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# DECOMPOSITION OF A UNITARY SCALAR OPERATOR INTO A PRODUCT OF ROOTS OF THE IDENTITY

D. YU. YAKYMENKO

Abstract. We prove that for all $m_1, m_2, m_3 \in \mathbb{N}$, $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \le 1$, every unitary scalar operator $\gamma I$ on a complex infinite-dimensional Hilbert space is a product $\gamma I = U_1 U_2 U_3$ where $U_i$ is a unitary operator such that $U_i^{m_i} = I$.

## 1. Introduction

Let $H$ be a complex Hilbert space, for $i = \overline{1, n}$, let $A_i$ be a self-adjoint operator with finite spectrum $\sigma(A_i)$. Let $I$ denote the identity operator on $H$. Consider the following equation:

$$A_1 + A_2 + \cdots + A_n = \lambda I, \quad \lambda \in \mathbb{C}. \tag{1}$$

In [6], [5] and related works the following problems were studied.

1) Describe the set of all possible values of $\lambda$ if $\sigma(A_i)$ are given.

2) Classify unitary nonequivalent tuples of operators $(A_i)_{i=1}^n$ that satisfy equation (1) if $\lambda$ and $\sigma(A_i)$ are given.

In this work we continue to study the multiplicative analog of the mentioned problems. It was known that every unitary operator on an infinite-dimensional Hilbert space $H$ is a product of four symmetries (see [2]), that is,

$$\forall U \in Uni(H)\ \exists U_i \in Uni(H):\ U = U_1 U_2 U_3 U_4, \quad U_i^2 = I,$$

(here $Uni(H)$ denotes the set of all unitary operators on $H$), and every $U \in Uni(H)$ is a product of three $n$-th roots of the identity if $n \ge 3$ (see [3]), that is,

$$\forall n \ge 3\ \forall U \in Uni(H)\ \exists U_i \in Uni(H):\ U = U_1 U_2 U_3, \quad U_i^n = I.$$

In recent papers we have proved that if $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \le 1$ and two numbers from $\{m_1, m_2, m_3\}$ are even then every unitary scalar operator is a product of three $m_i$-th roots of $I$ (see [9]), moreover, every unitary operator is a product of three $m_i$-th roots (see [10]). In the present paper using a technique different from [9] and [10] we prove the existence of decomposition of a scalar unitary operator without the condition on parity of $m_i$ (see Theorem 1).

## 2. Statements and proofs

The main result of this work is the following theorem.

**Theorem 1.** *For all $m_1, m_2, m_3 \in \mathbb{N}$, $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \le 1$, every unitary scalar operator $\gamma I$ on a complex infinite-dimensional Hilbert space is a product $\gamma I = U_1 U_2 U_3$ where $U_i$ is a unitary operator such that $U_i^{m_i} = I$.*

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*Proof.* From now on we suppose that $\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}\le 1$ so we will omit this condition in the statements.

Consider the following central extension of an ordinary triangle group

$$E \;=\; \langle x,y,z,q|x^{m_1}=y^{m_2}=z^{m_3}=1, xyz=q, xq=qx, yq=qy, zq=qz\rangle.$$

This group is infinite since an ordinary triangle group is infinite [7].

It is clear that if $\gamma I = U_1U_2U_3,\ U_i^{m_i}=I$ on $H$ then $\pi: E\to H$ with $\pi(q)=\gamma I,\ \pi(x)=U_1,\ \pi(y)=U_2,\ \pi(z)=U_3$ is a unitary representation of the group $E$. Also, every irreducible unitary representation of $E$ gives us a solution to $\gamma I=U_1U_2U_3$, $U_i^{m_i}=I$.

If every $\gamma I$ is a product of three $m_i$-th roots of $I$ then the element $q$ from $E$ is an element of infinite order. Our proof of Theorem 1 consists of two steps: to prove that the element $q$ is an element of infinite order in $E$ and to deduce Theorem 1 from it. Let us start with the easiest one.

**Proposition 1.** *If $q$ is an element of infinite order in $E$ then for every $\gamma\in\mathbb{C}, |\gamma|=1$, the operator $\gamma I$ on the infinite-dimensional Hilbert space $H$ is the product $\gamma I=U_1U_2U_3$ where $U_iU_i^*=I,\ U_i^{m_i}=I$.*

*Proof.* Since $q$ is an element of infinite order in $E$, the normal subgroup $Q=\langle q\rangle$ of $E$ is an infinite cyclic group and $E/Q$ is an ordinary triangle group. Let $\gamma\in\mathbb{C}, |\gamma|=1$ and $\pi(q)=\gamma$ be a 1-dimensional unitary representation of $Q$. Consider the induced unitary representation $\tau=Ind_Q^E\pi$ that acts on an infinite-dimensional Hilbert space $H$ since $E/Q$ is infinite. The element $q$ belongs to the center of $E$, therefore $\tau(q)=\gamma I$. We have $\tau(q)=\tau(x)\tau(y)\tau(z)$ hence $\gamma I=\tau(x)\tau(y)\tau(z)$ gives the decomposition we are looking for. $\square$

**Proposition 2.** *The element $q$ is an element of infinite order in $E$.*

*Proof.* To prove this fact we need the notion of a finite complete rewriting system [1], [4]. Let $A=\{a_1,\dots,a_n\}$ be a finite set. The set $A$ is called an alphabet and the elements of $A$ are called letters. The ordered sets of letters from $A$ are called words. The empty word is denoted by 1. The set of all words including the empty word with the concatenation operation form the free monoid $A^*$. Given a word $W\in A^*$, we will denote its length by $|W|$, defined as the numbers of letters in $W$.

A rewriting system $R$ over $A$ is a set of rules $U\to V$, $U,V\in A^*$, that is, $R\subset A^*\times A^*$. A word $W_1\in A^*$ is said to be rewritten to another word $W_2\in A^*$ by a one-step reduction induced by $R$, if $W_1=Z_1XZ_2, W_2=Z_1YZ_2$ for some rule $X\to Y$ in $R$. In this situation we write $W_1\to_R W_2$. The reflexive transitive closure and the reflexive symmetric transitive closure of $\to_R$ are denoted by $\to_R^*$ and $\leftrightarrow_R^*$, respectively. The relation $\leftrightarrow_R^*$ is defined to be a congruence on $A^*$ generated by $R$.

Let $Left(R)=\{X\in A^*: X\to Y\in R\}$ and $Irr(R)=A^*\setminus A^*Left(R)A^*$. That is, $Irr(R)$ is the set of all words from $A^*$ that can not be reduced by any rule from $R$. A word $W\in A^*$ is called an irreducible word if $W\in Irr(R)$. From now on we suppose that $1\notin Left(R)$ hence $1\in Irr(R)$.

We say $R$ is Noetherian if there is no infinite reduction sequence

$$W_1\to_R W_2\to_R W_3\to_R\cdots.$$

System $R$ is said to be confluent if whenever $U\to_R^* V$ and $U\to_R^* W$, then there is an $X\in A^*$ such that $V\to_R^* X$ and $W\to_R^* X$. If $R$ is both Noetherian and confluent, we say that $R$ is a complete rewriting system. $R$ is a finite complete rewriting system if additionally $R$ is a finite set.

The following fact is well known.

**Proposition 3.** *Suppose $R$ is a complete rewriting system for $A$. Then for each $W \in A^*$ there is a unique $W' \in Irr(R)$ such that $W \to_R^* W'$. The word $W'$ is denoted by $irr(W)$.*

It is clear that if $R$ is a complete rewriting system and $W \in A^*$, then to find $irr(W)$ we just need to apply rules from $R$ to $W$ in an arbitrary order till we stop. If $R$ is a finite system then this algorithm is computable and it computes $irr(W)$ in a finite number of steps.

Finite complete rewriting systems make a useful tool in solving word problem for groups. Suppose we have a finitely presented group

$$G \;=\; \langle A = \{a_1, \ldots, a_n\} \mid S = \{s_1, \ldots, s_m\} \subset A^{\pm}\rangle,$$

here $A$ is a set of generators, $A^{\pm}$ is the set of all words from the alphabet $\{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$ and $S$ is a set of relations, that is $S \subset A^{\pm}$. Words from $S$ determine relations $s_i = 1$ in $G$.

For a word $W \in A^{\pm}$ we denote by $W^{-1}$ the word constructed from $W$ by reversing the order of the letters and changing the sign of every letter to the opposite. The word $W^k$ denotes the concatenation of $k$ words $W$. Note that words from $A^{\pm}$ with the concatenation and the inverse operation form a free group $F_n$.

**Proposition 4.** *Let $R$ be a finite complete rewriting system on $A^{\pm}$ and suppose that*
*1) $1 \in Irr(R)$;*
*2) rules $a_i a_i^{-1} \to 1, a_i^{-1} a_i \to 1$ belong to $R$, $i = \overline{1,n}$;*
*3) if $U \to V \in R$ then $U = V$ in group $G$;*
*4) if $U \in S$ then $U \to_R^* 1$, $U^{-1} \to_R^* 1$.*
*Then $W = 1$ in $G$ if and only if $W \to_R^* 1$ (that is $irr(W) = 1$).*

Such a system $R$ is called a finite complete rewriting system for a group $G$. To check if $W = 1$ in $G$ we just need to find $irr(W)$ using $R$.

Let's go back to our group $E$. Note that $E$ has equal presentation

$$E \;=\; \langle x, y, q | x^{m_1} = y^{m_2} = 1, (xy)^{m_3} = q^{m_3}, xq = qx, yq = qy\rangle,$$

which can be obtained using Tits transformations.

For simplicity we construct a finite complete rewriting system for the subgroup $E'$ generated by $\langle x, y, q^{m_3}\rangle$ and which has the presentation

$$E' \;=\; \langle x, y, q | x^{m_1} = y^{m_2} = 1, (xy)^{m_3} = q, xq = qx, yq = qy\rangle.$$

It is clear that $|q| = \infty$ in $E$ if and only if $|q| = \infty$ in $E'$.

Our finite complete rewriting system $R$ for $E'$ depends on parity of the numbers $m_1, m_2, m_3$.

For simplicity we denote the letter $x^{-1}$ as $X$ and the letter $y^{-1}$ as $Y$.

**Finite complete rewriting system for $E'$ (alphabet $\{x, y, q, X, Y, q^{-1}\}$):**

**Case 1**: $(m_1, m_2, m_3) \equiv (0, 0, 0) \bmod 2$
**Rules**:
$\{xX \to 1,\ Xx \to 1,\ yY \to 1,\ Yy \to 1,\ qq^{-1} \to 1,\ q^{-1}q \to 1,$
$xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY,$
$xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y,$
$x^{\frac{m_1}{2}+1} \to X^{\frac{m_1}{2}-1},\ X^{\frac{m_1}{2}} \to x^{\frac{m_1}{2}},$
$y^{\frac{m_2}{2}+1} \to Y^{\frac{m_2}{2}-1},\ Y^{\frac{m_1}{2}} \to y^{\frac{m_1}{2}},$
$(xy)^{\frac{m_3}{2}}x \to q(YX)^{\frac{m_3}{2}-1}Y,\ (yx)^{\frac{m_3}{2}}y \to q(XY)^{\frac{m_3}{2}-1}X,$
$(YX)^{\frac{m_3}{2}} \to q^{-1}(xy)^{\frac{m_3}{2}},\ (XY)^{\frac{m_3}{2}} \to q^{-1}(yx)^{\frac{m_3}{2}},$
$(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1}{2}} \to q^{-1}(xy)^{\frac{m_3}{2}}X^{\frac{m_1}{2}-1},$
$X^{\frac{m_1}{2}-1}(yx)^{\frac{m_3}{2}} \to q^{-1}x^{\frac{m_1}{2}}(YX)^{\frac{m_3}{2}-1}Y,$

$(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2}{2}} \to q^{-1}(yx)^{\frac{m_3}{2}}Y^{\frac{m_2}{2}-1}$,
$Y^{\frac{m_1}{2}-1}(xy)^{\frac{m_3}{2}} \to qy^{\frac{m_2}{2}}(XY)^{\frac{m_3}{2}-1}X\}$.

**Case 2**: $(m_1, m_2, m_3) \equiv (0, 0, 1) \bmod 2$

**Rules**:

$\{xX \to 1,\ Xx \to 1,\ yY \to 1,\ Yy \to 1,\ qq^{-1} \to 1,\ q^{-1}q \to 1$,
$xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY$,
$xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y$,
$x^{\frac{m_1}{2}+1} \to X^{\frac{m_1}{2}-1},\ X^{\frac{m_1}{2}} \to x^{\frac{m_1}{2}}$,
$y^{\frac{m_2}{2}+1} \to Y^{\frac{m_2}{2}-1},\ Y^{\frac{m_1}{2}} \to y^{\frac{m_1}{2}}$,
$(xy)^{\frac{m_3+1}{2}} \to q(YX)^{\frac{m_3-1}{2}},\ (yx)^{\frac{m_3+1}{2}} \to q(XY)^{\frac{m_3-1}{2}}$,
$(YX)^{\frac{m_3-1}{2}}Y \to q^{-1}(xy)^{\frac{m_3-1}{2}}x,\ (XY)^{\frac{m_3-1}{2}}X \to q^{-1}(yx)^{\frac{m_3-1}{2}}y$,
$X^{\frac{m_1}{2}-1}y(xy)^{\frac{m_3-1}{2}} \to q^{-1}x^{\frac{m_1}{2}}(YX)^{\frac{m_3-1}{2}}$,
$(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1}{2}} \to q^{-1}(yx)^{\frac{m_3-1}{2}}yX^{\frac{m_1}{2}-1}$,
$Y^{\frac{m_1}{2}-1}x(yx)^{\frac{m_3-1}{2}} \to qy^{\frac{m_2}{2}}(XY)^{\frac{m_3-1}{2}}$,
$(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2}{2}} \to q^{-1}(xy)^{\frac{m_3-1}{2}}xY^{\frac{m_2}{2}-1}\}$.

**Case 3**: $(m_1, m_2, m_3) \equiv (1, 1, 0) \bmod 2$

**Rules**:

$\{xX \to 1,\ Xx \to 1,\ yY \to 1,\ Yy \to 1,\ qq^{-1} \to 1,\ q^{-1}q \to 1$,
$xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY$,
$xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y$,
$x^{\frac{m_1+1}{2}} \to X^{\frac{m_1-1}{2}},\ X^{\frac{m_1+1}{2}} \to x^{\frac{m_1-1}{2}}$,
$y^{\frac{m_2+1}{2}} \to Y^{\frac{m_2-1}{2}},\ Y^{\frac{m_1+1}{2}} \to y^{\frac{m_1-1}{2}}$,
$(xy)^{\frac{m_3}{2}}x \to q(YX)^{\frac{m_3}{2}-1}Y,\ (yx)^{\frac{m_3}{2}}y \to q(XY)^{\frac{m_3}{2}-1}X$,
$(YX)^{\frac{m_3}{2}} \to q^{-1}(xy)^{\frac{m_3}{2}},\ (XY)^{\frac{m_3}{2}} \to q^{-1}(yx)^{\frac{m_3}{2}}$,
$(xy)^{\frac{m_3}{2}}X^{\frac{m_1-1}{2}} \to q(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1-1}{2}},\ X^{\frac{m_1-1}{2}}(yx)^{\frac{m_3}{2}} \to qx^{\frac{m_1-1}{2}}(YX)^{\frac{m_3}{2}-1}Y$,
$(yx)^{\frac{m_3}{2}}Y^{\frac{m_2-1}{2}} \to q(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2-1}{2}},\ Y^{\frac{m_2-1}{2}}(xy)^{\frac{m_3}{2}} \to qy^{\frac{m_2-1}{2}}(XY)^{\frac{m_3}{2}-1}X$,
$X^{\frac{m_1-1}{2}}y(xy)^{\frac{m_3}{2}-1}X^{\frac{m_1-1}{2}} \to qx^{\frac{m_1-1}{2}}(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1-1}{2}}$,
$(xy)^{\frac{m_3}{2}}X^{\frac{m_1-3}{2}}(yx)^{\frac{m_3}{2}} \to q^2(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1-1}{2}}Y(XY)^{\frac{m_3}{2}-1}$,
$(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1-1}{2}}Y(XY)^{\frac{m_3}{2}-1}x^{\frac{m_1-1}{2}} \to q^{-2}(xy)^{\frac{m_3}{2}}X^{\frac{m_1-3}{2}}y(xy)^{\frac{m_3}{2}-1}X^{\frac{m_1-1}{2}}$,
$X^{\frac{m_1-1}{2}}y(xy)^{\frac{m_3}{2}-1}X^{\frac{m_1-3}{2}}(yx)^{\frac{m_3}{2}} \to q^2x^{\frac{m_1-1}{2}}Y(XY)^{\frac{m_3}{2}-1}x^{\frac{m_1-1}{2}}Y(XY)^{\frac{m_3}{2}-1}$,
$Y^{\frac{m_2-1}{2}}x(yx)^{\frac{m_3}{2}-1}Y^{\frac{m_2-1}{2}} \to qy^{\frac{m_2-1}{2}}(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2-1}{2}}$,
$(yx)^{\frac{m_3}{2}}Y^{\frac{m_2-3}{2}}(xy)^{\frac{m_3}{2}} \to q^2(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2-1}{2}}X(YX)^{\frac{m_3}{2}-1}$,
$(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2-1}{2}}X(YX)^{\frac{m_3}{2}-1}y^{\frac{m_2-1}{2}} \to q^{-2}(yx)^{\frac{m_3}{2}}Y^{\frac{m_2-3}{2}}x(yx)^{\frac{m_3}{2}-1}Y^{\frac{m_2-1}{2}}$,
$Y^{\frac{m_2-1}{2}}x(yx)^{\frac{m_3}{2}-1}Y^{\frac{m_2-3}{2}}(xy)^{\frac{m_3}{2}} \to q^2y^{\frac{m_2-1}{2}}X(YX)^{\frac{m_3}{2}-1}y^{\frac{m_2-1}{2}}X(YX)^{\frac{m_3}{2}-1}$.

**Case 4**: $(m_1, m_2, m_3) \equiv (1, 1, 1) \bmod 2$

**Rules**:

$\{xX \to 1,\ Xx \to 1,\ yY \to 1,\ Yy \to 1,\ qq^{-1} \to 1,\ q^{-1}q \to 1$,
$xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY$,
$xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y$,
$x^{\frac{m_1+1}{2}} \to X^{\frac{m_1-1}{2}},\ X^{\frac{m_1+1}{2}} \to x^{\frac{m_1-1}{2}}$,
$y^{\frac{m_2+1}{2}} \to Y^{\frac{m_2-1}{2}},\ Y^{\frac{m_1+1}{2}} \to y^{\frac{m_1-1}{2}}$,
$(xy)^{\frac{m_3+1}{2}} \to q(YX)^{\frac{m_3-1}{2}},\ (yx)^{\frac{m_3+1}{2}} \to q(XY)^{\frac{m_3-1}{2}}$,
$(YX)^{\frac{m_3-1}{2}}Y \to q^{-1}(xy)^{\frac{m_3-1}{2}}x,\ (XY)^{\frac{m_3-1}{2}}X \to q^{-1}(yx)^{\frac{m_3-1}{2}}y$,
$X^{\frac{m_1-1}{2}}y(xy)^{\frac{m_3-1}{2}} \to qx^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}},\ (yx)^{\frac{m_3-1}{2}}yX^{\frac{m_1-1}{2}} \to q(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}}$,
$Y^{\frac{m_2-1}{2}}x(yx)^{\frac{m_3-1}{2}} \to qy^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}},\ (xy)^{\frac{m_3-1}{2}}xY^{\frac{m_2-1}{2}} \to q(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}}$,
$X^{\frac{m_1-1}{2}}(yx)^{\frac{m_3-1}{2}}Y^{\frac{m_2-1}{2}} \to qx^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}}$,

$Y^{\frac{m_2-1}{2}}(xy)^{\frac{m_3-1}{2}}X^{\frac{m_1-1}{2}} \to qy^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}},$
$y(xy)^{\frac{m_3-1}{2}}X^{\frac{m_1-3}{2}}y(xy)^{\frac{m_3-1}{2}} \to q^2(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}},$
$x(yx)^{\frac{m_3-1}{2}}Y^{\frac{m_2-3}{2}}x(yx)^{\frac{m_3-1}{2}} \to q^2(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}},$
$X^{\frac{m_1-1}{2}}(yx)^{\frac{m_3-1}{2}}Y^{\frac{m_2-3}{2}}(xy)^{\frac{m_3-1}{2}}x \to q^2x^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}},$
$(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}} \to q^{-2}(yx)^{\frac{m_3-1}{2}}yX^{\frac{m_1-3}{2}}(yx)^{\frac{m_3-1}{2}}Y^{\frac{m_2-1}{2}},$
$Y^{\frac{m_2-1}{2}}(xy)^{\frac{m_3-1}{2}}X^{\frac{m_1-3}{2}}(yx)^{\frac{m_3-1}{2}}y \to q^2y^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}},$
$(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}} \to q^{-2}(xy)^{\frac{m_3-1}{2}}xY^{\frac{m_1-3}{2}}(xy)^{\frac{m_3-1}{2}}X^{\frac{m_1-1}{2}}\}.$

This system is constructed by a modification of the finite complete rewriting system $R'$ for an ordinary triangle group $\langle x, y | x^{m_1} = y^{m_2} = 1, (xy)^{m_3} = 1\rangle$ (see [8]). If we substitute every letter $q$ and $q^{-1}$ in the rules of $R$ by an empty word then we obtain $R'$ exactly.

**Proposition 5.** *The constructed system $R$ is a finite complete rewriting system for the group $E'$.*

*Proof.* It is suffice to prove the following:

a) the system $R$ is Noetherian;

b) the system $R$ is confluent;

c) the system $R$ is corresponding to the group $E'$, that is, the requirements of the proposition 4 are satisfied.

Let us impose, on the words from $\{x, y, q\}^{\pm}$, the following partial order. Let $x < y < X < Y$. On the words from $\{x, y\}^{\pm}$ we set the shortlex order. That is, $W_1 < W_2$ if $|W_1| < |W_2|$ and $t_1W_1 < t_2W_2$ for any words $W_1, W_2$ with $|W_1| = |W_2|$ and any letters $t_1, t_2$ with $t_1 < t_2$. Let $eraseQ$ is a function from $\{x, y, q\}^{\pm}$ to $\{x, y\}^{\pm}$ that erases all entries of the letters $q$ and $q^{-1}$ from the word. Set $W_1 < W_2$ if $eraseQ(W_1) < eraseQ(W_2)$. If $eraseQ(W_1) = eraseQ(W_2)$ we set $W_1 < W_2$ if $W_2$ can be obtained from $W_1$ using rules $xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY,\ xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y,\ qq^{-1} \to 1,\ q^{-1}q \to 1$. This set of rules is equivalent to shifting the letters $q, q^{-1}$ to the left and the erasing entries of $qq^{-1}, q^{-1}q$.

It is not hard to see that the constructed binary relation $<$ is actually a partial order on words from $\{x, y, q\}^{\pm}$ and there is no infinite descending chains with respect to this order.

It can be easily verified that if $(W_1 \to W_2) \in R$ then $W_2 < W_1$. Therefore if $W_1 \to_R W_2$ then $W_2 < W_1$. Hence $R$ is Noetherian.

Checking the requirements of Proposition 4 is also a simple, straightforward task.

The hardest part is to prove the confluence of $R$. This can be done using critical pairs analysis as in the Knuth-Bendix algorithm. This analysis for $R$ is essentially the same as for $R'$. There are different implementations of the Knuth-Bendix algorithm such as *kbmag* in the computer algebra system GAP which can be used to complete this task.

So $R$ is a finite complete rewriting system for the group $E'$. □

It is not hard to see that $\forall k \in \mathbb{N} :\ q^k \in Irr(R)$, hence $q$ has infinite order in $E$. End of proof of Proposition 2. Theorem 1 is a direct consequence of Propositions 1 and 2. □

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Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

*E-mail address*: dandan.ua@gmail.com

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