Methods of Functional Analysis and Topology
Vol. 19 (2013), no. 4, pp. 310–318

ON THE EXTREMAL EXTENSIONS OF A NON-NEGATIVE JACOBI OPERATOR

ALEKSANDRA ANANIEVA AND NATALY GOLOSHCHAPOVA

Dedicated with deep respect to Professor Ya. V. Mykytyuk on the occasion of his 60th birthday

Abstract. We consider the minimal non-negative Jacobi operator with $p\times p-$matrix entries. Using the technique of boundary triplets and the corresponding Weyl functions, we describe the Friedrichs and Krein extensions of the minimal Jacobi operator. Moreover, we parametrize the set of all non-negative extensions in terms of boundary conditions.

1. Introduction

Let A be a densely defined non-negative symmetric operator in the Hilbert space $\mathfrak{H}$. Since A is non-negative, by Friedrichs-Krein theorem, it admits non-negative self-adjoint extensions. A description of all non-negative self-adjoint extensions of A and also a criterion of uniqueness of a non-negative self-adjoint extension of A were first given by Krein in [16]. His results were generalized in numerous papers (see [3, 9, 11] and references therein).

Among all non-negative self-adjoint extensions of A, two (extremal) extensions are particularly interesting and important enough to have a name. The Friedrichs extension (the so-called "hard" extension) A_F is the "greatest" one in the sense of quadratic forms. It is given by the restriction of A^* to the domain

$$\mathrm{dom}\,(A_F)=\left\{\begin{array}{c} u\in\mathrm{dom}\,(A^*):\exists\ u_k\in\mathrm{dom}\,(A)\text{ such that }\|u-u_k\|_{\mathfrak{H}}\to 0\\ \text{as }k\to\infty\text{ and }\ (A(u_j-u_k),u_j-u_k)_{\mathfrak{H}}\to 0\text{ as }j,k\to\infty\end{array}\right\}.$$

In other words, A_F is a self-adjoint operator associated with the closure of the symmetric form

$$\mathfrak{t}[u,v]=(Au,v)_{\mathfrak{H}},\quad u,v\in\mathrm{dom}\,(A).$$

The Krein extension ("soft" extension) A_K is defined to be the restriction of A^* to the domain

$$\mathrm{dom}\,(A_K)=\left\{\begin{array}{c} u\in\mathrm{dom}\,(A^*):\exists\ u_k\in\mathrm{dom}\,(A)\text{ such that }\|A^*u-Au_k\|_{\mathfrak{H}}\to 0\\ \text{as }k\to\infty\text{ and }\ (u_j-u_k,A(u_j-u_k))_{\mathfrak{H}}\to 0\text{ as }j,k\to\infty\end{array}\right\}. \tag{1}$$

If A is positive definite, $A\geq\varepsilon I>0$, then (1) takes the form

$$\mathrm{dom}\,(A_K)=\mathrm{dom}\,(\overline{A})\dot{+}\ker(A^*). \tag{2}$$

Krein proved in [16] that all non-negative self-adjoint extensions $\widetilde{A}$ of A lie between A_F and A_K, i.e.,

$$((A_F+aI)^{-1}u,u)_{\mathfrak{H}}\leq((\widetilde{A}+aI)^{-1}u,u)_{\mathfrak{H}}\leq((A_K+aI)^{-1}u,u)_{\mathfrak{H}},\quad u\in\mathfrak{H},$$

2000 *Mathematics Subject Classification.* 47B36; 47B36.

Key words and phrases. Jacobi matrix, non-negative operator, Friedrichs and Krein extensions, boundary triplet, Weyl function.

The authors express their gratitude to Prof. M. Malamud for posing the problem and permanent attention to the work and to Prof. L. Oridoroga for useful discussions. The second author was supported by FAPESP grant 2012/50503-6.

for any $a > 0$.

In the present paper we are dealing with the problem of description of extremal extensions of a non-negative Jacobi operator to be defined below. Let $A_j, B_j \in \mathbb{C}^{p\times p}$. Moreover, we assume the that matrices A_j are self-adjoint and the matrices B_j are invertible for each $j \geq 0$ (see [5, Chapter VII, § 2]). We consider semi-infinite Jacobi matrix with matrix entries

$$\mathbf{J} = \begin{pmatrix} A_0 & B_0 & O_p & \dots \\ B_0^* & A_1 & B_1 & \dots \\ O_p & B_1^* & A_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where O_p is the zero $p \times p$ matrix. Given a sequence $u = (u_j),\ u_j \in \mathbb{C}^p$, $\mathbf{J}u$ is again a sequence of column vectors. If we set $B_{-1} = O_p$,

$$(\mathbf{J}u)_j = B_j u_{j+1} + A_j u_j + B_{j-1}^* u_{j-1}, \quad j \geq 0.$$

The maximal operator $T_{\max}$ is defined by

$$(T_{\max}u)_j = (\mathbf{J}u)_j, \quad j \geq 0 \tag{3}$$

on the domain

$$\operatorname{dom}(T_{\max}) = \{u \in l_p^2 :\ \mathbf{J}u \in l_p^2\}.$$

The minimal operator $T_{\min}$ is the closure in l_p^2 of the preminimal operator T which is the restriction of $T_{\max}$ to the domain

$$\operatorname{dom}(T) = \{u \in l_p^2 :\ u_j = 0 \text{ for all but a finite number of values of } j\}. \tag{4}$$

It is straightforward to see that $T_{\min}$ is a densely defined symmetric operator and

$$T_{\min}^* = T_{\max}, \quad T_{\max}^* = \overline{T} = T_{\min}.$$

Deficiency indices $n_\pm(T_{\min}) = \dim(\ker(T_{\max} \pm zI)),\ z \in \mathbb{C}_+$, satisfy the inequality $0 \leq n_+(T_{\min}), n_-(T_{\min}) \leq p$ (see [5, Chapter VII, § 2]). In the following, we shall assume that $n_\pm(T_{\min}) = p$ (completely indefinite case takes place) and T is non-negative. Note that non-negativity of T implies non-negativity of $A_j,\ j \geq 0$.

Examples of symmetric block Jacobi matrices generating symmetric operators with arbitrary possible values of the deficiency numbers were constructed in [13].

The problem of description of the extremal extensions T_F and T_K in the scalar case ($p = 1$) was studied in a number of papers. Description of the Friedrichs domain in terms of a weighted Dirichlet sum was obtained in [4].

In [20], Simon showed that certain matrix operators that approximate the Friedrichs and Krein extensions converge in the strong resolvent norm. In [7] Brown and Christiansen (assuming that T is *positive definite*) obtained a description of T_F and T_K using the concept of the so-called minimal solution (see also [18]).

The purpose of this work is to generalize at least partially the results obtained in [7] to the case of arbitrary $p \in \mathbb{N}$ and a *non-negative* operator T. We use an abstract description of extremal non-negative extensions obtained by V. Derkach and M. Malamud in the framework of boundary triplets and the corresponding Weyl functions approach (see [9, 14] and also Section 2 for precise definitions). In particular, we show that the mentioned results from [7] might be expressed in terms of boundary triplets theory.

Notation. In what follows $\mathbb{C}^{p\times p}$ denotes the set of $p \times p$ complex-valued matrices; l_p^2 denotes the Hilbert space of infinite sequences $u = (u_j),\ u_j \in \mathbb{C}^p$, equipped with the inner product $(u, v)_{l_p^2} = \sum_{j=0}^{\infty} v_j^* u_j$. The set of closed (bounded) operators in the Hilbert space $\mathcal{H}$ is denoted by $\mathcal{C}(\mathcal{H})$ (respectively, $\mathcal{B}(\mathcal{H})$).

2. Linear relations, boundary triplets and Weyl functions

Let A be a closed densely defined symmetric operator in a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_{\pm}(A) = \dim\ker(A^* \pm iI) \leq \infty$.

Definition 1. ([14]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a *boundary triplet* for the adjoint operator A^* of A if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 :\ \mathrm{dom}\,(A^*) \to \mathcal{H}$ are linear mappings such that

(i) the second Green identity,

$$(A^*f, g)_{\mathfrak{H}} - (f, A^*g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}},$$

holds for all $f, g \in \mathrm{dom}\,(A^*)$, and

(ii) the mapping $\Gamma := (\Gamma_0, \Gamma_1)^{\top} : \mathrm{dom}\,(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

With each boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ one associates two self-adjoint extensions, $A_j := A^* \upharpoonright \ker(\Gamma_j),\ j \in \{0, 1\}$.

Definition 2.

(i) A *closed linear relation* Θ in $\mathcal{H}$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. The *domain*, the *range*, and the *multivalued part* of Θ are defined as follows

$$\mathrm{dom}\,(\Theta) := \big\{f : \{f, f'\} \in \Theta\big\}, \quad \mathrm{ran}\,(\Theta) := \big\{f' : \{f, f'\} \in \Theta\big\},$$
$$\mathrm{mul}\,(\Theta) := \big\{f' : \{0, f'\} \in \Theta\big\}.$$

(ii) A linear relation Θ is *symmetric* if

$$(f', h)_{\mathcal{H}} - (f, h')_{\mathcal{H}} = 0 \quad \text{for all} \quad \{f, f'\}, \{h, h'\} \in \Theta.$$

(iii) The *adjoint* relation Θ^* is defined by

$$\Theta^* = \big\{\{h, h'\} : (f', h)_{\mathcal{H}} = (f, h')_{\mathcal{H}} \ \text{ for all } \ \{f, f'\} \in \Theta\big\}.$$

(iv) A closed linear relation Θ is called *self-adjoint* if both Θ and Θ^* are maximal symmetric, i.e., they do not admit symmetric extensions.

For the symmetric relation $\Theta \subseteq \Theta^*$ in $\mathcal{H}$ the multivalued part $\mathrm{mul}\,(\Theta)$ is orthogonal to $\mathrm{dom}\,(\Theta)$ in $\mathcal{H}$. Setting $\mathcal{H}_{\mathrm{op}} := \overline{\mathrm{dom}\,(\Theta)}$ and $\mathcal{H}_{\infty} := \mathrm{mul}\,(\Theta)$, one verifies that Θ can be rewritten as a direct orthogonal sum of a self-adjoint operator Θ_{op} (*operator part* of Θ) in the subspace $\mathcal{H}_{\mathrm{op}}$ and a "pure" relation $\Theta_{\infty} = \big\{\{0, f'\} : f' \in \mathrm{mul}\,(\Theta)\big\}$ in the subspace $\mathcal{H}_{\infty}$.

Proposition 1. ([9, 14]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping

$$\mathrm{Ext}_A \ni \widetilde{A} := A_{\Theta} \to \Theta := \Gamma(\mathrm{dom}\,(\widetilde{A})) = \big\{\{\Gamma_0 f, \Gamma_1 f\} :\ f \in \mathrm{dom}\,(\widetilde{A})\big\} \tag{5}$$

establishes a bijective correspondence between the set of all closed proper extensions Ext_A of A and the set of all closed linear relations $\widetilde{\mathcal{C}}(\mathcal{H})$ in $\mathcal{H}$. Furthermore, the following assertions hold.

(i) The equality $(A_{\Theta})^* = A_{\Theta^*}$ holds for any $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$.

(ii) The extension A_{Θ} in (5) is symmetric (self-adjoint) if and only if Θ is symmetric (self-adjoint).

(iii) If, in addition, the extensions A_{Θ} and A_0 are disjoint, i.e., $\mathrm{dom}\,(A_{\Theta}) \cap \mathrm{dom}\,(A_0) = \mathrm{dom}\,(A)$, then (5) takes the form

$$A_{\Theta} = A_B = A^* \upharpoonright \ker\big(\Gamma_1 - B\Gamma_0\big), \quad B \in \mathcal{C}(\mathcal{H}).$$

Remark 1. In the case where $n_{\pm}(A) = n < \infty$, any proper extension A_{Θ} of the operator A admits the representation (see [12])

$$A_{\Theta} := A_{C,D} = A^* \upharpoonright \ker\big(D\Gamma_1 - C\Gamma_0\big), \quad C, D \in \mathcal{B}(\mathcal{H}). \tag{6}$$

Moreover, according to the Rofe-Beketov theorem [19] (see also [2, Theorem 125.4]), $A_{C,D}$ is self-adjoint if and only if C, D, satisfy the following conditions

$$CD^* = DC^* \quad \text{and} \quad 0 \in \rho(CC^* + DD^*). \tag{7}$$

Definition 3. ([9]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator-valued function $M(\cdot) : \rho(A_0) \to \mathcal{B}(\mathcal{H})$ defined by

$$\Gamma_1 f_z = M(z)\Gamma_0 f_z, \quad \text{for all} \quad f_z \in \ker(A^* - zI), \quad z \in \rho(A_0),$$

is called *the Weyl function*, corresponding to the triplet Π.

Proposition 2. ([9, 11]). Let A be a densely defined nonnegative symmetric operator with finite deficiency indices in $\mathfrak{H}$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Let also $M(\cdot)$ be the corresponding Weyl function. Then the following assertions hold.

(i) There exist strong resolvent limits

$$M(0) := s - R - \lim_{x \uparrow 0} M(x), \quad M(-\infty) := s - R - \lim_{x \downarrow -\infty} M(x). \tag{8}$$

(ii) $M(0)$ and $M(-\infty)$ are self-adjoint linear relations in $\mathcal{H}$ associated with the semi-bounded below (above) quadratic forms

$$\mathrm{t}_0[f] = \lim_{x \uparrow 0}(M(x)f, f) \geq \beta ||f||^2, \quad \mathrm{t}_{-\infty}[f] = \lim_{x \downarrow -\infty}(M(x)f, f) \leq \alpha ||f||^2,$$

and

$$\mathrm{dom}\,(\mathrm{t}_0) = \left\{ f \in \mathcal{H} : \lim_{x \uparrow 0} |(M(x)f, f)| < \infty \right\} = \mathrm{dom}\,((M(0)_{\mathrm{op}} - \beta)^{1/2}),$$

$$\mathrm{dom}\,(\mathrm{t}_{-\infty}) = \left\{ f \in \mathcal{H} : \lim_{x \downarrow -\infty} |(M(x)f, f)| < \infty \right\} = \mathrm{dom}\,((\alpha - M(-\infty)_{\mathrm{op}})^{1/2}).$$

Moreover,

$$\mathrm{dom}\,(A_K) = \{ f \in \mathrm{dom}\,(A^*) : \ \{\Gamma_0 f, \Gamma_1 f\} \in M(0)\},$$
$$\mathrm{dom}\,(A_F) = \{ f \in \mathrm{dom}\,(A^*) : \ \{\Gamma_0 f, \Gamma_1 f\} \in M(-\infty)\}.$$

(iii) The extensions A_0 and A_K are disjoint (A_0 and A_F are disjoint) if and only if

$$M(0) \in \mathcal{C}(\mathcal{H}) \quad (M(-\infty) \in \mathcal{C}(\mathcal{H}), \quad \text{respectively}).$$

Moreover,

$$\mathrm{dom}\,(A_K) = \mathrm{dom}\,(A^*) \upharpoonright \ker(\Gamma_1 - M(0)\Gamma_0)$$
$$(\mathrm{dom}\,(A_F) = \mathrm{dom}\,(A^*) \upharpoonright \ker(\Gamma_1 - M(-\infty)\Gamma_0), \quad \text{respectively}).$$

(iv) $A_0 = A_K$ ($A_0 = A_F$) if and only if

$$\lim_{x \uparrow 0}(M(x)f, f) = +\infty, \quad f \in \mathcal{H} \setminus \{0\}$$
$$(\lim_{x \downarrow -\infty}(M(x)f, f) = -\infty, \quad f \in \mathcal{H} \setminus \{0\}, \quad \text{respectively}).$$

(v) If, in addition, $A_0 = A_F$ ($A_0 = A_K$), then the set of all non-negative self-adjoint extensions of A admits the parametrization (5), where Θ satisfies

$$\Theta - M(0) \geq 0 \quad (\Theta - M(-\infty) \leq 0, \quad \text{respectively}). \tag{9}$$

Moreover, if (9) does not hold, the number of negative eigenvalues of an arbitrary self-adjoint extension $\kappa_-(A_\Theta)$ is given by

$$\kappa_-(A_\Theta) = \kappa_-(\Theta - M(0)) \quad (\kappa_-(A_\Theta) = \kappa_-(M(-\infty) - \Theta), \quad \text{respectively}).$$

Remark 2. We should mention that the existence of the limits in (8) follows from finiteness of deficiency indices of A.

Remark 3. Note that if the lover bound of A is zero and the spectrum of A_F is purely discrete, then A_F and A_K are not disjoint. In this case $M(0)$ is a linear relation if $A_0 \geq 0$.

Corollary 1. *Let the assumptions of Proposition 2 hold and $A_0 = A_K$. Let also $A_{C,D}$ be a self-adjoint extension of A defined by* (6)*. Then $A_{C,D}$ is non-negative if and only if*

$$CD^* - DM(-\infty)D^* \leq 0. \tag{10}$$

Moreover, if (10) *does not hold, the number of negative eigenvalues of $A_{C,D}$ (counting multiplicities) coincides with the number of positive eigenvalues of the linear relation $CD^* - DM(-\infty)D^*$, i.e.,*

$$\kappa_-(A_{C,D}) = \kappa_+(CD^* - DM(-\infty)D^*).$$

Corollary 2. ([11]). *Suppose that A_F has purely discrete spectrum. Then the Krein extension A_K is given by*

$$\mathrm{dom}\,(A_K) = \mathrm{dom}\,(\overline{A}) \dotplus \ker(A^*).$$

Moreover, the spectrum of $A_K \upharpoonright \ker(A^*)^\perp$ is purely discrete.

3. Extremal extensions of $T_{\min}$

As usual, we denote by $(P_j(z))$ the solution to the matrix equation

$$(\mathbf{J}U)_j = zU_j, \quad j \geq 0,$$

with the initial conditions $P_0(z) = I_p$, $\quad P_1(z) = B_0^{-1}(zI_p - A_0)$. Here $I_p \in \mathbb{C}^{p\times p}$ is the identity matrix. Furthermore, we denote by $(Q_j(z))$ the solution to

$$(\mathbf{J}U)_j = zU_j, \quad j \geq 1$$

with $Q_0(z) = O_p$ and $Q_1(z) = B_0^{-1}$. The matrix functions $P_j(z)$ and $Q_j(z)$ are polynomials in the complex variable z of degree j and $j-1$, respectively, with matrix coefficients. We mention that $(P_j(z))$ and $(Q_j(z))$ are called matrix polynomials of the first and second kind, respectively.

Following [10], we define a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $T_{\max}$ by setting

$$\mathcal{H} = \mathbb{C}^p, \quad \Gamma_1 u = (Q(0))^* T_{\max} u - P_0 u, \quad \Gamma_0 u = (P(0))^* T_{\max} u, \tag{11}$$

where $u \in \mathrm{dom}\,(T_{\max})$ and P_0 is orthogonal projection in $\bigoplus_{j=0}^\infty \mathcal{H}_j$ onto $\mathcal{H}_0$, in which $\mathcal{H}_j = \mathbb{C}^p$.

We should note that the mappings $(P(0))^*$ and $(Q(0))^*$ act as infinite "$p\times\infty$" matrices, i.e., $(P(0))^*, (Q(0))^* : l_p^2 \to \mathbb{C}^p$. Each their row is "constructed" by the corresponding rows of $P_j^*(0)$ and $Q_j^*(0)$, respectively.

It is easily seen that

$$\Gamma_1(P_j(z)) = z\sum_{j=1}^\infty Q_j^*(0)P_j(z) - I_p, \quad \Gamma_0(P_j(z)) = z\sum_{j=0}^\infty P_j^*(0)P_j(z),$$

and, by Definition 3, we get

$$\begin{aligned} M(z) &= \Gamma_1(P_j(z))(\Gamma_0(P_j(z)))^{-1} \\ &= \left(z\sum_{j=1}^\infty Q_j^*(0)P_j(z) - I_p\right)\cdot\left(z\sum_{j=0}^\infty P_j^*(0)P_j(z)\right)^{-1}. \end{aligned}$$

Applying Proposition 2 to the operator $T_{\min}$, we obtain the following result.

Theorem 1. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $T_{\max}$ given by* (11) *and let $M(\cdot)$ be the corresponding Weyl function. Then the following assertions hold.*

(i) The Krein extension T_K coincides with T_0, i.e.,

$$\mathrm{dom}\,(T_K) = \mathrm{dom}\,(T_{\max}) \upharpoonright \ker(\Gamma_0) = \{u \in T_{\max} : (P(0))^* T_{\max} u = 0\}.$$

(ii) *The self-adjoint extension T_Θ is non-negative if and only if it admits the representation*

$$T_\Theta = T_{C,D} = T_{\max} \upharpoonright \ker(D\Gamma_1 - C\Gamma_0),$$

where C, D satisfy (7) *and* (10).

(iii) *The number of negative eigenvalues of $T_{C,D} = T^*_{C,D}$ (defined by* (6)*) coincides with the number of positive eigenvalues of the linear relation $CD^* - DM(-\infty)D^*$, i.e.,*

$$\kappa_-(T_{C,D}) = \kappa_+(CD^* - DM(-\infty)D^*).$$

Proof. The statement can be proved in at least two different ways.

1. Since $M(x)$ is holomorphic and increasing in $(-\varepsilon, 0)$ (see [11]), the strong limit $s - \lim_{x\uparrow 0}(M(x)+\gamma)^{-1}$ exists for any $\gamma > 0$. Namely,

$$\begin{aligned} M_\gamma(0) :=& s - \lim_{x\uparrow 0}\big(M(x)+\gamma\big)^{-1} = s - \lim_{x\uparrow 0}\left(x\sum_{j=0}^{\infty} P_j^*(0)P_j(x)\right) \\ &\times\left(x\sum_{j=1}^{\infty} Q_j^*(0)P_j(x) - I_p + \gamma x\sum_{j=0}^{\infty} P_j^*(0)P_j(x)\right)^{-1} = O_p. \end{aligned} \tag{12}$$

Indeed, since $n_\pm(T_{\min}) = p$ (see [15]), the series $\sum_{j=0}^{\infty}||P_j(z)||^2$ and $\sum_{j=0}^{\infty}||Q_j(z)||^2$ converge uniformly on each bounded subset $\mathbb{C}$ (see, for instance, [15, Theorem 1]). Therefore, we can pass to the limit in (12) under the sum sign as $x \to 0$. Hence, $M_\gamma^{-1}(0) = \{0, \mathcal{H}\} = \{\{0, f\} : f \in \mathcal{H}\}$. Since

$$\operatorname{dom}(T_0) = \{u \in \operatorname{dom}(T_{\max}) : \Gamma_0 u = 0\} = \{u \in \operatorname{dom}(T_{\max}) : \{\Gamma_0 u, \Gamma_1 u\} \in \{0, \mathcal{H}\}\},$$

by Proposition 2 *(ii)*, we arrive at $T_K = T_0$.

2. Suppose, in addition, that $T \geq \varepsilon I > 0$. Using the equality $\operatorname{dom}(T_0) = \operatorname{dom}(T_{\max}) \upharpoonright \ker(\Gamma_0)$, we easily get from (11) that $\ker(T_{\max}) \subset \operatorname{dom}(T_0)$. Therefore, (2) implies the inclusion $\operatorname{dom}(T_0) \supset \operatorname{dom}(T_K)$. Since T_0 and T_K are self-adjoint, we arrive at $T_K = T_0$.

(ii) and *(iii)* easily follow from Corollary 1 and assertion *(i)*. □

Assume now that $p = 1$ and A_j, B_j are positive real numbers. It is known that $T_{\min}$ is connected with some Stieltjes moment problem, see [1]. Briefly, a Stieltjes moment problem has a following description. Given a sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$ of reals. When is there a measure, $d\mu$ on $[0, \infty)$ so that

$$\gamma_n = \int_0^\infty x^n d\mu(x)$$

and if such a μ exists, is it unique?

The operator $T_{\min}$ is self-adjoint if and only if associated Stieltjes moment problem is determinate, i.e., it has a unique solution. Since $n_\pm(T_{\min}) = 1$, the determinacy does not take place and, therefore, the sequence $\frac{Q_j(0)}{P_j(0)}$ converges (see [1, Theorem 0.4, p. 293] or [6, Section 3]).

The existence of this limit is a key fact for the description of the Friedrichs extension which we are going to present. In particular, the limit

$$\alpha := \lim_{j\to\infty} \frac{Q_j(0)}{P_j(0)} \tag{13}$$

is negative. Indeed, since all zeros of the polynomials $P_j(x)$ and $Q_j(x)$ lie in the interval $[0, \infty)$ (see, [8, Chapter I]), $P_j(\cdot)$ and $Q_j(\cdot)$ do not change the sign in $(-\infty, 0)$. Noticing that $P_j(x)/Q_j(x) < 0$ for $x < 0$ large enough (the degrees of $P_j(\cdot)$ and $Q_j(\cdot)$ are j and $j-1$, respectively, and leading coefficients equal $B_{j-1}^{-1} \cdot \cdots \cdot B_0^{-1} > 0$), we get the negativity of α.

Theorem 2. *Assume $p = 1$. Let also $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $T_{\max}$ defined by* (11) *and $M(\cdot)$ be the corresponding Weyl function. The domain of the Friedrichs extension is given by*

$$\operatorname{dom}(T_F) = \{u \in T_{\max} : (\Gamma_1 - \alpha\Gamma_0)u = 0\}, \tag{14}$$

where α is defined by (13).

Proof. To prove the statement we use Proposition 2(*iii*). Namely, it is sufficient to show that

$$M(-\infty) = \lim_{x\downarrow -\infty} M(x) = \lim_{x\downarrow -\infty} \frac{x\sum_{j=1}^{\infty} Q_j(0)P_j(x) - 1}{x\sum_{j=0}^{\infty} P_j(0)P_j(x)} = \alpha. \tag{15}$$

Since the orthogonal polynomials do not change the sign in $(-\infty, 0)$,

$$P_j(0)P_j(x) = |P_j(0)P_j(x)| \quad \text{and} \quad Q_j(0)P_j(x) = -|Q_j(0)P_j(x)|. \tag{16}$$

Thus, the sequence $\sum_{j=0}^{n} P_j(0)P_j(x) = \sum_{j=0}^{n} |P_j(0)P_j(x)|$ increases as $n \to \infty$ and, therefore,

$$\lim_{x\downarrow -\infty} M(x) = \lim_{x\downarrow -\infty} \frac{x\sum_{j=1}^{\infty} Q_j(0)P_j(x) - 1}{x\sum_{j=0}^{\infty} P_j(0)P_j(x)} = -\lim_{x\downarrow -\infty} \frac{\sum_{j=1}^{\infty} |Q_j(0)P_j(x)|}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|}.$$

It follows from (13) that for any small $\delta > 0$ there exists $N = N(\delta)$ such that the estimate

$$\alpha - \delta < \frac{Q_j(0)}{P_j(0)} < \alpha + \delta$$

holds for $j > N(\delta)$. Combining this inequality with (16), we get

$$\alpha - \delta < \frac{Q_j(0)P_j(x)}{|P_j(0)P_j(x)|} < \alpha + \delta, \quad x \in (-\infty, 0), \quad j \geq N(\delta).$$

The latter is equivalent to

$$\begin{gathered} (\alpha - \delta)|P_j(0)P_j(x)| < Q_j(0)P_j(x) < (\alpha + \delta)|P_j(0)P_j(x)|, \\ x \in (-\infty, 0), \quad j \geq N(\delta). \end{gathered} \tag{17}$$

Hence,

$$\begin{aligned} (\alpha - \delta)\frac{\sum_{j=N(\delta)}^{\infty} |P_j(0)P_j(x)|}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|} &< \frac{\sum_{j=1}^{\infty} Q_j(0)P_j(x)}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|} - \frac{\sum_{j=1}^{N(\delta)-1} Q_j(0)P_j(x)}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|} \\ &< (\alpha + \delta)\frac{\sum_{j=N(\delta)}^{\infty} |P_j(0)P_j(x)|}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|}, \quad x \in (-\infty, 0), \quad j \geq N(\delta). \end{aligned} \tag{18}$$

Since

$$\sum_{j=0}^{\infty} P_j(x)P_j(0) = \sum_{j=0}^{\infty} |P_j(x)P_j(0)| > |P_{N(\delta)}(x)P_{N(\delta)}(0)|$$

and P_j is a polynomial of degree j, we get

$$0 \leq \lim_{x\to -\infty} \frac{\sum_{j=1}^{N(\delta)-1} |P_j(0)P_j(x)|}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|} < \lim_{x\to -\infty} \frac{\sum_{j=1}^{N(\delta)-1} |P_j(0)P_j(x)|}{|P_{N(\delta)}(x)P_{N(\delta)}(0)|} = 0. \tag{19}$$

Similarly we obtain

$$\lim_{x\to -\infty} \frac{\sum_{j=1}^{N(\delta)-1} Q_j(0)P_j(x)}{\sum_{j=0}^{\infty} |P_j(0)P_j(x)|} = 0. \tag{20}$$

Taking into account that

$$\sum_{j=0}^{\infty} |P_j(0)P_j(x)| = \sum_{j=N(\delta)}^{\infty} |P_j(0)P_j(x)| + \sum_{j=0}^{N(\delta)-1} |P_j(0)P_j(x)|,$$

we get from (18)–(20) the following inequality:

$$\alpha-\delta\leq\lim_{x\to-\infty}\frac{\sum_{j=1}^{\infty}Q_j(0)P_j(x)}{\sum_{j=0}^{\infty}|P_j(0)P_j(x)|}\leq\alpha+\delta$$

for any arbitrary small δ. Thus, equality (15) takes place.

□

Remark 4. We should mention that the description of the Krein and Friedrichs extensions given in Theorems 1 and 2 in the scalar case coincides with one obtained earlier by Brown and Christiansen in [7].

Remark 5.

(i) The condition $B_j>0$ can be dropped. Indeed, it is easy to show that a scalar Jacobi matrix with arbitrary real B_j is unitarily equivalent to a Jacobi matrix with positive B_j. The unitary equivalence is established by the diagonal matrix with 1 and -1 on the diagonal. Besides, if $B_j<0$, then 1 and -1 have to stand next to each other in the same rows as B_j.

(ii) The fact that all zeroes of $P_j(\cdot)$ belong to $[0,\infty)$ might also be derived using that the Weyl function corresponding to another boundary triplet is holomorphic on $(-\infty,0)$ (see [11, Proposition 10.1(2)]).

(iii) In [7], the authors have obtained the description (14) by a different method. Namely, they essentially used the fact that the minimal (or principal) solution $u=(u_j)$ of the equation $(\mathbf{J}u)_j=0,\ j\geq 1$, has the form $u=(u_j)=(P_j(0)-\alpha Q_j(0))$ and belongs to the domain $\mathrm{dom}\,(T_F)$.

(iv) In [17, Chapter 5, § 3] (see also [7]), it was noted that all solutions $\widetilde{\mu}$ of the Stieltjes moment problem associated with $T_{\min}$ lie between the solutions μ_K and μ_F coming from the Friedrichs and Krein extensions in the following sense:

$$\int_0^\infty\frac{d\mu_K(t)}{x-t}\leq\int_0^\infty\frac{d\widetilde{\mu}(t)}{x-t}\leq\int_0^\infty\frac{d\mu_F(t)}{x-t},\quad x<0.$$

Proposition 2(v) leads to a description of all non-negative self-adjoint extensions of T in the scalar case.

Corollary 3. *Assume $p=1$. Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be the boundary triplet for $T_{\max}$ given by* (11) *and let $M(\cdot)$ be the corresponding Weyl function. The set of all non-negative self-adjoint extensions T_h of the operator $T_{\min}$ is parametrized as follows:*

$$\mathrm{dom}\,(T_h)=\big\{u\in\mathrm{dom}\,(T_{\max}):(\Gamma_1-h\Gamma_0)u=0\big\},\tag{21}$$

$h\in[-\infty;\alpha]$, *where α is defined by* (13). *In particular,*

$$\mathrm{dom}\,(T_{-\infty})=\big\{u\in\mathrm{dom}\,(T_{\max}):\Gamma_0u=0\big\}.$$

Proof. First note that for $h=\alpha$ and $h=-\infty$ the statement was proved above. Indeed,

$$\mathrm{dom}\,(T_\alpha)=\{u\in\mathrm{dom}\,(T_{\max}):(\Gamma_1-\alpha\Gamma_0)u=0\}=\mathrm{dom}\,(T_F)$$

and

$$\mathrm{dom}\,(T_{-\infty})=\mathrm{dom}\,(T_{\max})\upharpoonright\ker(\Gamma_0)=\mathrm{dom}\,(T_K).$$

Thus, it remains to prove that for $h<\alpha$ formula (21) defines a non-negative self-adjoint extension. The result is implied by combining Proposition 2(v) with Theorem 2.

Indeed, consider a new boundary triplet, $\widetilde{\Pi}=\{\widetilde{\mathcal{H}},\widetilde{\Gamma_0},\widetilde{\Gamma_1}\}$

$$\widetilde{\mathcal{H}}=\mathbb{C},\quad\widetilde{\Gamma_0}f=\Gamma_1f-\alpha\Gamma_0f,\quad\widetilde{\Gamma_1}f=-\Gamma_0f,$$

where Γ_0,Γ_1 are given by (11). One easily obtains that the corresponding Weyl function is $\widetilde{M}(z)=(\alpha-M(z))^{-1}$. Taking into account the above information about "limit values" of the Weyl function $M(\cdot)$, we get that $\widetilde{M}(-\infty)$ is a "pure" linear relation, i.e.,

$\widetilde{M}(-\infty) = \{0, \mathcal{H}\}$, and $\widetilde{M}(0) = 0$. Hence, by Proposition 2(ii), $T_0 := T_{\max} \upharpoonright \ker(\widetilde{\Gamma_0}) = T_F$. Equation (21) in terms of the new boundary triplet takes the form

$$\operatorname{dom}(T_h) = \Big\{u \in \operatorname{dom}(T_{\max}) : \Big(\widetilde{\Gamma}_1 - \frac{1}{\alpha - h}\widetilde{\Gamma}_0\Big)u = 0\Big\}.$$

Applying Proposition 2(v), we get that $T_h \geq 0$ if and only if $\frac{1}{\alpha-h} > \widetilde{M}(0) = 0$ or $h < \alpha$. □

References

1. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner Publishing Co., New York, 1965.
2. N. I. Akhiezer, I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Vol. II, Pitman, Boston, 1981.
3. Yu. Arlinskii, E. Tsekanovskii, *The von Neumann problem for nonnegative symmetric operators*, Integr. Equ. Oper. Theory **51** (2005), 319–356.
4. M. Benammar, W. D. Evans, *On the Friedrichs extension of semi-bounded difference operators*, Math. Proc. Cambridge Phil. Soc. **116** (1994), no. 1, 167–177.
5. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965).
6. C. Berg, *Markov's theorem revisited*, J. Approx. Theory **78** (1994), no. 2, 260–275.
7. B. Malkolm Brown, J. S. Christiansen, *On the Krein and Friedrichs extension of a positive Jacobi operator*, Expo. Math. **23** (2005), 176–186.
8. T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Mathematics and its Applications, Vol. 13, Gordon and Breach Science Publishers, New York—London—Paris, 1978.
9. V. A. Derkach, M. M. Malamud, *Generalized rezolvent and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), no. 1, 1–95.
10. V. A. Derkach, M. M. Malamud, *Characteristic functions of almost solvable extensions of Hermitian operators*, Ukrain. Mat. Zh. **44** (1992), no. 4, 435–459. (Russian); English transl. Ukrainian Math. J. **44** (1992), no. 4, 379–401.
11. V. A. Derkach, M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. (New York) **73** (1995), 141–242.
12. V. A. Derkach, S. Hassi, M. M. Malamud and H. S. V. de Snoo, *Generalized resolvents of symmetric operators and admissibility*, Methods Funct. Anal. Topology **6** (2000), no. 3, 24–55.
13. Y. M. Dyukarev, *Examples of block Jacobi matrices generating symmetric operators with arbitrary possible values of the deficiency numbers*, Sbornik: Mathematics **201** (2010), no. 12, 1791–1800.
14. V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984)
15. A. G. Kostyuchenko, K. A. Mirzoev, *Three-term recurrence relations with matrix coefficients. The completely indefinite case*, Mat. Zametki **63** (1998), no. 5, 709–716. (Russian); English transl. Math. Notes **63** (1998), no. 5, 624–630.
16. M. G. Kreĭn, *The theory of selfadjoint extensions of semibounded Hermitian transformations and its applications.* I, Mat. Sbornik **20** (1947), no. 3, 431–495. (Russian)
17. M. G. Krein, A. A. Nudelman, *The Markov moment problem and extremal problems*, Nauka, Moscow, 1973. (Russian)
18. M. Marletta, A. Zettl, *The Friedrichs extension of singular differential operators*, J. Diff. Equat. **160** (2000), no. 2, 404–421.
19. F. S. Rofe-Beketov, *Self-adjoint extensions of differential operators in the space of vector-valued functions*, Teor. Funkcii Funkcional. Anal. Prilozhen. **8** (1969), 3–24. (Russian)
20. B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Advances Math. **137** (1998), no. 1, 182–203.

R. Luxemburg, 74, Donetsk, 83114, Ukraine
E-mail address: ananeva89@gmail.com

R. Luxemburg, 74, Donetsk, 83114, Ukraine
E-mail address: natalygoloshchapova@gmail.com

Received 02/04/2013; Revised 14/06/2013