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ON KONDRATIEV SPACES OF TEST FUNCTIONS IN THE NON-GAUSSIAN INFINITE-DIMENSIONAL ANALYSIS

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The paper is dedicated to Professor Yu. G. Kondratiev to his sixtieth anniversary

Abstract. A blanket version of the non-Gaussian analysis under the so-called biorthogonal approach uses the Kondratiev spaces of test functions with orthogonal bases given by a generating function $Q \times H \ni (x, \lambda) \mapsto h(x; \lambda) \in \mathbb{C}$, where Q is a metric space, H is some complex Hilbert space, h satisfies certain assumptions (in particular, $h(\cdot; \lambda)$ is a continuous function, $h(x; \cdot)$ is a holomorphic at zero function). In this paper we consider the construction of the Kondratiev spaces of test functions with orthogonal bases given by a generating function $\gamma(\lambda)h(x; \alpha(\lambda))$, where $\gamma : H \to \mathbb{C}$ and $\alpha : H \to H$ are holomorphic at zero functions, and study some properties of these spaces. The results of the paper give a possibility to extend an area of possible applications of the above mentioned theory.

Introduction

The theory of generalized functions of infinitely many variables with special spaces of test and generalized functions and with the pairing generated by the Gaussian measure was developed by Yu. G. Kondratiev [26, 25, 24], see also [28, 27, 29] (afterwords the said spaces were called the *Kondratiev spaces*), and independently by T. Hida [15, 16] (the corresponding spaces are called the *Hida spaces*). On the other hand, at the same time Yu. M. Berezansky with colleagues has developed a general theory of generalized functions of infinitely many variables, this theory is less detailed than the Kondratiev's or Hida's one, but with the pairing generated by a non-Gaussian measure, generally speaking (see, e.g., [9, 33, 6]). In this connection a natural wish to develop a *detailed* (as far as possible) theory with a *general* (again, as far as possible) pairing has arisen. Among the first works in this direction were the papers of Y. Ito and I. Kubo [17, 18], in which some results of the Gaussian theory were extended to the case when the pairing between test and generalized functions is generated by the Poisson measure.

A consequent development of a non-Gaussian theory of generalized functions occurred in different directions. One of these directions is based on the idea of Yu. G. Kondratiev to use as orthogonal bases in spaces of test and generalized functions so-called *biorthogonal systems* (generalized Appell polynomials and dual to them functions) that were introduced by Yu. L. Daletsky [13]. This idea is realized by Yu. G. Kondratiev and his colleagues first for so-called smooth twice analytic measures (that generate the pairing between test and generalized functions instead of the Gaussian measure) [2, 31, 1], afterwords in a more general case of analytic non-degenerate measures [32, 30]. Later on different investigations in the framework of the "biorthogonal analysis" were executed by many specialists, in particular, by G. F. Us [36], by Yu. M. Berezansky and Yu. G. Kondratiev [7], by Yu. M. Berezansky [3, 5, 4], by the author [19, 21, 20], by Yu. M. Berezansky and V. A. Tesko [12, 11], by V. A. Tesko [35], by E. Yablonsky [37]

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and by others. Note that Yu. M. Berezansky offers to generalize appreciably the initial biorthogonal systems: instead of generalized Appell polynomials he uses as orthogonal bases in the spaces of test functions some special functions given by a generating function $Q \times H \ni (x, \lambda) \mapsto h(x; \lambda) \in \mathbb{C}$, where Q is a metric space, H is some complex Hilbert or nuclear space, h satisfies certain assumptions (in particular, $h(\cdot; \lambda)$ is a continuous function, $h(x; \cdot)$ is a holomorphic at zero function). The sequent natural step consists in using of a generating function $\gamma(\lambda)h(x; \alpha(\lambda))$, where $\gamma : H \to \mathbb{C}$ and $\alpha : H \to H$ are holomorphic at zero functions, by analogy with the case of generalized Appell-like polynomials [30, 19]. The possibility to realize this idea was declared by the author in the short paper [22], but without a detailed presentation.

In this paper we consider the construction of the Kondratiev spaces of test functions with orthogonal bases given by a generating function $\gamma(\lambda)h(x; \alpha(\lambda))$, where h satisfies assumptions accepted in [12] (this case essentially differs from the case described in [22]), and study some properties of these spaces. The results of the paper give a possibility to extend an area of possible applications of results [12].

1. The Kondratiev spaces of test functions

Let $\mathcal{H}_p$, $p \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ be a family of real separable Hilbert spaces such that

- for each $p \in \mathbb{Z}_+$ $\mathcal{H}_{p+1}$ is densely and continuously embedded into $\mathcal{H}_p$ (it is convenient to suppose that for each $p \in \mathbb{Z}_+$ $\|\cdot\|_{\mathcal{H}_{p+1}} \geq \|\cdot\|_{\mathcal{H}_p}$, the general case can be reduced to this one [8]);
- the embeddings $\mathcal{H}_2 \hookrightarrow \mathcal{H}_1$ and $\mathcal{H}_3 \hookrightarrow \mathcal{H}_2$ are *quasinuclear*, i.e., the corresponding embedding operators are of Hilbert-Schmidt type.

We consider a chain (a rigging of $\mathcal{H}_0$)

$$\mathcal{N}' \supset \cdots \supset \mathcal{H}_{-p} \supset \cdots \supset \mathcal{H}_0 \supset \cdots \supset \mathcal{H}_p \supset \cdots \supset \mathcal{N}, \tag{1.1}$$

where $\mathcal{N} = \mathrm{pr\,lim}_{p \in \mathbb{Z}_+} \mathcal{H}_p$ is the projective limit of the sequence of spaces $\{\mathcal{H}_p\}_{p \in \mathbb{Z}_+}$ (it means that $\mathcal{N} = \bigcap_{p \in \mathbb{Z}_+} \mathcal{H}_p$ with a topology of the projective limit—the weaker topology such that for each $p \in \mathbb{Z}_+$ the embedding of $\mathcal{N}$ into $\mathcal{H}_p$ is continuous, see, e.g., [8, 10] for details), $\mathcal{H}_{-p}$ are Hilbert spaces dual of $\mathcal{H}_p$ with respect to $\mathcal{H}_0$, $\mathcal{N}' = \bigcup_{p \in \mathbb{Z}_+} \mathcal{H}_{-p}$ (often it can be convenient to introduce on $\mathcal{N}'$ a topology of inductive limit—the strongest topology such that for each $p \in \mathbb{Z}_+$ the embedding of $\mathcal{H}_{-p}$ into $\mathcal{N}'$ is continuous, in this case one writes $\mathcal{N}' = \mathrm{ind\,lim}_{p \in \mathbb{Z}_+} \mathcal{H}_{-p}$). In some versions of the white noise analysis it can be necessary to assume in addition that *chain* (1.1) *is nuclear* (or, which is the same, that the space $\mathcal{N}$ is nuclear), it means that for each $p \in \mathbb{Z}_+$ there exists $p' \in \mathbb{N}$ such that the embedding $\mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is quasinuclear. But for study of Kondratiev spaces in the context of the present paper we do not need this assumption—the premises accepted above are sufficient.

Denote by a subscript $\mathbb{C}$ complexifications of spaces, by $\widehat{\otimes}$ the symmetric tensor product. Let Q be a metric space (for example, in the Gaussian white noise analysis and in some its generalizations one uses $\mathcal{H}_{-p}$ as a space Q; also sometimes it is necessary to accept some additional assumptions on Q, for example, to assume that Q *is a separable space*, see [12, 35]). Denote by $C(Q)$ a linear topological space of complex bounded on balls in Q continuous functions on Q, the convergence in $C(Q)$ by definition is uniform on every ball in Q.

Let U_0 be a neighborhood of zero in $\mathcal{H}_{1,\mathbb{C}}$, and $Q \times U_0 \ni (x, \lambda) \mapsto h(x; \lambda) \in \mathbb{C}$ be a function satisfying the assumptions:

- for each $x \in Q$ the function $h(x; \cdot)$ is holomorphic at $0 \in U_0$;
- for each $\lambda \in U_0$ the function $h(\cdot; \lambda) \in C(Q)$;
- for an arbitrary ball $\mathcal{U} \subset Q$ and an arbitrary closed ball $\overline{U} \subset U_0$ there exists a constant $c = c(\mathcal{U}, \overline{U}) > 0$ such that $h(x, \lambda) < c$ if $x \in \mathcal{U}$ and $\lambda \in \overline{U}$.

Using properties of holomorphic functions (e.g.,[14]) and the kernel theorem (e.g., [8, 10]) it is shown in [12] that for each $x \in Q$ there exists an expansion

$$\begin{gathered} h(x;\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle h_n(x), \lambda^{\otimes n} \rangle, \\ h_n(x) \in \mathcal{H}_{-2,\mathbb{C}}^{\widehat{\otimes} n}, \quad \lambda \in B_x := \{\lambda \in \mathcal{H}_{2,\mathbb{C}} \cap U_0 : |\lambda|_2 < R_x,\ R_x > 0\}, \quad \lambda^{\otimes 0} := 1, \end{gathered} \tag{1.2}$$

here and below we denote by $\langle \cdot, \cdot \rangle$ the dual pairings in tensor powers of the complexification of chain (1.1), by $|\cdot|_p$ the norms in tensor powers of $\mathcal{H}_{p,\mathbb{C}}$, $p \in \mathbb{Z}$, $\mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$. Note that series (1.2) converges uniformly in every closed ball from B_x. We assume in addition that

- $B := \cap_{x \in Q} B_x$ is a nonempty open set.

Let $K > 1$ be some constant (the exact value of K is not essential for the considerations of the present paper, for example, one can assign $K = 2$), $p \in \mathbb{N} \backslash \{1\}$, $q \in \mathbb{N}$.

Definition. A Hilbert space of formal series

$$\begin{gathered} (\mathcal{H}_p)_q := \Big\{ f(x) = \sum_{n=0}^{\infty} \langle h_n(x), f^{(n)} \rangle,\ f^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n},\ x \in Q : \\ \|f\|^2_{(\mathcal{H}_p)_q} := \sum_{n=0}^{\infty} (n!)^2 K^{qn} |f^{(n)}|_p^2 < \infty \Big\} \end{gathered} \tag{1.3}$$

with the corresponding to $\|\cdot\|_{(\mathcal{H}_p)_q}$ scalar product is called the Kondratiev space of test functions.

Together with the spaces $(\mathcal{H}_p)_q$ one can consider the spaces of test functions $(\mathcal{H}_p) := \operatorname{pr}\lim_{q \in \mathbb{N}} (\mathcal{H}_p)_q$ and $(\mathcal{N}) := \operatorname{pr}\lim_{q \in \mathbb{N}, p \in \mathbb{N} \backslash \{1\}} (\mathcal{H}_p)_q$, these spaces also are called the Kondratiev ones.

Remark. One can also introduce and study the *parametrized Kondratiev spaces of test functions* $(\mathcal{H}_p)_q^{\beta}$ ($\beta \in [0, 1]$) that are defined by analogy with (1.3), but with $(n!)^{1+\beta}$ instead of $(n!)^2$ in the definition of the norm; or even consider more general spaces of test functions, by analogy with [34]. But the "payment" for such generalizations consists in an essential deteriorating of properties of mentioned spaces and of the corresponding dual spaces in comparison with the case of the spaces $(\mathcal{H}_p)_q$. The detailed study of the mentioned generalizations goes beyond of the present paper.

Important properties of the Kondratiev spaces of test functions are described in the following statement.

Proposition. ([12]) *There exists $q_0 = q_0(h) \in \mathbb{N}$ such that for arbitrary $p \in \mathbb{N} \backslash \{1, 2\}$ and $q \geq q_0$ the first series in* (1.3) *converges uniformly on every ball from Q to a function from $C(Q)$. Moreover, for each ball $\mathcal{U} \subset Q$ there exists a constant $c = c(\mathcal{U}) > 0$ such that*

$$|f(x)| \leq c \|f\|_{(\mathcal{H}_p)_q}, \quad x \in \mathcal{U}, \quad f \in (\mathcal{H}_p)_q. \tag{1.4}$$

Nevertheless, now we can not assert that $(\mathcal{H}_p)_q$ are function spaces because it is possible, generally speaking, that $f \in (\mathcal{H}_p)_q$, for each $x \in Q$ $f(x) = 0$, but $\|f\|_{(\mathcal{H}_p)_q} > 0$. So, we accept the following additional assumption on h:

- the system $\big(h_n(x)\big)_{n=0}^{\infty}$ is *minimal with respect to the spaces* $(\mathcal{H}_p)_q$ in the sense that if $f \in (\mathcal{H}_p)_q$ and for each $x \in Q$ $f(x) = 0$ then $f = 0$ in $(\mathcal{H}_p)_q$ (now $p \in \mathbb{N} \backslash \{1, 2\}$ and $q \geq q_0$).

Remark. It is easy to see that in order to verify the minimality of the system $\big(h_n(x)\big)_{n=0}^{\infty}$ it is sufficient to consider the case $p = 3$ and $q = q_0$.

Example. If $Q = \mathcal{H}_{-p}$ with some $p \in \mathbb{N}\backslash\{1,2\}$, and $h(x;\lambda) = \chi(\langle x,\lambda\rangle)$, where $\chi : \mathbb{C} \to \mathbb{C}$ is an entire function such that in the decomposition $\chi(u) = \sum_{n=0}^{\infty} \frac{1}{n!}\chi_n u^n$ $\chi_n \neq 0$ for all $n \in \mathbb{Z}_+$, then the system $\big(h_n(x) = \chi_n x^{\otimes n}\big)_{n=0}^{\infty}$ is minimal with respect to the corresponding spaces of test functions (see [23] for details).

From the proposition and the minimality condition the following statement follows.

Theorem. ([12]) *The spaces* $(\mathcal{H}_p)_q$, $p \in \mathbb{N}\backslash\{1,2\}$, $q \geq q_0$, *consist of continuous bounded on balls from* Q *functions, i.e.,* $(\mathcal{H}_p)_q \subset C(Q)$*. Moreover, these embeddings are continuous.*

Let $\gamma : \mathcal{H}_{1,\mathbb{C}} \to \mathbb{C}$ be a *holomorphic at zero* function, $\gamma(0) \neq 0$ (from the "technical point of view" it is convenient to accept the normalization $\gamma(0) = 1$). Then by analogy with (1.2) we have the decomposition

$$\gamma(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \gamma_n, \lambda^{\otimes n}\rangle,$$

$$\gamma_n \in \mathcal{H}_{-2,\mathbb{C}}^{\widehat{\otimes}n}, \quad \lambda \in B_\gamma := \{\lambda \in \mathcal{H}_{2,\mathbb{C}} : |\lambda|_2 < R_\gamma,\ R_\gamma > 0\},$$

and the series converges uniformly in every closed ball from B_γ. By analogy

$$\frac{1}{\gamma(\lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \gamma_n^{-1}, \lambda^{\otimes n}\rangle,$$

$$\gamma_n^{-1} \in \mathcal{H}_{-2,\mathbb{C}}^{\widehat{\otimes}n}, \quad \lambda \in B_{1/\gamma} := \{\lambda \in \mathcal{H}_{2,\mathbb{C}} : |\lambda|_2 < R_{1/\gamma},\ R_{1/\gamma} > 0\}.$$

Define $h^\gamma(x;\lambda) := \gamma(\lambda)h(x;\lambda)$. As is easy to see, for each $x \in Q$ the function $\mathcal{H}_{1,\mathbb{C}} \supset U_0 \ni \lambda \mapsto h^\gamma(x;\lambda) \in \mathbb{C}$ is holomorphic at $0 \in \mathcal{H}_{1,\mathbb{C}}$ and admits the representation

$$h^\gamma(x;\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle h_n^\gamma(x), \lambda^{\otimes n}\rangle, \quad h_n^\gamma(x) \in \mathcal{H}_{-2,\mathbb{C}}^{\widehat{\otimes}n}, \quad \lambda \in B_x \cap B_\gamma.$$

As is easy to calculate, the kernels $h_n^\gamma(x)$ and $h_n(x)$ are connected as follows:

$$\begin{aligned} h_n^\gamma(x) &= \sum_{m=0}^{n} C_n^m h_m(x)\widehat{\otimes}\gamma_{n-m}, \\ h_n(x) &= \sum_{m=0}^{n} C_n^m h_m^\gamma(x)\widehat{\otimes}\gamma_{n-m}^{-1}, \end{aligned} \tag{1.5}$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

Now one can define the Kondratiev spaces of test functions $(\mathcal{H}_p)_q^\gamma$ by analogy with the definition of the spaces $(\mathcal{H}_p)_q$, but using the kernels $h_n^\gamma(x)$ instead of $h_n(x)$. The following statement is proved in [12], an alternative proof consists in direct calculation with using of formulas (1.5), by analogy with [32, 19, 23].

Theorem. *There exists* $q_0' = q_0'(h,\gamma) \in \mathbb{N}$ *such that for all* $p \in \mathbb{N}\backslash\{1,2\}$ *and* $q \geq q_0'$ *the spaces* $(\mathcal{H}_p)_q^\gamma$ *are function ones (in particular, the system* $\big(h_n^\gamma(x)\big)_{n=0}^{\infty}$ *is minimal with respect to these spaces); and there exist* $q_1 = q_1(q), q_2 = q_2(q) \in \mathbb{N}$ *such that* $(\mathcal{H}_p)_q \supset (\mathcal{H}_p)_{q+q_1}^\gamma \supset (\mathcal{H}_p)_{q+q_1+q_2}$*, where the embeddings are dense and continuous.*

Let $\alpha : \mathcal{H}_{2,\mathbb{C}} \to \mathcal{H}_{2,\mathbb{C}} \cap U_0$ be a *holomorphic at zero* function, $\alpha(0) = 0$. Then the expansion

$$\alpha(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!}\widehat{d^n\alpha(0)}(\lambda), \tag{1.6}$$

$$\lambda \in U_\alpha := \{\lambda \in \mathcal{H}_{2,\mathbb{C}} : |\lambda|_2 < r_\alpha,\ r_\alpha > 0\}$$

holds, here $\widehat{d^n\alpha(0)}(\lambda)$ are n-homogeneous polynomials ([14]). Now our goal is to consider the Kondratiev spaces of test functions constructed by the kernels with a generating function

$$h^{\gamma,\alpha}(x;\lambda) = \gamma(\lambda)h(x;\alpha(\lambda)). \tag{1.7}$$

In order to obtain an intensional theory, we have to accept an additional assumption on α (an analog of the local boundedness in the theory connected with generalized Appell polynomials [30, 19, 23]):

- there exists $\rho \in (0, r_\alpha)$ such that $\sup_{|\lambda'|_2=\rho} |\alpha(\lambda')|_3 < \infty$.

Let $\xi_n \in \mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{3,\mathbb{C}}$, $\eta_n \in \mathcal{H}_{3,\mathbb{C}}^{\widehat{\otimes}n}$, $n \in \mathbb{N}$. We define a generalized pairing $\langle \xi_n, \eta_n \rangle \in \mathcal{H}_{3,\mathbb{C}}$ by setting for all $\theta \in \mathcal{H}_{-3,\mathbb{C}}$

$$\langle \theta, \langle \xi_n, \eta_n \rangle \rangle \equiv \langle \xi_n, \eta_n \otimes \theta \rangle. \tag{1.8}$$

Since $|\langle \xi_n, \eta_n \otimes \theta \rangle| \le \|\xi_n\|_{\mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{3,\mathbb{C}}} \|\eta_n\|_{\mathcal{H}_{3,\mathbb{C}}^{\widehat{\otimes}n}} \|\theta\|_{\mathcal{H}_{-3,\mathbb{C}}}$, an element $\langle \xi_n, \eta_n \rangle$ is well-definite and $\|\langle \xi_n, \eta_n \rangle\|_{\mathcal{H}_{3,\mathbb{C}}} \le \|\xi_n\|_{\mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{3,\mathbb{C}}} \|\eta_n\|_{\mathcal{H}_{3,\mathbb{C}}^{\widehat{\otimes}n}}$.

Proposition. *Let α satisfy the assumptions accepted above. Then there exist kernels $\alpha_n \in \mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{3,\mathbb{C}}$, $n \in \mathbb{N}$, such that*

$$\alpha(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \langle \alpha_n, \lambda^{\otimes n} \rangle, \quad \lambda \in B_\alpha := \Big\{ \lambda \in \mathcal{H}_{3,\mathbb{C}} : |\lambda|_3 < \frac{\rho}{e\|O_{3,2}\|_{HS}} \Big\},$$

and

$$\|\alpha_n\|_{\mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{3,\mathbb{C}}} \le n! \Big(\frac{e}{\rho} \|O_{3,2}\|_{HS} \Big)^n \sup_{|\lambda'|_2=\rho} |\alpha(\lambda')|_3, \tag{1.9}$$

where $O_{3,2} : \mathcal{H}_{3,\mathbb{C}} \to \mathcal{H}_{2,\mathbb{C}}$ is the embedding operator, $\|O_{3,2}\|_{HS}$ is its Hilbert-Schmidt norm.

Proof. Let $f_0 \in \mathcal{H}_{-2,\mathbb{C}}$. Then the function

$$\mathcal{H}_{2,\mathbb{C}} \ni \lambda \mapsto \langle f_0, \alpha(\lambda) \rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \langle f_0, \widehat{d^n\alpha(0)}(\lambda) \rangle \in \mathbb{C}$$

(see (1.6)) is holomorphic at zero, therefore by the Cauchy inequality [14] (see also [32, 30])

$$\Big| \frac{1}{n!} \langle f_0, \widehat{d^n\alpha(0)}(\lambda) \rangle \Big| \le \frac{1}{\rho^n} \sup_{|\lambda'|_2=\rho} |\langle f_0, \alpha(\lambda') \rangle| |\lambda|_2^n \le \frac{1}{\rho^n} \sup_{|\lambda'|_2=\rho} |\alpha(\lambda')|_3 |f_0|_{-3} |\lambda|_2^n.$$

Denote by B_n the n-linear symmetric $\mathcal{H}_{2,\mathbb{C}}$-valued function generated by the polynomial $\widehat{d^n\alpha(0)}$, i.e., $B_n(\underbrace{\lambda, \dots, \lambda}_{n}) = \widehat{d^n\alpha(0)}(\lambda)$. By the polarization [14] we have

$$\Big| \frac{1}{n!} \langle f_0, B_n(\lambda_1, \dots, \lambda_n) \rangle \Big| \le \Big(\frac{e}{\rho} \Big)^n \sup_{|\lambda'|_2=\rho} |\alpha(\lambda')|_3 |f_0|_{-3} \prod_{k=1}^{n} |\lambda_k|_2.$$

Now we need the following statement.

Lemma. *Let a be an $n+1$-linear symmetric by first n arguments continuous form on $\underbrace{\mathcal{H}_{2,\mathbb{C}} \oplus \cdots \oplus \mathcal{H}_{2,\mathbb{C}}}_{n} \oplus \mathcal{H}_{-3,\mathbb{C}}$, so, there exists $c > 0$ such that for $f_1, \dots, f_n \in \mathcal{H}_{2,\mathbb{C}}$, $f_0 \in \mathcal{H}_{-3,\mathbb{C}}$, the estimate $|a(f_1, \dots, f_n; f_0)| \le c|f_0|_{-3} \prod_{k=1}^{n} |f_k|_2$ is fulfilled. Then there exists a kernel $A \in \mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{3,\mathbb{C}}$ such that for $u_1, \dots, u_n \in \mathcal{H}_{3,\mathbb{C}}$, $f_0 \in \mathcal{H}_{-3,\mathbb{C}}$*

$$a(u_1, \dots, u_n; f_0) = \langle A, u_1 \widehat{\otimes} \cdots \widehat{\otimes} u_n \otimes f_0 \rangle, \tag{1.10}$$

and

$$\|A\|_{\mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{3,\mathbb{C}}} \le c\|O_{3,2}\|_{HS}^n. \tag{1.11}$$

Proof. We consider the chain

$$\mathcal{H}_{-3,\mathbb{C}}^{(2)} \supset \mathcal{H}_{2,\mathbb{C}} \supset \mathcal{H}_{3,\mathbb{C}}. \tag{1.12}$$

By the kernel theorem [8, 10] there exists a kernel $B \in \mathcal{H}_{-3,\mathbb{C}}^{(2)\widehat{\otimes}n} \otimes \mathcal{H}_{-3,\mathbb{C}}$ such that for $u_1, \dots, u_n \in \mathcal{H}_{3,\mathbb{C}}$, $f_0 \in \mathcal{H}_{-3,\mathbb{C}}$, $a(u_1, \dots, u_n; f_0) = (B, u_1\widehat{\otimes}\cdots\widehat{\otimes}u_n \otimes f_0)_{\mathcal{H}_{2,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{-3,\mathbb{C}}}$, and $\|B\|_{\mathcal{H}_{-3,\mathbb{C}}^{(2)\widehat{\otimes}n}\otimes\mathcal{H}_{-3,\mathbb{C}}} \le c\|O_{3,2}\|_{HS}^n$, here $(\cdot,\cdot)_{\mathcal{H}_{2,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{-3,\mathbb{C}}}$ denotes the dual pairing generated by the scalar product in $\mathcal{H}_{2,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{-3,\mathbb{C}}$. Let $\mathbf{I}^{(2)}, \mathbf{I}$ be the canonical isomorphisms of chains (1.12) and $\mathcal{H}_{-3,\mathbb{C}} \supset \mathcal{H}_{0,\mathbb{C}} \supset \mathcal{H}_{3,\mathbb{C}}$ respectively, and denote by $\mathbf{1}$ the identical operator. Set $A := (\mathbf{I}^{-1\otimes n} \otimes \mathbf{1})(\mathbf{I}^{(2)\otimes n} \otimes \mathbf{I})B$. Then, obviously, A satisfies equality (1.10) and estimate (1.11). □

Applying the lemma to the forms $\langle f_0, B_n\rangle$, $n \in \mathbb{N}$, we obtain the representation

$$\langle f_0, \alpha(\lambda)\rangle = \sum_{n=1}^{\infty} \frac{1}{n!}\langle \alpha_n, \lambda^{\otimes n} \otimes f_0\rangle, \quad \lambda \in B_\alpha,$$

with $\alpha_n \in \mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{3,\mathbb{C}}$ and satisfying estimates (1.9), whence the result of the proposition follows. □

Now by analogy with [30] it is easy to calculate that for $m \in \mathbb{N}$

$$\alpha(\lambda)^{\otimes m} = \sum_{n=m}^{\infty} \frac{1}{n!}\langle A_n^m, \lambda^{\otimes n}\rangle, \quad \lambda \in B_\alpha,$$

where

$$A_n^m = 1_{\{n\ge m\}} \sum_{\substack{l_1,\dots,l_m\in\mathbb{N},\\ l_1+\cdots+l_m=n}} \frac{n!}{l_1!\dots l_m!}\alpha_{l_1}\overline{\otimes}\cdots\overline{\otimes}\alpha_{l_m} \in \mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{3,\mathbb{C}}^{\widehat{\otimes}m}, \tag{1.13}$$

here 1_E is the indicator of an event E; $\overline{\otimes}$ denotes a tensor product symmetrized by the generalized and test components, for example, for $F_1, F_2 \in \mathcal{H}_{-3,\mathbb{C}}$ and $f_1, f_2 \in \mathcal{H}_{3,\mathbb{C}}$ $(F_1 \otimes f_1)\overline{\otimes}(F_2 \otimes f_2) = (F_1\widehat{\otimes}F_2) \otimes (f_1\widehat{\otimes}f_2)$; the generalized pairing $\langle\cdot,\cdot\rangle$ is defined by analogy with (1.8).

Consider now a function $h^\alpha(x;\lambda) := h(x;\alpha(\lambda))$, $x \in Q$, $\lambda \in \mathcal{H}_{2,\mathbb{C}}$. For each x this function is holomorphic at zero, and, moreover, by virtue of the assumptions accepted above the representation

$$h^\alpha(x;\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle h_n^\alpha(x), \lambda^{\otimes n}\rangle,$$

$$h_n^\alpha(x) \in \mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n}, \quad \lambda \in B_{\alpha,x} := \{\lambda \in \mathcal{H}_{3,\mathbb{C}} : |\lambda|_3 < R_{\alpha,x},\ R_{\alpha,x} > 0\}$$

holds. By analogy with [30, 19, 23] one can calculate that for $n \in \mathbb{N}$

$$h_n^\alpha(x) = \sum_{m=1}^{n} \frac{1}{m!}\langle h_m(x), A_n^m\rangle, \tag{1.14}$$

$h_0^\alpha(x) = h_0(x)$. Similarly, for function (1.7) we have

$$h^{\gamma,\alpha}(x;\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle h_n^{\gamma,\alpha}(x), \lambda^{\otimes n}\rangle,$$

$$h_n^{\gamma,\alpha}(x) \in \mathcal{H}_{-3,\mathbb{C}}^{\widehat{\otimes}n}, \quad \lambda \in B_{\gamma,\alpha,x} := \{\lambda \in \mathcal{H}_{3,\mathbb{C}} : |\lambda|_3 < R_{\gamma,\alpha,x},\ R_{\gamma,\alpha,x} > 0\},$$

and, obviously, the kernels $h_n^{\gamma,\alpha}(x)$ are connected with the kernels $h_n^{\alpha}(x)$ by formulas (1.5) with $h_n^{\gamma,\alpha}(x)$, $h_n^{\alpha}(x)$ instead of $h_n^{\gamma}(x)$, $h_n(x)$ respectively.

Now one can define the Kondratiev spaces of test functions $(\mathcal{H}_p)_q^{\gamma,\alpha}$ by analogy with the definition of the spaces $(\mathcal{H}_p)_q$, but using the kernels $h_n^{\gamma,\alpha}(x)$ instead of $h_n(x)$. So, elements of $(\mathcal{H}_p)_q^{\gamma,\alpha}$ are formal series of the form

$$f(x) = \sum_{n=0}^{\infty} \langle h_n^{\gamma,\alpha}(x), f^{(n)} \rangle \tag{1.15}$$

such that

$$\|f\|^2_{(\mathcal{H}_p)_q^{\gamma,\alpha}} = \sum_{n=0}^{\infty} (n!)^2 K^{qn} |f^{(n)}|_p^2 < \infty.$$

Using representations (1.5) with $h_n^{\gamma,\alpha}(x)$, $h_n^{\alpha}(x)$, (1.14), estimates (3.7), (5.7) from [12] and (1.9), by analogy with [12] one can show that an estimate of type (1.4) holds true for the spaces $(\mathcal{H}_p)_q^{\gamma,\alpha}$. More exactly, we have the following statement.

Proposition. *There exists $q_0'' = q_0''(h, \gamma, \alpha) \in \mathbb{N}$ such that for arbitrary $p \in \mathbb{N}\backslash\{1,2\}$ and $q \geq q_0''$ series* (1.15) *converges uniformly on every ball from Q to a function from $C(Q)$. Moreover, for each ball $\mathcal{U} \subset Q$ there exists a constant $c' = c'(\mathcal{U}) > 0$ such that*

$$|f(x)| \leq c' \|f\|_{(\mathcal{H}_p)_q^{\gamma,\alpha}}, \quad x \in \mathcal{U}, \quad f \in (\mathcal{H}_p)_q^{\gamma,\alpha}.$$

Furthermore, now for $q \in \mathbb{N}$ sufficiently large there exists $\widetilde{q} = \widetilde{q}(h, \gamma, \alpha, q) \in \mathbb{N}$ such that for $f \in (\mathcal{H}_3)_{q+\widetilde{q}}^{\gamma,\alpha}$ the estimate $\|f\|_{(\mathcal{H}_3)_q} \leq k \|f\|_{(\mathcal{H}_3)_{q+\widetilde{q}}^{\gamma,\alpha}}$ holds, here $\|f\|_{(\mathcal{H}_3)_q}$ is the $(\mathcal{H}_3)_q$-norm of the sum of series (1.15) for f, $k = k(q, \widetilde{q}) > 0$ is some constant. Nevertheless, in contrast to the spaces $(\mathcal{H}_p)_q$, $(\mathcal{H}_p)_q^{\gamma}$, the spaces $(\mathcal{H}_p)_q^{\gamma,\alpha}$ can be not functional ones because the system $\left(h_n^{\gamma,\alpha}(x)\right)_{n=0}^{\infty}$ can be a not minimal one (with respect to these spaces). But if we accept in addition that the *system $\left(h_n^{\gamma,\alpha}(x)\right)_{n=0}^{\infty}$ is minimal one with respect to the spaces* $(\mathcal{H}_p)_q^{\gamma,\alpha}$ then these spaces will be function ones, $(\mathcal{H}_3)_{q+\widetilde{q}}^{\gamma,\alpha}$ will be continuously embedded into $(\mathcal{H}_3)_q$, and the interconnection between the spaces $(\mathcal{H}_p)_q^{\gamma,\alpha}$ and $(\mathcal{H}_p)_q^{1,\alpha}$(corresponding to $\gamma \equiv 1$) is quite analogous to the interconnection between the spaces $(\mathcal{H}_p)_q^{\gamma}$ and $(\mathcal{H}_p)_q$. Finally, if

- the function α is invertible and its inverse function satisfies the assumptions accepted for α

then by analogy with [23] we obtain the following statement.

Theorem. *The system $\left(h_n^{\gamma,\alpha}(x)\right)_{n=0}^{\infty}$ is minimal one with respect to the spaces $(\mathcal{H}_p)_q^{\gamma,\alpha}$; for $p \in \mathbb{N}\backslash\{1,2\}$ and $q \geq q_0''$ these spaces are function ones; and for $\widehat{q} \in \mathbb{N}$ sufficiently large there exist $q_1 = q_1(\widehat{q}), q_2 = q_2(\widehat{q}) \in \mathbb{N}$ such that $(\mathcal{H}_3)_{\widehat{q}} \supset (\mathcal{H}_3)_{\widehat{q}+q_1}^{\gamma,\alpha} \supset (\mathcal{H}_3)_{\widehat{q}+q_1+q_2}$, where the embeddings are dense and continuous.*

Corollary. *The space $(\mathcal{H}_3)^{\gamma,\alpha} :=$ pr $\lim_{q\in\mathbb{N}} (\mathcal{H}_3)_q^{\gamma,\alpha}$ does not depend on γ and α as a topological one.*

Note that now we have no similar results for the spaces $(\mathcal{H}_p)_q^{\gamma,\alpha}$ with $p > 3$ because the kernels A_n^m (see (1.13)) do not belong to the spaces $\mathcal{H}_{-p,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} m}$, generally speaking. But it is possible to overcome this problem if one considers a function $\alpha : \mathcal{N}_{\mathbb{C}} \to \mathcal{N}_{\mathbb{C}}$ ($\alpha(0) = 0$) that is invertible, holomorphic at zero and locally bounded, and the inverse function has the same properties, by analogy with [30, 19, 23].

Remark. The results of [12] connected with the Kondratiev spaces of test functions and the results of this paper can be easily generalized to the case $p \in \mathbb{N}\backslash\{1\}$; and, actually, it is sufficient to assume that only the embedding operator $O_{2,1} : \mathcal{H}_2 \to \mathcal{H}_1$ is a quasinuclear one. In fact, the assumption that the embedding operator $O_{3,2} : \mathcal{H}_3 \to \mathcal{H}_2$ is a quasinuclear one is used in [12] in order to obtain estimates for $|h_n(x)|_{-3}$ and $|h_n^\gamma(x)|_{-3}$, but similar estimates for $|h_n(x)|_{-2}$ and $|h_n^\gamma(x)|_{-2}$ immediately follow from the kernel theorem (cf. [32]).

Although, generally speaking, the function $h^{\gamma,\alpha}(x;\lambda)$ does not satisfy all assumptions accepted for $h(x;\lambda)$, the spaces $(\mathcal{H}_p)_q^{\gamma,\alpha}$ can be used in order to construct a version of a non-Gaussian analysis by analogy with [12, 35], for study of pseudodifferential operators (in particular, of generalized translation operators), operators of stochastic integration and differentiation, etc.

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