matrices of the operator $T_{q,U}$ using Definition 2.1 and the norming matrices of the operator $S_{V,U}$ using Definition 3.1. However, one can also define the norming matrices of the operator $S_{V,U}$ similarly as in Definition 2.1. Namely, let $\mathbf{Y}_V(\cdot, \zeta) \in W_2^1((0, 1), \mathcal{M}_{4r})$ be a $4r \times 4r$ matrix-valued solution of the Cauchy problem

$$(3.11) \quad \mathbf{J} \frac{d}{dx} \mathbf{Y} + \mathbf{V} \mathbf{Y} = \zeta \mathbf{Y}, \quad \mathbf{Y}(0, \zeta) = I_{4r}.$$

For every $j \in \mathbb{Z}$, set

$$\mathbf{M}_j := \frac{1}{2} \int_0^1 \mathbf{Y}_V(s, \zeta_j)^* \mathbf{Y}_V(s, \zeta_j) ds, \quad \zeta_j := \zeta_j(V, U).$$

Observe that $\mathbf{M}_j = \mathbf{M}_j^* > 0$ and denote by $\mathbf{P}_j : \mathbb{C}^{4r} \rightarrow \mathbb{C}^{4r}$ the orthogonal projector onto $\mathcal{E}_j := \ker[\mathbf{A} \mathbf{Y}_V(0, \zeta_j) + \mathbf{B} \mathbf{Y}_V(1, \zeta_j)]$, where

$$\mathbf{A} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -I & I \end{pmatrix}, \quad \mathbf{B} := \frac{1}{\sqrt{2}} \begin{pmatrix} I & -U \\ 0 & 0 \end{pmatrix}, \quad I := I_{2r}.$$

Next, define the positive self-adjoint operator $\mathbf{B}_j : \mathcal{E}_j \rightarrow \mathcal{E}_j$ via $\mathbf{B}_j := (\mathbf{P}_j \mathbf{M}_j \mathbf{P}_j)|_{\mathcal{E}_j}$ and the operator $\mathbf{D}_j : \mathbb{C}^{4r} \rightarrow \mathbb{C}^{4r}$ by setting $\mathbf{D}_j := \mathbf{B}_j^{-1} \mathbf{P}_j$. As in Definition 2.1, one may call $\mathbf{D}_j$ the norming matrix of the operator $S_{V,U}$. However, it turns out that $\mathbf{D}_j$ is of the same rank as the norming matrix C_j from Definition 3.1 and, moreover, there are simple formulas relating C_j and $\mathbf{D}_j$:

$$(3.12) \quad C_j = -\frac{1}{2} a \mathbf{J} \mathbf{D}_j \mathbf{J} a^*, \quad \mathbf{D}_j = -2 \mathbf{J} a^* C_j a \mathbf{J},$$

where

$$(3.13) \quad a := \frac{1}{\sqrt{2}} (I, -I), \quad I := I_{2r}.$$

Thus there is no essential difference between defining the norming matrices of $S_{V,U}$ as described in Definition 3.1 or as described in this remark. However, using Definition 3.1 is more convenient in this paper. The proof of formulas (3.12) will follow from the proof of Lemma 3.1 below.

4. PROOFS

In this section, we prove the main results of this paper.

4.1. Proof of Lemma 2.1. Let $q \in \mathcal{Q}_p$, $p \in [1, \infty)$. We start by proving Lemma 2.1 providing a parametrization of all self-adjoint extensions of the minimal operator T_q^0 (see Sect. 2). Although the proof essentially uses the concept of boundary triplets (see, e.g., [9, Chap. 3]), we omit the terminology and reduce it to straightforward manipulations:

Proof of Lemma 2.1. Let the operators

$$E_j : W_2^1((-1, 1), \mathbb{C}^{2r}) \rightarrow \mathbb{C}^{2r}, \quad j = 1, 2$$

act by the formulas

$$E_1 f := (f_1(1), f_2(-1))^\top, \quad E_2 f := (f_1(-1), f_2(1))^\top,$$

where f_1 and f_2 are $\mathbb{C}^r$ -valued functions composed of the first r and the last r components of f , respectively. Define the operator $E : W_2^1((-1, 1), \mathbb{C}^{2r}) \rightarrow \mathcal{G} := \mathbb{C}^{2r} \times \mathbb{C}^{2r}$ by the formula

$$E f := (E_1 f, E_2 f)^\top.$$

Then a direct verification shows that for every $f, h \in D(T_q)$,

$$(4.1) \quad (T_q f | h)_{\mathcal{H}} - (f | T_q h)_{\mathcal{H}} = -i(E_1 f | E_1 h)_{\mathbb{C}^{2r}} + i(E_2 f | E_2 h)_{\mathbb{C}^{2r}} = (\mathbf{J} E f | E h)_{\mathcal{G}},$$

where T_q is the maximal operator (see Sect. 2) and $\mathbf{J} := -i \operatorname{diag}(I_{2r}, -I_{2r})$.

Now let $\mathcal{T}$ stand for the set of all linear operators $T : \mathcal{H} \rightarrow \mathcal{H}$ such that $T_q^0 \subset T \subset T_q$ and denote by $\mathcal{T}_s$ the set of all *self-adjoint* operators $T \in \mathcal{T}$. For every $T \in \mathcal{T}$, set

$$F_T := \{E f \mid f \in D(T)\}.$$

It is then easily seen from (4.1) that every operator $T \in \mathcal{T}$ is related to its adjoint T^* via

$$F_{T^*} = (\mathbf{J} F_T)^\perp.$$

Hence, the operator $T \in \mathcal{T}$ belongs to $\mathcal{T}_s$ if and only if

$$\dim F_T = 2r, \quad \mathbf{J} F_T \perp F_T.$$

It now follows from (4.1) that for every $T \in \mathcal{T}_s$ and $f \in D(T)$, $\|E_1 f\| = \|E_2 f\|$, and thus we find that the operator $T \in \mathcal{T}$ belongs to $\mathcal{T}_s$ if and only if there exists a unitary matrix $U \in \mathcal{U}_{2r}$ such that

$$D(T) = \{f \in D(T_q) \mid (E_1 + U E_2)f = 0\},$$

i.e. $D(T) = \ker(E_1 + U E_2)$. Finally, to complete the proof it only remains to observe that for an arbitrary $f \in W_2^1((-1, 1), \mathbb{C}^{2r})$ one has $f \in \ker(E_1 + U E_2)$ if and only if

$$A_U f(-1) + B_U f(1) = 0,$$

where A_U and B_U are given by formula (2.4). $\square$

4.2. Basic properties of the operators $S_{V,U}$. Before proving Lemma 3.1 allowing us to reduce the inverse problem for the operators $T_{q,U}$ to the one for the operators $S_{V,U}$, we need to list some properties of the latter.

We start by introducing the Weyl–Titchmarsh function of the operator $S_{V,U}$ (see [4]). Let $V \in \mathfrak{Q}_p$. For an arbitrary $\zeta \in \mathbb{C}$, let $\varphi_V(\cdot, \zeta)$ and $\psi_V(\cdot, \zeta)$ be a $4r \times 2r$ matrix-valued solutions of the Cauchy problems

$$(4.2) \quad \mathbf{J} \frac{d}{dx} \varphi + \mathbf{V} \varphi = \zeta \varphi, \quad \varphi(0, \zeta) = \mathbf{J} a^*,$$

and

$$\mathbf{J} \frac{d}{dx} \psi + \mathbf{V} \psi = \zeta \psi, \quad \psi(0, \zeta) = a^*,$$

respectively, where $\mathbf{J}$ and $\mathbf{V}$ are given by formula (3.1) and a is given by formula (3.13). Set $c_{V,U}(\zeta) := b_U \psi_V(1, \zeta)$ and $s_{V,U}(\zeta) := b_U \varphi_V(1, \zeta)$, where

$$(4.3) \quad b_U := \frac{1}{\sqrt{2}} (U^{-1/2}, -U^{1/2})$$

and the square root of U is taken so that if $e^{2i\gamma_k}$, $\gamma_k \in [0, \pi)$, are all distinct eigenvalues of U , then $e^{i\gamma_k}$ are all distinct eigenvalues of $U^{1/2}$. Then the function

$$(4.4) \quad M_{V,U}(\zeta) := -s_{V,U}(\zeta)^{-1} c_{V,U}(\zeta)$$

will be called the *Weyl–Titchmarsh function* of the operator $S_{V,U}$.

The following proposition is proved in [18]:

Proposition 4.1. *For every $V \in \mathfrak{Q}_p$, there exists a unique function $K_V \in G_p^+(\mathcal{M}_{4r})$ (see Appendix A) such that for all $\zeta \in \mathbb{C}$ and $x \in [0, 1]$,*

$$(4.5) \quad \varphi_V(x, \zeta) = \varphi_0(x, \zeta) + \int_0^x K_V(x, s) \varphi_0(s, \zeta) ds,$$

where $\varphi_0(x, \zeta) = \frac{1}{\sqrt{2i}} \begin{pmatrix} e^{i\zeta x} I \\ e^{-i\zeta x} I \end{pmatrix}$, $I := I_{2r}$, is a solution of (4.2) in the free case $\mathbf{V} = 0$.

For an arbitrary $\zeta \in \mathbb{C}$, we define the operator $\Phi_V(\zeta) : \mathbb{C}^{2r} \rightarrow \mathbb{H}$ by setting

$$(4.6) \quad [\Phi_V(\zeta)c](x) := \varphi_V(x, \zeta)c, \quad x \in [0, 1].$$

It then follows from (4.5) that for every $\zeta \in \mathbb{C}$,

$$(4.7) \quad \Phi_V(\zeta) = (I + K_V)\Phi_0(\zeta),$$

where $K_V : \mathbb{H} \rightarrow \mathbb{H}$ is the integral operator with kernel K_V and I is the identity operator in $\mathbb{H}$. Note that for every $V \in \mathfrak{Q}_p$, K_V is a Volterra operator so that $I + K_V$ is invertible. Furthermore, it follows that for every $V \in \mathfrak{Q}_p$ and $\zeta \in \mathbb{C}$,

$$(4.8) \quad \ker \Phi_V(\zeta) = \{0\}, \quad \text{Ran } \Phi_V^*(\zeta) = \mathbb{C}^{2r}.$$

Now we are ready to state the basic properties of the operators $S_{V,U}$:

Proposition 4.2. *For every $V \in \mathfrak{Q}_p$ and $U \in \mathcal{U}_{2r}$,*

- (i) *the operator $S_{V,U}$ is self-adjoint;*
- (ii) *the spectrum $\sigma(S_{V,U})$ of the operator $S_{V,U}$ consists of countably many isolated real eigenvalues of finite multiplicity; moreover,*

$$\sigma(S_{V,U}) = \{\zeta \in \mathbb{C} \mid \ker s_{V,U}(\zeta) \neq \{0\}\};$$

- (iii) *for every $j \in \mathbb{Z}$, let $P_j : \mathbb{H} \rightarrow \mathbb{H}$ be the orthogonal projector onto $\ker(S_{V,U} - \zeta_j I)$, $\zeta_j := \zeta_j(V, U)$ be eigenvalues of the operator $S_{V,U}$ and $C_j := C_j(V, U)$ be the corresponding norming matrices; then for every $j \in \mathbb{Z}$ one has $C_j \geq 0$ and*

$$(4.9) \quad P_j = \Phi_V(\zeta_j)C_j\Phi_V^*(\zeta_j).$$

The proof of Proposition 4.2 repeats the proof of Theorem 2.1 in [16].

Since $S_{V,U}$ is a self-adjoint operator with discrete spectrum, it also follows that

$$(4.10) \quad \sum_{j=-\infty}^{\infty} P_j = I;$$

by virtue of the relations (4.9) and (4.10), the operators $\Phi_V(\zeta)$ will play an important role in this investigation.

4.3. Proof of Lemma 3.1. Now we are ready to prove Lemma 3.1 allowing us to reduce the inverse spectral problem for the operators $T_{q,U}$ to the one for the operators $S_{V,U}$:

Proof of Lemma 3.1. Let $q \in \mathfrak{Q}_p$, $U \in \mathcal{U}_{2r}$ and $\mathfrak{a}_{q,U} = ((\lambda_j, A_j))_{j \in \mathbb{Z}}$ be the spectral data of the operator $T_{q,U}$. Consider the unitary transformation $\mathcal{V} : \mathcal{H} \rightarrow \mathbb{H}$ given by the formula

$$(4.11) \quad (\mathcal{V}y)(x) = (y_1(x), y_2(-x), y_1(-x), y_2(x))^{\top}, \quad x \in (0, 1),$$

where y_1 and y_2 are $\mathbb{C}^r$ -valued functions composed of the first r and the last r components of y , respectively. Then a direct verification shows that

$$T_{q,U} = \mathcal{V}^{-1}S_{V,U}\mathcal{V},$$

where the potential V is given by formula (3.2). In particular, it then follows that the spectra of the operators $T_{q,U}$ and $S_{V,U}$ coincide and for every $j \in \mathbb{Z}$,

$$(4.12) \quad \lambda_j = \zeta_j(V, U).$$

Thus it only remains to prove that for every $j \in \mathbb{Z}$,

$$(4.13) \quad A_j = C_j(V, U).$$

For an arbitrary $\lambda \in \mathbb{C}$, define the operator $\Psi_q(\lambda) : \mathbb{C}^{2r} \rightarrow \mathcal{H}$ by the formula

$$[\Psi_q(\lambda)c](x) := \frac{1}{\sqrt{2i}}Y_q(x, \lambda)c, \quad x \in [-1, 1],$$

where Y_q is a solution of the Cauchy problem (2.6). For every $j \in \mathbb{Z}$, let $\mathcal{P}_j : \mathcal{H} \rightarrow \mathcal{H}$ be the orthogonal projector onto the eigenspace $\ker(T_{q,U} - \lambda_j \mathcal{I})$, where $\mathcal{I}$ is the identity operator in $\mathcal{H}$. Then (4.13) will be proved if we show that

$$(4.14) \quad \mathcal{P}_j = \Psi_q(\lambda_j) A_j \Psi_q^*(\lambda_j).$$

Indeed, observe that for every $j \in \mathbb{Z}$,

$$(4.15) \quad \mathcal{P}_j = \mathcal{V}^{-1} P_j \mathcal{V},$$

where $P_j : \mathbb{H} \rightarrow \mathbb{H}$ is the orthogonal projector onto $\ker(S_{V,U} - \lambda_j I)$ and $\mathcal{V}$ is the unitary transformation (4.11). Furthermore, a direct verification shows that for every $\lambda \in \mathbb{C}$,

$$(4.16) \quad \Phi_V(\lambda) = \mathcal{V} \Psi_q(\lambda).$$

We then obtain from (4.14)–(4.16), (4.12) and (4.9) that for every $j \in \mathbb{Z}$,

$$\Phi_V(\lambda_j)(A_j - C_j(V, U))\Phi_V^*(\lambda_j) = 0.$$

Since for every $\lambda \in \mathbb{C}$, $\ker \Phi_V(\lambda) = \{0\}$ and $\text{Ran } \Phi_V^*(\lambda) = \mathbb{C}^{2r}$, this proves (4.13).

Thus it only remains to prove (4.14). For this purpose, note that for every $j \in \mathbb{Z}$, the operator $\tilde{\mathcal{P}}_j := \Psi_q(\lambda_j) A_j \Psi_q^*(\lambda_j)$ is self-adjoint and

$$\text{Ran } \tilde{\mathcal{P}}_j = \Psi_q(\lambda_j) \mathcal{E}_j = \ker(T_{q,U} - \lambda_j \mathcal{I}),$$

where $\mathcal{E}_j := \ker[A_U Y_q(-1, \lambda_j) + B_U Y_q(1, \lambda_j)]$. Therefore, in order to prove that $\tilde{\mathcal{P}}_j = \mathcal{P}_j$ it suffices to prove that $\tilde{\mathcal{P}}_j^2 = \tilde{\mathcal{P}}_j$. To this end, recall Remark 2.1 and verify that

$$A_j \Psi_q^*(\lambda_j) \Psi_q(\lambda_j) A_j = A_j \left\{ \frac{1}{2} \int_{-1}^1 Y_q(s, \lambda_j)^* Y_q(s, \lambda_j) ds \right\} A_j = A_j M_j A_j = A_j.$$

Therefore, $\tilde{\mathcal{P}}_j^2 = \tilde{\mathcal{P}}_j$ follows and the proof is complete. $\square$

The following important corollary now follows from Lemma 3.1:

Corollary 4.1. *Every sequence $\mathbf{a}$ from the class $\mathfrak{A}_p$ belongs to the class $\mathfrak{B}_p$ and is the spectral data of an operator $S_{V,U}$ with some potential V of the form (3.2) and $U := U_a$ given by (3.7).*

Remark 4.1. The proof of formulas (3.12) providing a relations between differently defined norming matrices of the operator $S_{V,U}$ (see Remark 3.3) also follows from the proof of Lemma 3.1. Namely, let $\mathbf{Y}_V(\cdot, \zeta)$ be a $4r \times 4r$ matrix-valued solution of the Cauchy problem (3.11) and $\Psi_V(\zeta) : \mathbb{C}^{4r} \rightarrow \mathbb{H}$, where $\zeta \in \mathbb{C}$, be an operator defined by the formula

$$[\Psi_V(\zeta)c](x) := \frac{1}{\sqrt{2i}} \mathbf{Y}_V(x, \zeta)c, \quad x \in [0, 1].$$

As in the proof of Lemma 3.1, it can be shown that $P_j = \Psi_V(\zeta_j) \mathbf{D}_j \Psi_V^*(\zeta_j)$, where P_j is the eigenprojector of the operator $S_{V,U}$ and $\mathbf{D}_j$ is as in Remark 3.3. Next, since $\varphi_V(x, \zeta) = \mathbf{Y}_V(x, \zeta) \mathbf{J} a^*$ (see (4.2)), it follows that $\Phi_V(\zeta) = \sqrt{2i} \Psi_V(\zeta) \mathbf{J} a^*$ (see (4.6)); taking into account also (4.9) we obtain that

$$\Psi_V(\zeta_j) (\mathbf{D}_j + 2 \mathbf{J} a^* C_j a \mathbf{J}) \Psi_V^*(\zeta_j) = 0,$$

where $C_j := C_j(V, U)$ is as in Definition 3.1. Since for an arbitrary $\zeta \in \mathbb{C}$ one has $\ker \Psi_V(\zeta) = \{0\}$ and $\text{Ran } \Psi_V^*(\zeta) = \mathbb{C}^{4r}$, this proves the second relation in (3.12). The first one follows from the relations $\mathbf{J}^2 = -I_{4r}$ and $aa^* = I_{2r}$.

4.4. Proof of Lemmas 3.2–3.4. We now proceed to solve the inverse spectral problem for the operators $S_{V,U}$. The proof of Lemmas 3.2–3.4 is based on the connection between the operators $S_{V,U}$ and $S_{V,I}$, where $I := I_{2r}$ is the identity matrix. We then use the results of [18] where the direct and inverse spectral problems for the operators $S_{V,I}$ were solved.

Let $V \in \mathfrak{Q}_p$ and $U \in \mathcal{U}_{2r}$. We start from the following observation:

Lemma 4.1. *Let $\mathfrak{a} := ((\lambda_j, A_j))_{j \in \mathbb{Z}}$ be the spectral data of the operator $S_{V,U}$ and $\mu := \mu^{\mathfrak{a}}$ be its spectral measure. Then for every $f \in \mathcal{S}^{2r}$ such that $\text{supp } f \subset (-1, 1)$,*

$$(H_\mu, f) = (H_\nu, f),$$

where ν is the spectral measure of the operator $S_{V,I}$.

Proof. For every $j \in \mathbb{Z}$, let $P_j : \mathbb{H} \rightarrow \mathbb{H}$ be the orthogonal projector onto the eigenspace $\ker(S_{V,U} - \lambda_j I)$, where I is the identity operator in $\mathbb{H}$. We then find from (4.9) and (4.10) that

$$\sum_{j=-\infty}^{\infty} \Phi_V(\lambda_j) A_j \Phi_V^*(\lambda_j) = I,$$

where the series on the left hand side converges in the strong operator topology. Recalling also (4.7), we observe that

$$(4.17) \quad \sum_{j=-\infty}^{\infty} \Phi_0(\lambda_j) A_j \Phi_0^*(\lambda_j) = (I + K_V)^{-1} (I + K_V^*)^{-1},$$

where $K_V : \mathbb{H} \rightarrow \mathbb{H}$ is the integral operator with kernel K_V from Proposition 4.1. Note that the right hand side of (4.17) depends *only* on the potential V of the operator $S_{V,U}$, while the left hand side of (4.17) depends *only* on the spectral data.

Now let ν be the spectral measure of the operator $S_{V,I}$ and $H := H_\nu$. It then follows from the results of [18] that $H \in L_p((-1, 1), \mathcal{M}_{2r})$ and

$$(4.18) \quad (I + K_V)^{-1} (I + K_V^*)^{-1} = I + F_H,$$

where $F_H : \mathbb{H} \rightarrow \mathbb{H}$ is the integral operator with kernel

$$(4.19) \quad F_H(x, t) := \frac{1}{2} \begin{pmatrix} H\left(\frac{x-t}{2}\right) & H\left(\frac{x+t}{2}\right) \\ H\left(-\frac{x+t}{2}\right) & H\left(-\frac{x-t}{2}\right) \end{pmatrix}, \quad 0 \leq x, t \leq 1.$$

Therefore, we find from (4.17) and (4.18) that

$$(4.20) \quad \sum_{j=-\infty}^{\infty} \Phi_0(\lambda_j) A_j \Phi_0^*(\lambda_j) = I + F_H, \quad H := H_\nu.$$

The lemma will follow directly from this relation.

Indeed, set $\tilde{\mathcal{H}} := L_2((0, 1), \mathbb{C}^{2r})$ and consider the unitary transformation $\mathcal{W} : \tilde{\mathcal{H}} \rightarrow \mathbb{H}$ acting by the formula

$$(\mathcal{W}g)(x) := \frac{1}{\sqrt{2}} \left(g\left(\frac{1+x}{2}\right), g\left(\frac{1-x}{2}\right) \right)^\top, \quad g \in \tilde{\mathcal{H}}.$$

Then a direct verification shows that $F_H = \mathcal{W}H\mathcal{W}^{-1}$, where $H : \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$ is the integral operator given by the formula

$$(Hg)(x) = \int_0^1 H(x-s)g(s) ds.$$

Furthermore, it also follows that $\text{ie}^{i\lambda} \Phi_0(\lambda) = \mathcal{W} \Upsilon_0(\lambda)$, where for an arbitrary $\lambda \in \mathbb{C}$, the operator $\Upsilon_0(\lambda) : \mathbb{C}^{2r} \rightarrow \tilde{\mathcal{H}}$ acts by the formula

$$[\Upsilon_0(\lambda)c](x) := \text{e}^{2i\lambda x} c.$$

Therefore, (4.20) is reduced to the equality

$$(4.21) \quad \sum_{j=-\infty}^{\infty} \Upsilon_0(\lambda_j) A_j \Upsilon_0^*(\lambda_j) = \tilde{\mathcal{I}} + H,$$

where $\tilde{\mathcal{I}}$ is the identity operator in $\tilde{\mathcal{H}}$. In particular, in the free case $V = 0$ (4.21) reads

$$(4.22) \quad \sum_{n=-\infty}^{\infty} \Upsilon_0(\lambda_n^0) A_n^0 \Upsilon_0^*(\lambda_n^0) = \tilde{\mathcal{I}},$$

where λ_n^0 and A_n^0 , $n \in \mathbb{Z}$, are eigenvalues and norming matrices of the free operator $S_{0,U}$, respectively. Since for an arbitrary $f \in \mathcal{S}^{2r}$ such that $\text{supp } f \subset (-1, 1)$ one has $(H, f) = \int_{-1}^1 H(s) f(s) ds$,

$$(\widehat{\mu}, f) = \sum_{j=-\infty}^{\infty} \int_{-1}^1 e^{2i\lambda_j s} A_j f(s) ds, \quad (\widehat{\mu_0}, f) = \sum_{n=-\infty}^{\infty} \int_{-1}^1 e^{2i\lambda_n^0 s} A_n^0 f(s) ds,$$

substituting (4.22) into (4.21) and using the formulas for $\Upsilon_0(\lambda)$ and H one can easily find that

$$(H_\mu, f) := (\widehat{\mu} - \widehat{\mu_0}, f) = (H, f),$$

as desired. $\square$

We now use the results of [18] to obtain the following corollary:

Corollary 4.2. *For an arbitrary $V \in \mathfrak{Q}_p$ and $U \in \mathcal{U}_{2r}$, the spectral data of the operator $S_{V,U}$ satisfy the conditions (C_3) and (C_4) from Lemma 3.2.*

Proof. It is proved in [18] that for an arbitrary $V \in \mathfrak{Q}_p$ it holds $H_\nu \in \mathfrak{H}_p$, where ν is the spectral measure of the operator $S_{V,I}$. It then follows from Lemma 4.1 that for an arbitrary $V \in \mathfrak{Q}_p$ and $U \in \mathcal{U}_{2r}$ it holds $H_\mu \in \mathfrak{H}_p$, where $\mu := \mu^{\mathfrak{a}}$ is the spectral measure of the operator $S_{V,U}$ and $\mathfrak{a} := ((\lambda_j, A_j))_{j \in \mathbb{Z}}$ is its spectral data. Therefore, we immediately obtain that the spectral data of the operator $S_{V,U}$ satisfy the condition (C_4) .

In order to prove the condition (C_3) , observe that by virtue of (4.20) one has

$$\ker(I + F_H) = \ker \left(\sum_{j=-\infty}^{\infty} \Phi_0(\lambda_j) A_j \Phi_0^*(\lambda_j) \right) = \bigcap_{j=-\infty}^{\infty} \ker A_j \Phi_0^*(\lambda_j) = \tilde{\mathcal{W}} \mathcal{X}^\perp,$$

where $H := H_\nu$, $\mathcal{X} := \{e^{i\lambda_j t} d \mid j \in \mathbb{Z}, d \in \text{Ran } A_j\}$ and $\tilde{\mathcal{W}} : L_2((-1, 1), \mathbb{C}^{2r}) \rightarrow \mathbb{H}$ is the unitary mapping acting by the formula $(\tilde{\mathcal{W}}f)(x) = (f(x), f(-x))^\top$, $x \in (0, 1)$. Since $H \in \mathfrak{H}_p$, it follows from the results of [18] that $I + F_H > 0$ and thus $\ker(I + F_H) = \{0\}$. Therefore, $\mathcal{X}^\perp = \{0\}$, which proves the condition (C_3) . $\square$

Remark 4.2. It is proved in [18, Lemma 4.2] that for an arbitrary sequence $\mathfrak{a}$ satisfying the conditions (C_3) and (C_4) one has $H_\mu \in \mathfrak{H}_p$, where $\mu := \mu^{\mathfrak{a}}$.

Next, since eigenvalues of the operator $S_{V,U}$ are zeros of the entire function $\tilde{s}_{V,U}(\lambda) := \det s_{V,U}(\lambda)$ (see Proposition 4.2), the standard technique based on Rouché's theorem implies that eigenvalues of $S_{V,U}$ satisfy the asymptotics (3.4) and the condition (3.5). Furthermore, since

$$\|M_{V,U}(\lambda) - M_{0,U}(\lambda)\| = o(1)$$

as $\lambda \rightarrow \infty$ within the domain $\mathcal{O}_\varepsilon := \{\lambda \in \mathbb{C} \mid \forall m \in \mathbb{Z} : |\lambda - \zeta_m^0| \geq \varepsilon\}$ for some $\varepsilon > 0$, one can easily prove (3.6) and obtain that the spectral data of the operator $S_{V,U}$ satisfy the condition (C_1) .

Therefore, so far we proved that the spectral data of the operator $S_{V,U}$ satisfy the conditions (C_1) , (C_3) and (C_4) .

For an arbitrary sequence $\mathbf{a} := ((\lambda_j, A_j))_{j \in \mathbb{Z}}$ satisfying the conditions (C_1) , (C_3) and (C_4) , we set $\mu := \mu^{\mathbf{a}}$, $H := H_\mu$ and $V := \Theta(H)$ (see Remark 4.2). For every $j \in \mathbb{Z}$, we then define the operator $P_{\mathbf{a},j} : \mathbb{H} \rightarrow \mathbb{H}$ by the formula

$$(4.23) \quad P_{\mathbf{a},j} := \Phi_V(\lambda_j) A_j \Phi_V^*(\lambda_j).$$

Proposition 4.3. *Let $\mathbf{a}$ be an arbitrary sequence satisfying the conditions (C_1) , (C_3) and (C_4) . Then:*

- (i) *the series $\sum_{j \in \mathbb{Z}} P_{\mathbf{a},j}$ converges to the identity operator $\mathbb{H} \rightarrow \mathbb{H}$ in the strong operator topology;*
- (ii) *a sequence $\mathbf{a}$ satisfies the condition (C_2) if and only if $\{P_{\mathbf{a},j}\}_{j \in \mathbb{Z}}$ is a system of pairwise orthogonal projectors in $\mathbb{H}$.*

The proof of Proposition 4.3 can be obtained by a straightforward modification of the proof of Proposition 3.3 and Lemma 4.5 in [18]; the proof uses the factorization of integral operators and the vector analogue of Kadec's 1/4-theorem.

We now use Proposition 4.3 to prove Lemmas 3.2 and 3.4:

Proof of Lemmas 3.2 and 3.4. Let $\mathbf{a} \in \mathfrak{B}_p$ be the spectral data of the operator $S_{V,U}$. It then follows from the above that $\mathbf{a}$ satisfies the conditions (C_1) , (C_3) and (C_4) . Now observe that by virtue of Proposition 4.2, the operators $P_{\mathbf{a},j}$, $j \in \mathbb{Z}$, coincide with eigenprojectors of the operator $S_{V,U}$. Proposition 4.3 then implies that $\mathbf{a}$ satisfies the condition (C_2) . This is the necessity part of Lemma 3.2.

Now let $\mathbf{a} := ((\lambda_j, A_j))_{j \in \mathbb{Z}}$ be an arbitrary sequence satisfying the conditions (C_1) – (C_4) ; set $\mu := \mu^{\mathbf{a}}$, $H := H_\mu$ and $V := \Theta(H)$. Define the operators $P_{\mathbf{a},j}$, $j \in \mathbb{Z}$, by formula (4.23). It then follows from Proposition 4.3 that $\{P_{\mathbf{a},j}\}_{j \in \mathbb{Z}}$ is a complete system of pairwise orthogonal projectors in $\mathbb{H}$. Then the same arguments as in [18] will imply that $\mathbf{a}$ coincides with the spectral data of the operator $S_{V,U}$ with $U := U_{\mathbf{a}}$.

Namely, let $\mathbf{b} := ((\zeta_j, C_j))_{j \in \mathbb{Z}}$ be the spectral data of the operator $S_{V,U}$. As in [18], we observe that it only suffices to prove the inclusion

$$(4.24) \quad \text{Ran } P_{\mathbf{a},j} \subset \ker(S_{V,U} - \lambda_j I), \quad j \in \mathbb{Z}.$$

Indeed, taking into account completeness of $\{P_{\mathbf{a},j}\}_{j \in \mathbb{Z}}$, we immediately conclude from (4.24) that $\lambda_j = \zeta_j$ for every $j \in \mathbb{Z}$. From this equality and from (4.24) we then obtain that for every $j \in \mathbb{Z}$, $P_j - P_{\mathbf{a},j} \geq 0$, where P_j are eigenprojectors of the operator $S_{V,U}$. However, taking into account completeness of the systems $\{P_{\mathbf{a},j}\}_{j \in \mathbb{Z}}$ and $\{P_j\}_{j \in \mathbb{Z}}$, we observe that $\sum_{j \in \mathbb{Z}} (P_j - P_{\mathbf{a},j}) = 0$ and thus $P_j - P_{\mathbf{a},j} = 0$ for every $j \in \mathbb{Z}$. Therefore, recalling the representation (4.9) for P_j , we find that

$$\Phi_V(\lambda_j) \{C_j - A_j\} \Phi_V^*(\lambda_j) = 0, \quad j \in \mathbb{Z},$$

Taking into account (4.8) we then obtain that $A_j = C_j$. Together with $\lambda_j = \zeta_j$, this implies that $\mathbf{a} = \mathbf{b}$, as desired.

Thus it only remains to prove (4.24). Since $P_{\mathbf{a},j} = \Phi_V(\lambda_j) A_j \Phi_V^*(\lambda_j)$ and $\text{Ran } \Phi_V^*(\lambda) = \mathbb{C}^{2r}$ for an arbitrary $\lambda \in \mathbb{C}$, we find that for every $j \in \mathbb{Z}$,

$$\text{Ran } P_{\mathbf{a},j} = \{\varphi_V(\cdot, \lambda_j) A_j c \mid c \in \mathbb{C}^{2r}\}.$$

Since for every $\lambda \in \mathbb{C}$, $\varphi_V(\cdot, \lambda)$ is a solution of the Cauchy problem

$$(4.25) \quad \mathbf{J} \frac{d}{dx} \varphi + \mathbf{V} \varphi = \lambda \varphi, \quad \varphi(0, \lambda) = \mathbf{J} a^*,$$

we then find that for every $f \in \text{Ran } P_{\mathbf{a},j}$ it holds $\mathfrak{S}_V(f) = \lambda_j f$ and $f_1(0) = f_2(0)$. Therefore, it only remains to prove that for every $f \in \text{Ran } P_{\mathbf{a},j}$ one has $f_1(1) = U f_2(1)$ with $U := U_{\mathbf{a}}$. The latter reads that for every $i \in \mathbb{Z}$,

$$(4.26) \quad b_U \varphi_V(1, \lambda_i) A_i = 0,$$

where b_U is given by formula (4.3).

So let us prove (4.26). To this end, recalling that $\varphi_V(\cdot, \lambda)$ is a solution of the Cauchy problem (4.25) and integrating by parts, we obtain that for all $i, j \in \mathbb{Z}$ and $c, d \in \mathbb{C}^{2r}$,

$$\lambda_i(\Phi_V(\lambda_i)c \mid \Phi_V(\lambda_j)d) = (\mathbf{J}\varphi_V(1, \lambda_i)c \mid \varphi_V(1, \lambda_j)d) + \lambda_j(\Phi_V(\lambda_i)c \mid \Phi_V(\lambda_j)d)$$

and thus

$$(4.27) \quad (\lambda_i - \lambda_j)\Phi_V(\lambda_j)^*\Phi_V(\lambda_i) = \varphi_V(1, \lambda_j)^*\mathbf{J}\varphi_V(1, \lambda_i).$$

Since $P_{a,i}P_{a,j} = 0$ as $i \neq j$ and for all $\lambda \in \mathbb{C}$ it holds $\ker \Phi_V(\lambda) = \{0\}$ and $\text{Ran } \Phi_V^*(\lambda) = \mathbb{C}^{2r}$, we find that $A_j\Phi_V^*(\lambda_j)\Phi_V(\lambda_i)A_i = 0$, $i \neq j$. Therefore, we obtain from (4.27) that

$$(4.28) \quad A_j\varphi_V(1, \lambda_j)^*\mathbf{J}\varphi_V(1, \lambda_i)A_i = 0, \quad i \neq j.$$

Let $j \in \mathbb{Z}$. Taking into account (4.28), we find that

$$(4.29) \quad \left\{ \sum_{k=1}^s (-1)^{n+1} \sum_{\lambda_j \in \Delta_{ns+k}} \mathbf{J}\varphi_V(1, \lambda_j)A_j \right\}^* \varphi_V(1, \lambda_i)A_i = 0$$

for large values of $n \in \mathbb{Z}$. If we show that

$$(4.30) \quad \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^s (-1)^{n+1} \sum_{\lambda_j \in \Delta_{ns+k}} \mathbf{J}\varphi_V(1, \lambda_j)A_j \right\}^* = b_U,$$

then passing to the limit $n \rightarrow \infty$ in (4.29) would yield (4.26). In order to prove (4.30), taking into account the Riemann-Lebesgue lemma, (3.4) and (3.6) we find that for every $k \in \{1, \dots, s\}$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\{ (-1)^{n+1} \sum_{\lambda_j \in \Delta_{ns+k}} \mathbf{J}\varphi_V(1, \lambda_j)A_j \right\}^* \\ &= \lim_{n \rightarrow \infty} \left\{ (-1)^{n+1} \mathbf{J}\varphi_0(1, \zeta_{ns+k}^0) \sum_{\lambda_j \in \Delta_{ns+k}} A_j \right\}^* \\ &= \frac{1}{\sqrt{2}} \left\{ \lim_{n \rightarrow \infty} \sum_{\lambda_j \in \Delta_{ns+k}} A_j \right\} (e^{-i\gamma_k}, -e^{i\gamma_k}) = \frac{1}{\sqrt{2}} (e^{-i\gamma_k} P_k^0, -e^{i\gamma_k} P_k^0). \end{aligned}$$

Since

$$\sum_{k=1}^s \left\{ \frac{1}{\sqrt{2}} (e^{-i\gamma_k} P_k^0, -e^{i\gamma_k} P_k^0) \right\} = \frac{1}{\sqrt{2}} (U^{-1/2}, -U^{1/2}) = b_U,$$

(4.30) follows and thus we have proved that $\mathbf{a} = \mathbf{b}$, where $\mathbf{b}$ is the spectral data of the operator $S_{V,U}$ with $V = \Theta(H)$ and $U = U_a$. This is the sufficiency part of Lemma 3.2 and the proof of Lemma 3.4. $\square$

The proof of Lemma 3.3 repeats the proof of Theorem 1.3 in [18] and therefore we omit it in this paper.

4.5. Proof of Theorems 3.1–3.3. We now use Lemmas 3.1–3.4 to prove Theorems 3.1–3.3 and thus solve the inverse spectral problem for the operators $T_{q,U}$.

Let

$$T := \{T_{q,U} : \mathcal{H} \rightarrow \mathcal{H} \mid q \in \mathcal{Q}_p, U \in \mathcal{U}_{2r}\}, \quad S := \{S_{V,U} : \mathbb{H} \rightarrow \mathbb{H} \mid V \in \mathfrak{Q}_p, U \in \mathcal{U}_{2r}\}.$$

Recall that for every operator $T_{q,U} \in T$ we introduce the *associated operator* $S_{V,U} \in S$ with the potential V given by formula (3.2). Taking into account that the mapping

$$\mathcal{Q}_p \ni q \mapsto V(x) := \begin{pmatrix} 0 & q(x) \\ q(-x)^* & 0 \end{pmatrix} \in \{V \in \mathfrak{Q}_p \mid V(x)J = -JV(x) \text{ a.e. on } (0,1)\}$$

is bijective, we arrive at the following obvious remark:

Remark 4.3. Let $S_{V,U} \in S$. Then there exists an operator $T_{q,U} \in T$ such that $S_{V,U}$ is associated to $T_{q,U}$ if and only if the potential V of the operator $S_{V,U}$ satisfies the anti-commutative relation

$$V(x)J = -JV(x)$$

a.e. on $(0, 1)$. In this case, such operator $T_{q,U}$ is unique and its potential q can be found from V by formula (3.10).

Now we are ready to prove Theorem 3.1 providing a complete description of the class $\mathfrak{A}_p$ of the spectral data of the operators $T_{q,U}$:

Proof of Theorem 3.1. Necessity. Let $\mathfrak{a} \in \mathfrak{A}_p$ be the spectral data of the operator $T_{q,U}$ and $S_{V,U}$ be the associated operator. It then follows from Lemma 3.1 that $\mathfrak{a}$ is the spectral data of the operator $S_{V,U}$. From Lemmas 3.2 and 3.4 we then obtain that $\mathfrak{a}$ satisfies the conditions $(C_1) - (C_4)$ and that $V = \Theta(H)$, where $H := H_\mu$ and $\mu := \mu^{\mathfrak{a}}$. Since the operator $S_{V,U}$ is associated to $T_{q,U}$, by virtue of Remark 4.3 we also obtain that $\mathfrak{a}$ satisfies the condition (C_5) .

Sufficiency. Let $\mathfrak{a}$ be an arbitrary sequence satisfying the conditions $(C_1) - (C_5)$. From Lemmas 3.2–3.4 we then obtain that $\mathfrak{a}$ is the spectral data of the unique operator $S_{V,U}$ with $V = \Theta(H)$ and $U = U_{\mathfrak{a}}$, where $H := H_\mu$ and $\mu := \mu^{\mathfrak{a}}$. Since $\mathfrak{a}$ satisfies the condition (C_5) , by virtue of Remark 4.3 we then obtain that there exists a unique operator $T_{q,U}$ such that $S_{V,U}$ is associated to $T_{q,U}$. By virtue of Lemma 3.1, the spectral data of the operator $T_{q,U}$ coincide with $\mathfrak{a}$. This proves the sufficiency part of Theorem 3.1. $\square$

Next, we prove Theorem 3.2 claiming that the spectral data of the operator $T_{q,U}$ determine the potential q and the unitary matrix U uniquely:

Proof of Theorem 3.2. Let $\mathfrak{a} \in \mathfrak{A}_p$ be the spectral data of the operator $T_{q,U}$ and $\tilde{\mathfrak{a}} \in \mathfrak{A}_p$ be the spectral data of the operator $T_{\tilde{q},\tilde{U}}$. Assume that $\mathfrak{a} = \tilde{\mathfrak{a}}$. Let $S_{V,U}$ be the associated operator to $T_{q,U}$ and $S_{\tilde{V},\tilde{U}}$ be the associated operator to $T_{\tilde{q},\tilde{U}}$. It then follows from Lemma 3.1 that $\mathfrak{a}$ is the spectral data of $S_{V,U}$ and $\tilde{\mathfrak{a}}$ is the spectral data of $S_{\tilde{V},\tilde{U}}$. Since $\mathfrak{a} = \tilde{\mathfrak{a}}$, it then follows from Lemma 3.3 that $S_{V,U} = S_{\tilde{V},\tilde{U}}$. From Remark 4.3 we then obtain that $T_{q,U} = T_{\tilde{q},\tilde{U}}$. $\square$

Finally, we prove Theorem 3.3 suggesting how to find the potential q and the unitary matrix U from the spectral data of the operator $T_{q,U}$:

Proof of Theorem 3.3. Let $\mathfrak{a} \in \mathfrak{A}_p$ be a putative spectral data of the operator $T_{q,U}$. It then follows from Lemma 3.1 that $\mathfrak{a}$ is the spectral data of the associated operator $S_{V,U}$. From Lemmas 3.2–3.4 we then obtain that such operator $S_{V,U}$ is determined by its spectral data uniquely and that $V = \Theta(H)$ and $U = U_{\mathfrak{a}}$, where $H := H_\mu$ and $\mu := \mu^{\mathfrak{a}}$. Since $\mathfrak{a}$ satisfies the condition (C_5) , from Remark 4.3 we then obtain that there exists a unique operator $T_{q,U}$ such that $S_{V,U}$ is associated to $T_{q,U}$ and that the potential q of the operator $T_{q,U}$ can be found by formula (3.10). $\square$

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APPENDIX A. SPACES

In this appendix, we introduce some spaces that are used in this paper.

For an arbitrary Banach space X , we denote by $L_p((a, b), X)$, $p \in [1, \infty)$, the Banach space of all strongly measurable functions $f : (a, b) \rightarrow X$ for which the norm

$$\|f\|_{L_p} := \left(\int_a^b \|f(t)\|_X^p dt \right)^{1/p}$$

is finite. We denote by $C^k([a, b], X)$ the Banach space of all k times continuously differentiable functions $[a, b] \rightarrow X$ with the standard supremum norm. We write $W_p^1((a, b), X)$, $p \in [1, \infty)$, for the Sobolev space that is the completion of the linear space $C^1([a, b], X)$ by the norm

$$\|f\|_{W_p^1} := \left(\int_a^b \|f(t)\|_X^p dt \right)^{1/p} + \left(\int_a^b \|f'(t)\|_X^p dt \right)^{1/p};$$

every function $f \in W_p^1((a, b), X)$ has the derivative f' belonging to $L_p((a, b), X)$.

As mentioned in *Notations*, we write $\mathcal{M}_r$ for the Banach algebra of all $r \times r$ matrices with complex entries and identify it with the Banach algebra of all linear operators $\mathbb{C}^r \rightarrow \mathbb{C}^r$ endowed with the standard norm.

We denote by $G_p(\mathcal{M}_r)$, $p \in [1, \infty)$, the set of all measurable functions $K : [0, 1]^2 \rightarrow \mathcal{M}_r$ such that for all $x, t \in [0, 1]$, the functions $K(x, \cdot)$ and $K(\cdot, t)$ belong to $L_p((0, 1), \mathcal{M}_r)$ and, moreover, the mappings

$$[0, 1] \ni x \mapsto K(x, \cdot) \in L_p((0, 1), \mathcal{M}_r), \quad [0, 1] \ni t \mapsto K(\cdot, t) \in L_p((0, 1), \mathcal{M}_r)$$

are continuous. The set $G_p(\mathcal{M}_r)$ becomes a Banach space upon introducing the norm

$$\|K\|_{G_p} = \max \left\{ \max_{x \in [0, 1]} \|K(x, \cdot)\|_{L_p}, \max_{t \in [0, 1]} \|K(\cdot, t)\|_{L_p} \right\}.$$

We denote by $G_p^+(\mathcal{M}_r)$ the set of all functions $K \in G_p(\mathcal{M}_r)$ such that $K(x, t) = 0$ a.e. in $\Omega^- := \{(x, t) \mid 0 < x < t < 1\}$.

Finally, we denote by $\mathcal{S}$ the Schwartz space of all smooth functions $f \in C^\infty(\mathbb{R})$ whose derivatives (including the function itself) decay at infinity faster than any power of $|x|^{-1}$, i.e.

$$\mathcal{S} := \{f \in C^\infty(\mathbb{R}) \mid x^\alpha D^\beta f(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \alpha, \beta \in \mathbb{N} \cup \{0\}\}.$$

We set $\mathcal{S}^r := \{(f_1, \dots, f_r)^\top \mid f_j \in \mathcal{S}, j = 1, \dots, r\}$.

APPENDIX B. THE KREIN ACCELERANTS

Here, we recall some facts concerning the notion of the Krein accelerants (see, e.g., [16, 17, 18]).

Definition B.1. A function $H \in L_1((-1, 1), \mathcal{M}_r)$ is called an *accelerant* if $H(-x) = H(x)^*$ a.e. on $(-1, 1)$ and for every $a \in (0, 1]$, the integral equation

$$f(x) + \int_0^a H(x-t)f(t) dt = 0, \quad x \in (0, a),$$

has only zero solution in $L_2((0, a), \mathbb{C}^r)$.

Throughout this paper, we denote by $\mathfrak{H}_p := \mathfrak{H}_p(\mathcal{M}_{2r})$, $p \in [1, \infty)$, the set of all $2r \times 2r$ matrix-valued accelerants belonging to the space $L_p((-1, 1), \mathcal{M}_{2r})$; we endow $\mathfrak{H}_p$ with the metric of $L_p((-1, 1), \mathcal{M}_{2r})$.

It is known (see, e.g., [1]) that a function $H \in L_p((-1, 1), \mathcal{M}_{2r})$ belongs to $\mathfrak{H}_p$ if and only if the Krein equation

$$(B.1) \quad R(x, t) + H(x-t) + \int_0^x R(x, s)H(s-t) ds = 0, \quad 0 \leq t \leq x \leq 1,$$

is solvable in $G_p^+(\mathcal{M}_{2r})$ (see Appendix A). In this case, a solution of (B.1) is unique and we denote it by $R_H(x, t)$. We then define the Krein mapping $\Theta : \mathfrak{H}_1 \rightarrow \mathfrak{Q}_1$ (see *Notations*) by the formula

$$(B.2) \quad [\Theta(H)](x) := iR_H(x, 0), \quad x \in (0, 1).$$

It is proved in [18] that for every $p \in [1, \infty)$, the Krein mapping acts from $\mathfrak{H}_p$ to $\mathfrak{Q}_p$ and, moreover, appears to be a homeomorphism between $\mathfrak{H}_p$ and $\mathfrak{Q}_p$.

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