SPECTRAL FUNCTIONS OF THE SIMPLEST EVEN ORDER ORDINARY DIFFERENTIAL OPERATOR

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Dedicated with deep respect to Professor Ya. V. Mykytyuk on the occasion of his 60th birthday

ABSTRACT. We consider the minimal differential operator A generated in $L^2(0, \infty)$ by the differential expression $l(y) = (-1)^n y^{(2n)}$. Using the technique of boundary triplets and the corresponding Weyl functions, we find explicit form of the characteristic matrix and the corresponding spectral function for the Friedrichs and Krein extensions of the operator A.

1. INTRODUCTION

Let $\mathcal{P}$ be the minimal symmetric operator, generated in $L^2(0, \infty)$ by a differential expression

$$\sum_{k=0}^{n} (-1)^k \left(p_{n-k}(x) y^{(k)} \right)^{(k)}. \tag{1}$$

Assume that its deficiency indices are $n_\pm(\mathcal{P}) = n$. It is well-known [8, Theorem VI.21.2], [5, Theorem II.9.1] that any its proper self-adjoint extension $\widetilde{\mathcal{P}}$ is unitary equivalent to the multiplication operator Λ_σ in the space $L^2_\sigma(\mathbb{R})$, where $\Lambda_\sigma : f(x) \to xf(x)$, $f \in L^2_\sigma(\mathbb{R})$, and $\sigma(\cdot)$ is a non-decreasing left-continuous self-adjoint $n \times n$ matrix-function. The matrix-function $\sigma(\cdot)$ is called a spectral function of the operator $\widetilde{\mathcal{P}}$ and coincides with the spectral function of the characteristic matrix of $\widetilde{\mathcal{P}}$, which, in turn, can be found by the Green function of the operator $\widetilde{\mathcal{P}}$ (see [8, VI.21.4]).

The purpose of this paper is to find an explicit form of the spectral function for the Friedrichs extension (so-called "hard" extension) A_F and for the Krein extension A_K (see [1, § 109] for precise definitions) of the minimal symmetric operator A generated in $L^2(0, \infty)$ by the differential expression

$$l(y) := (-1)^n y^{(2n)}(\cdot). \tag{2}$$

Explicit form of the spectral function of some selfadjoint extension $\widetilde{A}$ of A plays an important role when the general selfadjoint differential operator is treated as a perturbation of $\widetilde{A}$. It is well-known that the Friedrichs extension A_F of the operator A is defined by the boundary conditions $y(0) = y'(0) = \cdots = y^{(n-1)} = 0$ and we show that the Krein extension A_K is defined by the boundary conditions $y^{(n)}(0) = \cdots = y^{(2n-1)} = 0$.

We will exploit the technique of boundary triplets and the corresponding Weyl functions (see Section 2 for precise definitions) to find the spectral function. This new approach to extension theory of symmetric operators has been appeared and elaborated

during the last three decades (see [4, 2, 3] and references therein). It is well-known [3] that the characteristic matrix of the selfadjoint extension $\widetilde{A}$ of A coincides with the Weyl function of the corresponding boundary triplet. This allows us to find the characteristic matrix and its spectral function easier than by classical method.

Let us formulate the main results of the paper.

Theorem 1. *The characteristic matrix (the Weyl function) of the Friedrichs extension A_F of the operator A is given by*

$$(3) \qquad M_F(\lambda) = \left(\frac{-C_j \cdot C_k}{\sin((j+k+1)\alpha)} \cdot \left(\sqrt[2n]{-\lambda} \right)^{j+k+1} \right)_{j,k=0}^{n-1}, \quad \operatorname{Im}\lambda > 0.$$

where

$$(4) \qquad C_0 := 1, \quad C_k := \prod_{p=1}^{k} \operatorname{ctg}(p\alpha), \quad \alpha = \frac{\pi}{2n}, \quad k \in \{1, \ldots, n-1\},$$

and

$$(5) \qquad \sqrt[2n]{-\lambda} := \sqrt[2n]{r} \cdot e^{\frac{i(\varphi - \pi)}{2n}}, \quad \lambda = re^{i\varphi}, \quad 0 < \varphi < \pi.$$

The corresponding spectral function is

$$(6) \qquad \sigma_F(t) = \frac{2n}{\pi} \left(\frac{C_j \cdot C_k}{2n+1+j+k} \cdot t^{\frac{2n+1+j+k}{2n}} \right)_{j,k=0}^{n-1}, \quad t \geqslant 0,$$

$$(7) \qquad \sigma_F(t) = 0, \quad t < 0.$$

Theorem 2. *The Krein extension A_K of the operator A is defined by the boundary conditions*

$$(8) \qquad y^{(n)}(0) = y^{(n+1)}(0) = \cdots = y^{(2n-1)} = 0.$$

Its characteristic matrix is

$$(9) \qquad M_K(\lambda) = \left(\frac{-C_j \cdot C_k}{\sin((j+k+1)\alpha)} \cdot \left(\frac{-1}{\sqrt[2n]{-\lambda}} \right)^{j+k+1} \right)_{j,k=0}^{n-1}, \quad \operatorname{Im}\lambda > 0.$$

The corresponding spectral function is

$$(10) \qquad \sigma_K(t) = \frac{2n}{\pi} \left((-1)^{j+k} \frac{C_j \cdot C_k}{2n-1-j-k} \cdot t^{\frac{2n-1-j-k}{2n}} \right)_{j,k=0}^{n-1}, \quad t \geqslant 0,$$

$$(11) \qquad \sigma_K(t) = 0, \quad t < 0.$$

2. Preliminaries

2.1. **R-functions.** Let $F(z)$ be an $n \times n$ matrix-function defined in $\mathbb{C}_+ := \{\lambda : \operatorname{Im}\lambda > 0\}$. It is called R-function (or Nevanlinna function) if it is holomorphic in $\mathbb{C}_+$ and $\operatorname{Im} F(z) \geqslant 0$, $z \in \mathbb{C}_+$. Each R-function admits the following integral representation

$$(12) \qquad F(z) = A + zB + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t), \quad z \in \mathbb{C}_+,$$

where $A, B \in \mathbb{C}^{n \times n}$ are selfadjoint matrices, $B \geqslant 0$ and $\sigma(t)$ is a non-decreasing left-continuous selfadjoint $n \times n$ matrix-function such that the matrix integral

$$(13) \qquad \int_{-\infty}^{+\infty} \frac{d\sigma(t)}{1+t^2}$$

converges. The matrix-function $\sigma(\cdot)$ is called the spectral function of $F(\cdot)$. Note that the spectral function $\sigma(\cdot)$ of $F(\cdot)$ can be obtained by the Stieltjes inversion formula,

$$(14) \quad \frac{1}{2}(\sigma(t+0) + \sigma(t)) - \frac{1}{2}(\sigma(s+0) + \sigma(s)) = \frac{1}{\pi} \lim_{y \downarrow 0} \int_s^t \mathrm{Im}(F(x+iy))dx, \quad s, t \in \mathbb{R}.$$

2.2. Boundary triplets and Weyl functions.
Let A be a closed symmetric operator in a Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_+(A) = n_-(A)$.

Definition 3. ([4]). *A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings*

$$(15) \qquad\qquad \Gamma_j : \mathrm{dom}(A^*) \longrightarrow \mathcal{H}, \quad j \in \{0, 1\},$$

is called a boundary triplet for the adjoint operator A^* of A if the following two conditions are satisfied:
(i) The second Green's formula

$$(16) \qquad (A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \mathrm{dom}(A^*),$$

takes place and
(ii) the mapping

$$(17) \qquad\qquad \Gamma : \mathrm{dom}(A^*) \longrightarrow \mathcal{H} \oplus \mathcal{H}, \quad \Gamma f := \{\Gamma_0 f, \Gamma_1 f\},$$

is surjective.

It is easily seen that for each self-adjoint extension $\widetilde{A}$ of A there exists a (non-unique) boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that

$$\mathrm{dom}(\widetilde{A}) = \ker(\Gamma_0).$$

We say in this case that the triplet Π corresponds to $\widetilde{A}$.

Definition 4. ([2, 3]). *Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator A^* and $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. The Weyl function of A corresponding to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is the unique mapping $M(\cdot) : \rho(A_0) \longrightarrow [\mathcal{H}]$ satisfying*

$$(18) \qquad \Gamma_1 f_z = M(z) \Gamma_0 f_z, \quad f_z \in \mathfrak{N}_z := \ker(A^* - zI), \quad z \in \rho(A_0).$$

It is well known (see [2]) that the above implicit definition of the Weyl function is correct and the Weyl function $M(\cdot)$ is an R-function obeying $0 \in \rho(\mathrm{Im}(M(i)))$. Therefore, if $\dim \mathcal{H} < \infty$, it admits the integral representation (12), where $\sigma_M(\cdot)$ can be found by (14).

3. PROOFS OF THE MAIN RESULTS

Lemma 5. *Let $\mathrm{Im}\lambda > 0$ and $\lambda = re^{i\varphi}$, $0 < \varphi < \pi$. Then*

$$(19) \qquad\qquad \mathfrak{N}_\lambda = \mathrm{span}\{y_k(\cdot, \lambda)\}_{k=0}^{n-1}, \quad y_k(x, \lambda) := e^{\omega_k \rho x},$$

where $\rho := i \sqrt[2n]{\lambda} := \sqrt[2n]{r} \cdot e^{\frac{(\pi n + \varphi)i}{2n}}$ and $\omega_k := e^{\frac{i\pi k}{n}}$.

Proof. The system $\{y_k(\cdot, \lambda)\}_{k=0}^{2n-1}$ forms a fundamental system of solutions of the equation $(-1)^n y^{(2n)} = \lambda y$ for $\lambda \neq 0$. For $k \in \{0, 1, \dots, n-1\}$ we have

$$(20) \qquad\qquad \mathrm{Re}(\omega_k \rho) = \sqrt[2n]{r} \cos\left(\frac{\pi}{2} + \frac{\varphi}{2n} + \frac{\pi k}{n}\right) < 0.$$

Hence $y_k(\cdot, \lambda) \in \mathfrak{N}_\lambda$, $k \in \{0, 1, \dots, n-1\}$. Since $\dim \mathfrak{N}_\lambda = n$, we are done. $\square$

Let $x_0, \ldots, x_{n-1} \in \mathbb{C}$. Put

$$(21) \qquad \mathrm{Vand}(x_0, \ldots, x_{n-1}) := (x_k^{n-1-j})_{j,k=0}^{n-1} = \begin{pmatrix} x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ x_0 & x_1 & \cdots & x_{n-1} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

The determinant of this matrix coincides with the Vandermonde determinant:

$$(22) \qquad \det(\mathrm{Vand}(x_0, \ldots, x_{n-1})) = \det\left((x_k^{n-1-j})_{j,k=0}^{n-1}\right) = \prod_{0 \leqslant j < k < n} (x_j - x_k).$$

Next put

$$(23) \qquad \mathrm{codiag}(x_0, x_1, \ldots, x_n) = \mathrm{codiag}(x_j)_{j=0}^{n-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & x_0 \\ 0 & 0 & \cdots & x_1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & x_{n-2} & \cdots & 0 & 0 \\ x_{n-1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It is clear that

$$(24) \qquad \mathrm{codiag}(x_j)_{j=0}^{n-1} \cdot (a_{j,k})_{j,k=0}^{n-1} = (x_j a_{n-1-j,k})_{j,k=0}^{n-1},$$

$$(25) \qquad (a_{j,k})_{j,k=0}^{n-1} \cdot \mathrm{codiag}(x_k)_{k=0}^{n-1} = (a_{j,n-1-k} x_{n-1-k})_{j,k=0}^{n-1}.$$

Proof of Theorem 1. The triplet $\Pi = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ with

$$(26) \qquad \Gamma_0 y := \mathrm{col}(y^{(n-1)}(0), \ldots, y'(0), y(0)),$$

$$(27) \qquad \Gamma_1 y := \mathrm{col}(y^{(n)}(0), -y^{(n+1)}(0), \ldots, (-1)^{n-1} y^{(2n-1)}(0))$$

is a boundary triplet for the adjoint operator A^* (see [2]). Clearly it corresponds to A_F. Hence the characteristic matrix of A_F coincides with the Weyl function $M_F(\lambda)$ of A that corresponds to the triplet Π.

It follows from $y_k^{(j)}(0, \lambda) = (\rho \cdot \omega_k)^j$ that

$$(28) \qquad N_0(\lambda) := \begin{pmatrix} \Gamma_0 y_0 & \cdots & \Gamma_0 y_{n-1} \end{pmatrix} = \left((\rho \cdot \omega_k)^{n-1-j}\right)_{j,k=0}^{n-1},$$

$$(29) \qquad N_1(\lambda) := \begin{pmatrix} \Gamma_1 y_0 & \cdots & \Gamma_1 y_{n-1} \end{pmatrix} = \left((-1)^j (\rho \cdot \omega_k)^{n+j}\right)_{j,k=0}^{n-1}.$$

Put

$$(30) \qquad V := (v_{jk})_{j,k=0}^{n-1} := (\omega_k^{n-1-j})_{j,k=0}^{n-1} = \mathrm{Vand}(\omega_0, \ldots, \omega_{n-1}).$$

Since numbers $\omega_0, \ldots, \omega_{n-1}$ are distinct, it follows from (22) that V is non-singular matrix. Put $V^{-1} =: (\tilde{v}_{jk})_{j,k=0}^{n-1}$. Then by Lemma 5, for the Weyl function $M_F(\lambda)$ we have

$$M_F(\lambda) = N_1(\lambda) \cdot N_0^{-1}(\lambda) = \left((-1)^j (\rho \cdot \omega_p)^{n+j}\right)_{j,p=0}^{n-1} \cdot \left(\rho^{k+1-n} \cdot \tilde{v}_{pk}\right)_{p,k=0}^{n-1}$$

$$(31) \qquad\qquad = \left((-1)^j \rho^{j+k+1} \sum_{p=0}^{n-1} \omega_p^{n+j} \cdot \tilde{v}_{pk}\right)_{j,k=0}^{n-1}.$$

Let V_{jk} be the cofactor of the element v_{jk} of the matrix V. Combining Cramer's rule with the expansion of the determinant according to the k-th row yields

$$(32) \qquad \sum_{p=0}^{n-1} \omega_p^{n+j} \cdot \tilde{v}_{pk} = \frac{1}{\det(V)} \sum_{p=0}^{n-1} \omega_p^{n+j} V_{kp} = \frac{\det\left(V_j^{(k)}\right)}{\det(V)},$$

where the matrix $V_j^{(k)}$ is obtained from the matrix V by replacing the row $(\omega_p^{n-1-k})_{p=0}^{n-1}$ by the row $(\omega_p^{n+j})_{p=0}^{n-1}$. Since $\omega_p^q = e^{\frac{\pi i p q}{n}} = \omega_q^p$, then $V_j^{(k)}$ is symmetric to the matrix $\mathrm{Vand}(\omega_0, \ldots, \omega_{n-k-2}, \omega_{n+j}, \omega_{n-k}, \ldots, \omega_{n-1})$ with respect to the off-diagonal. Hence

$$(33) \qquad \det\left(V_j^{(k)}\right) = \det\left(\mathrm{Vand}(\omega_0, \ldots, \omega_{n-k-2}, \omega_{n+j}, \omega_{n-k}, \ldots, \omega_{n-1})\right).$$

Combining (22) with (33) yields

$$\frac{\det\left(V_j^{(k)}\right)}{\det(V)} = \prod_{\substack{p=0 \\ p \neq n-1-k}}^{n-1} \frac{\omega_{n+j} - \omega_p}{\omega_{n-1-k} - \omega_p}.$$

Since $\omega_q - \omega_p = 2i\varepsilon^{p+q} \sin((q-p)\alpha)$, where $\alpha = \frac{\pi}{2n}$ and $\varepsilon = e^{i\alpha}$, then

$$
\begin{aligned}
(34) \qquad \frac{\det\left(V_j^{(k)}\right)}{\det(V)} &= \varepsilon^{(j+k+1)(n-1)} \cdot \prod_{\substack{p=0 \\ p \neq n-1-k}}^{n-1} \frac{\sin((n+j-p)\alpha)}{\sin((n-1-k-p)\alpha)} \\
&= \frac{\varepsilon^{(j+k+1)(n-1)}}{\sin((j+k+1)\alpha)} \cdot \frac{\prod_{p=0}^{n-1} \cos((j-p)\alpha)}{\prod_{p=1}^{n-1-k} \sin p\alpha \cdot \prod_{p=1}^{k}(-\sin p\alpha)} \\
&= \frac{(-1)^k \varepsilon^{(j+k+1)(n-1)}}{\sin((j+k+1)\alpha)} \cdot \frac{\prod_{p=1}^{j} \cos p\alpha \cdot \prod_{p=1}^{n-1-j} \cos p\alpha}{\prod_{p=1}^{k} \sin p\alpha \cdot \prod_{p=1}^{n-1-k} \sin p\alpha} \\
&= \frac{(-1)^k \varepsilon^{(j+k+1)(n-1)}}{\sin((j+k+1)\alpha)} \cdot \prod_{p=1}^{k} \mathrm{ctg}\, p\alpha \cdot \prod_{p=1}^{j} \mathrm{ctg}\, p\alpha.
\end{aligned}
$$

The last step is implied by the identity

$$(35) \qquad \prod_{p=1}^{j} \cos p\alpha \cdot \prod_{p=1}^{n-1-j} \sin p\alpha = \prod_{p=1}^{n-1} \cos p\alpha = \prod_{p=1}^{n-1} \sin p\alpha, \quad j \in \{0, 1, \ldots, n-1\}.$$

Inserting formulas (32), (34) into (31) and taking into account the identity $-\varepsilon^{n-1}\rho = \sqrt[2n]{r} \cdot e^{\frac{i(\varphi-\pi)}{2n}}$ we get the desired formula (3) for $M_F(\lambda)$.

Now let's prove formulas (6)–(7). Since $M_F(\lambda)$ is continuous function of λ in the closed upper half-plane, Stieltjes inversion formula (14) and Lebesgue limit theorem yields

$$(36) \qquad \sigma_F(t) = \frac{1}{\pi} \int_0^t \mathrm{Im}\left(\lim_{y \downarrow 0} M_F(x+iy)\right) dx, \quad t \in \mathbb{R}.$$

Note that if $\lambda = x + iy$ with $x \in \mathbb{R}$, $y > 0$, then (5) implies

$$(37) \qquad \lim_{y \downarrow 0} \sqrt[2n]{-\lambda} = \begin{cases} \sqrt[2n]{x} \cdot e^{-i\alpha}, & x \geqslant 0, \\ \sqrt[2n]{-x}, & x < 0. \end{cases}$$

Combining (3) with (37) yields

$$(38) \qquad \lim_{y \downarrow 0} M_F(x+iy) = \begin{cases} \left(-C_j \cdot C_k \cdot x^{\frac{j+k+1}{2n}} \cdot \frac{e^{-i(j+k+1)\alpha}}{\sin((j+k+1)\alpha)}\right)_{j,k=0}^{n-1}, & x \geqslant 0, \\ \left(-C_j \cdot C_k \cdot (-x)^{\frac{j+k+1}{2n}} \cdot \frac{1}{\sin((j+k+1)\alpha)}\right)_{j,k=0}^{n-1}, & x < 0. \end{cases}$$

Hence

$$(39) \qquad \mathrm{Im}\left(\lim_{y \downarrow 0} M_F(x+iy)\right) = \begin{cases} \left(C_j \cdot C_k \cdot x^{\frac{j+k+1}{2n}}\right)_{j,k=0}^{n-1}, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

Combining (36) with (39) yields (6)–(7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 6. *Calculation similar to* (31)–(34) *was made in the proof of Theorem 1 and Corollary 1 in* [7] *in connection with sharp constants in inequalities for intermediate derivatives. Moreover, it is curious to note that these constants are connected with diagonal entries of the Weyl functions* $M_F(\lambda)$ *and* $M_K(\lambda)$. *Namely, if* $A_{n,j}$, $j \in \{0, 1, \ldots, n-1\}$, *is the sharp constant in the following inequality*

$$(40) \qquad\qquad |f^j(0)| \leqslant A_{n,j} \cdot \left(\|f\|_2^2 + \|f^{(n)}\|_2^2\right)^{1/2}, \quad f \in W^{n,2}[0, \infty),$$

then formula (1.4) from [7] *and formulas* (38), (58) *imply*

$$(41) \qquad\qquad A_{n,j}^2 = [M_K(-1)]_{jj} = -[M_F(-1)]_{jj}.$$

Remark 7. *Formula* (3) *could be also proved using explicit formula for the inverse matrix* V^{-1} *from* [6] *and some auxiliary trigonometric identity from* [6]. *But this way is quite cumbersome.*

Example 8. *For* $n = 1$ *the Weyl function* $M_F(\lambda)$ *and its spectral function* $\sigma_F(t)$ *are well-known (see* [1, § 132], [8]) *and given by*

$$(42) \qquad\qquad M_F(\lambda) = i\sqrt{\lambda}, \quad \sigma_F(t) = \frac{2}{3\pi}t^{3/2}, \quad t > 0,$$

which coincides with formulas (3), (6) *for* $n = 1$. *For* $n = 2$ *these formulas turn into*

$$(43) \quad M_F(\lambda) = \begin{pmatrix} (i-1)\lambda^{1/4} & i\lambda^{1/2} \\ i\lambda^{1/2} & (i+1)\lambda^{3/4} \end{pmatrix}, \quad \sigma_F(t) = \frac{1}{\pi}\begin{pmatrix} \frac{4}{5}t^{5/4} & \frac{2}{3}t^{3/2} \\ \frac{2}{3}t^{3/2} & \frac{4}{7}t^{7/4} \end{pmatrix}, \quad t > 0,$$

while for $n = 3$ *we have*

$$(44) \quad M_F(\lambda) = \begin{pmatrix} \left(i-\sqrt{3}\right)\lambda^{1/6} & \left(-1+i\sqrt{3}\right)\lambda^{1/3} & i\lambda^{1/2} \\ \left(-1+i\sqrt{3}\right)\lambda^{1/3} & 3i\lambda^{1/2} & \left(1+i\sqrt{3}\right)\lambda^{2/3} \\ i\lambda^{1/2} & \left(1+i\sqrt{3}\right)\lambda^{2/3} & \left(i+\sqrt{3}\right)\lambda^{5/6} \end{pmatrix},$$

$$(45) \quad \sigma_F(t) = \frac{1}{\pi}\begin{pmatrix} \frac{6}{7}t^{7/6} & \frac{3\sqrt{3}}{4}t^{4/3} & \frac{2}{3}t^{3/2} \\ \frac{3\sqrt{3}}{4}t^{4/3} & 2t^{3/2} & \frac{3\sqrt{3}}{5}t^{5/3} \\ \frac{2}{3}t^{3/2} & \frac{3\sqrt{3}}{5}t^{5/3} & \frac{6}{11}t^{11/6} \end{pmatrix}, \quad t > 0.$$

Proof of Theorem 2. By [2, Proposition 5], $\mathrm{dom}(A_K) = \ker(\Gamma_1 - M_F(0)\Gamma_0)$, where $M_F(0) = s\text{-}\lim_{x\uparrow 0} M_F(x)$ and Γ_0, Γ_1 are given by (26)–(27). In view of (3), $M_F(0) = 0$. Hence $\mathrm{dom}(A_K) = \ker(\Gamma_1)$ and the boundary triplet $\Pi' := \{\mathbb{C}^n, \Gamma_0', \Gamma_1'\} := \{\mathbb{C}^n, \Gamma_1, -\Gamma_0\}$ corresponds to A_K. Definition of Γ_1 (see (27)) implies that A_K is defined by the boundary conditions (8). Also note that

$$(46) \qquad\qquad M_K(\lambda) = -N_0(\lambda)N_1^{-1}(\lambda) = -M_F^{-1}(\lambda).$$

It follows from (28) and (29) that

$$(47) \qquad\qquad N_0(\lambda) = D_0(\lambda)V, \quad N_1(\lambda) = D_1(\lambda) \cdot (\omega_k^j)_{j,k=0}^{n-1} \cdot D,$$

where

$$(48) \qquad D_0(\lambda) := \mathrm{diag}(\rho^{n-1-j})_{j=0}^{n-1}, \quad D_1(\lambda) := \mathrm{diag}((-1)^j \rho^{n+j})_{j=0}^{n-1},$$

$$(49) \qquad\qquad D := \mathrm{diag}(\omega_k^n)_{k=0}^{n} = \mathrm{diag}((-1)^k)_{k=0}^{n}.$$

Combining (24) with (30) yields

$$(\omega_k^j)_{j,k=0}^{n-1} = R \cdot V, \quad R = \mathrm{codiag}(1, \ldots, 1).$$

Therefore,

$$(50) \qquad\qquad N_1(\lambda) = D_1(\lambda) \cdot R \cdot V \cdot D.$$

Combining (31) with (47) and (50) and taking into account that $D = D^{-1}$ and $R = R^{-1}$ we get

$$(51) \qquad M_F(\lambda) = \ N_1(\lambda)N_0^{-1}(\lambda) \ = D_1(\lambda)R \cdot VDV^{-1} \cdot D_0^{-1}(\lambda),$$

$$(52) \qquad M_K(\lambda) = \ \ -M_F^{-1}(\lambda) \ \ = -D_0(\lambda) \cdot VDV^{-1} \cdot RD_1^{-1}(\lambda).$$

Expressing VDV^{-1} from (51) and inserting it to (52) we arrive at

$$(53) \qquad M_K(\lambda) = -D_0(\lambda)RD_1^{-1}(\lambda) \cdot M_F(\lambda) \cdot D_0(\lambda)RD_1^{-1}(\lambda).$$

Definition of $D_0(\lambda)$ and $D_1(\lambda)$ (see (48)) and formulas (24)–(25) implies

$$D_0(\lambda)RD_1^{-1}(\lambda) \ = \ \operatorname{codiag}(\rho^{n-1-j})_{j=0}^{n-1} \cdot \operatorname{diag}((-1)^j \rho^{-n-j})_{j=0}^{n-1}$$

$$(54) \qquad\qquad\qquad\qquad = \ \rho^{-n} \operatorname{codiag}((-1)^{n-1-j})_{j=0}^{n-1}.$$

Combining (53), (54), (24), (25) and (3) yields

$$(55)
\begin{aligned}
M_K(\lambda) &= \left(\rho^{-2n}(-1)^{n-1-j+k} \frac{C_{n-1-j} \cdot C_{n-1-k}}{\sin((2n-1-j-k)\alpha)} \left(\sqrt[2n]{-\lambda} \right)^{2n-1-j-k} \right)_{j,k=0}^{n-1} \\
&= \left(-\lambda \cdot \frac{i^{-2n}}{\lambda} \cdot (-1)^n \cdot \frac{C_{n-1-j} \cdot C_{n-1-k}}{\sin((j+k+1)\alpha)} \left(\frac{-1}{\sqrt[2n]{-\lambda}} \right)^{j+k+1} \right)_{j,k=0}^{n-1}.
\end{aligned}$$

It follows from (35) that $C_j = C_{n-1-j}$, $j \in \{0, 1, \ldots, n-1\}$. In view of this, (55) implies the desired formula (9) for $M_K(\lambda)$.

Now let's prove formulas (10)–(11). It follows from (9) that for $j, k \in \{0, 1, \ldots, n-1\}$

$$(56) \qquad \left| [M_K(x+iy)]_{jk} \right| \leqslant C \left(|x|^{-1+\frac{1}{2n}} + |x|^{-\frac{1}{2n}} \right), \quad x \in \mathbb{R} \setminus \{0\}, \quad y > 0,$$

for some $C > 0$. Hence Stieltjes inversion formula (14) and Lebesgue limit theorem yields

$$(57) \qquad \sigma_K(t) = \frac{1}{\pi} \int_0^t \operatorname{Im} \left(\lim_{y \downarrow 0} M_K(x+iy) \right) dx, \quad t \in \mathbb{R}.$$

Combining (9) with (37) we arrive at

$$(58) \ \lim_{y \downarrow 0} M_K(x+iy) = \begin{cases} \left((-1)^{j+k} \cdot C_j \cdot C_k \cdot x^{-\frac{j+k+1}{2n}} \cdot \frac{e^{i(j+k+1)\alpha}}{\sin((j+k+1)\alpha)} \right)_{j,k=0}^{n-1}, & x > 0, \\ \left((-1)^{j+k} \cdot C_j \cdot C_k \cdot (-x)^{-\frac{j+k+1}{2n}} \cdot \frac{1}{\sin((j+k+1)\alpha)} \right)_{j,k=0}^{n-1}, & x < 0. \end{cases}$$

Hence

$$(59) \qquad \operatorname{Im} \left(\lim_{y \downarrow 0} M_K(x+iy) \right) = \begin{cases} \left((-1)^{j+k} C_j \cdot C_k \cdot x^{-\frac{j+k+1}{2n}} \right)_{j,k=0}^{n-1}, & x > 0, \\ 0, & x < 0. \end{cases}$$

Combining (57) with (59) yields (10)–(11). $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 9. *Formulas (3), (9) and (46) lead to the following curious identity*

$$(60) \qquad \sum_{p=0}^{n-1} \frac{(-1)^{p+k} \cdot C_j \cdot C_p^2 \cdot C_k}{\sin((j+p+1)\alpha) \sin((p+k+1)\alpha)} = \delta_{jk}, \quad j, k \in \{0, 1, \ldots, n-1\}.$$

It seems non-trivial to prove it directly.

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