Methods of Functional Analysis and Topology
Vol. 19 (2013), no. 1, pp. 73–79

ON STABLE $\mathcal{C}$-SYMMETRIES FOR A CLASS OF $\mathcal{PT}$-SYMMETRIC OPERATORS

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Abstract. Recently, much attention is paid to the consideration of physical models described by $\mathcal{PT}$-symmetric Hamiltonians. In this paper, we establish a necessary and sufficient condition for existence of a stable $\mathcal{C}$-symmetry for a class of $\mathcal{PT}$-symmetric extensions of a symmetric operator S with deficiency indices $\langle 2, 2\rangle$.

1. Introduction

Recently, the models using $\mathcal{PT}$-symmetric Hamiltonians instead of self-adjoint ones became very popular in physics. Such models are studied by so-called $\mathcal{PT}$-symmetric quantum mechanics [1].

In order that a $\mathcal{PT}$-symmetric operator A can be used to construct such model, it is necessary that A belongs to the domain of an exact $\mathcal{PT}$-symmetry, i.e., it must have the property of $\mathcal{C}$-symmetry (see Definition 4.1). If a $\mathcal{PT}$-symmetric operator A is a proper extension of a symmetric operator S, the concept of stable $\mathcal{C}$-symmetry looks more natural (instead of more general concept of $\mathcal{C}$-symmetry) [2]. In the paper, we establish a necessary and sufficient condition of existence of a stable $\mathcal{C}$-symmetry for a class of $\mathcal{PT}$-symmetric extensions of a symmetric operator S with deficiency indices $\langle 2, 2\rangle$.

2. Preliminaries

Let $\mathfrak{H}$ be the Hilbert space with scalar product $(\cdot,\cdot)$.

Definition 2.1. *A unitary involution is an operator $\mathcal{P}$, $\mathcal{D}(\mathcal{P}) = \mathfrak{H}$, which satisfies the conditions*

$$\mathcal{P}^2 = I, \quad (\mathcal{P}f, \mathcal{P}g) = (f, g), \quad \forall f, g \in \mathfrak{H}.$$

Definition 2.2. *A conjugation operator is an operator $\mathcal{T}$, $\mathcal{D}(\mathcal{T}) = \mathfrak{H}$, which satisfies the conditions*

$$\mathcal{T}^2 = I, \quad (\mathcal{T}f, \mathcal{T}g) = (g, f), \quad \forall f, g \in \mathfrak{H}.$$

In what follows, we assume that the operators $\mathcal{P}$ and $\mathcal{T}$ commute, that is $\mathcal{PT} = \mathcal{TP}$.

Definition 2.3. *A closed linear operator A densely defined in the space $\mathfrak{H}$ is called $\mathcal{PT}$-symmetric if the equality*

$$\mathcal{PT}Af = A\mathcal{PT}f$$

holds for all elements f from the domain of definition $\mathcal{D}(A)$ of the operator A.

2010 *Mathematics Subject Classification.* Primary 47A57; Secondary 47A10, 47B25, 47B50, 47A55, 81Q15.

Key words and phrases. $\mathcal{PT}$-symmetric Hamiltonians, exact $\mathcal{PT}$-symmetry, real spectrum, stable $\mathcal{C}$-symmetry, extension theory of symmetric operators, boundary triplets, Krein spaces, Clifford algebra.

Suppose there exists a unitary involution $\mathcal{R}$ satisfying the following conditions:

$$\mathcal{PR} = -\mathcal{RP}, \quad \mathcal{RT} = \mathcal{TR}. \tag{2.1}$$

It follows form the first relation in (2.1) that the unitary involutions $\mathcal{P}$ and $\mathcal{R}$ can be considered as generating elements of the complex Clifford algebra [3],

$$\mathcal{C}l_2(\mathcal{P}, \mathcal{R}) := \text{span}\,\{I, \mathcal{P}, \mathcal{R}, i\mathcal{RP}\}.$$

Let S be a densely defined symmetric operator in $\mathfrak{H}$. We assume that S satisfies the conditions

$$S\mathcal{P}f = \mathcal{P}Sf, \quad S\mathcal{R}f = \mathcal{R}Sf, \quad S\mathcal{T}f = \mathcal{T}Sf, \quad \forall f \in \mathcal{D}(S). \tag{2.2}$$

The operator S is $\mathcal{PT}$-symmetric due to the first and the third relations in (2.2).

The subject of our studies is a special class of proper $\mathcal{PT}$-symmetric extensions of S

$$S \subset A \subset S^*, \quad \mathcal{PT}A = A\mathcal{PT}.$$

To describe these extensions we will use the technique of boundary triplets [4, 5]. In our case, it is convenient to use boundary triplets $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of the operator S^* satisfying the following condition (such boundary triplets are called *admissible*).

Condition I. *The relations*

$$\mathcal{P}_\mathcal{H}\Gamma_j = \Gamma_j\mathcal{P}, \quad \mathcal{R}_\mathcal{H}\Gamma_j = \Gamma_j\mathcal{R}, \quad \mathcal{T}_\mathcal{H}\Gamma_j = \Gamma_j\mathcal{T}, \quad j = 0, 1 \tag{2.3}$$

correctly determine the unitary involutions $\mathcal{P}_\mathcal{H}$ and $\mathcal{R}_\mathcal{H}$, and a conjugation operator $\mathcal{T}_\mathcal{H}$ in $\mathcal{H}$ with the following properties:

$$\mathcal{P}_\mathcal{H}\mathcal{R}_\mathcal{H} = -\mathcal{R}_\mathcal{H}\mathcal{P}_\mathcal{H}, \quad \mathcal{P}_\mathcal{H}\mathcal{T}_\mathcal{H} = \mathcal{T}_\mathcal{H}\mathcal{P}_\mathcal{H}, \quad \mathcal{R}_\mathcal{H}\mathcal{T}_\mathcal{H} = \mathcal{T}_\mathcal{H}\mathcal{R}_\mathcal{H}.$$

Roughly speaking, the admissible boundary triplets should provide the images $\mathcal{P}_\mathcal{H}$, $\mathcal{R}_\mathcal{H}$, and $\mathcal{T}_\mathcal{H}$ of operators $\mathcal{P}$, $\mathcal{R}$, and $\mathcal{T}$ in the auxiliary space $\mathcal{H}$ with the preservation of properties of the original operators.

Remark 2.1. *Admissible boundary triplets exist if the operator S has a real point of regular type*[1] [6, Lemma 3.1].

Our main attention is concentrated on the class $\{A_T\}$ of proper extensions, which are defined as the restrictions of S^* onto the domains

$$\mathcal{D}(A_T) = \{f \in \mathcal{D}(S^*) \mid T\Gamma_0 f = \Gamma_1 f\}, \tag{2.4}$$

where T is a bounded operator in $\mathcal{H}$ and $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is an admissible boundary triplet of S^*.

Lemma 2.1. ([6, Proposition 3.1]). *Let A_T be determined by (2.4). Then A_T is $\mathcal{PT}$-symmetric if and only if the corresponding operator T is a $\mathcal{P}_\mathcal{H}\mathcal{T}_\mathcal{H}$-symmetric in $\mathcal{H}$.*

As a rule, we will consider symmetric operators S with deficiency indices $\langle 2, 2\rangle$. In that case $\dim \mathcal{H} = 2$, and hence, the action of the operator T can be specified by the matrix $\mathbf{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$, $t_{ij} \in \mathbb{C}$. The matrix $\mathbf{T}$ depends on the choice of an orthonormal basis in $\mathcal{H}$, but such quantities as the determinant and the trace

$$\det T = t_{11}t_{22} - t_{12}t_{21}, \quad \text{Tr } T = t_{11} + t_{22}$$

are constants (invariants) of the operator T.

It follows from Lemma 2.1 and the results of [8] that if A_T is $\mathcal{PT}$-symmetric, then the determinant $\det T$ and the trace $\text{Tr } T$ of the corresponding operator T in (2.4) are real numbers.

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[1]A number r is called *a point of regular type* of an operator A, if there exists $c = c(r) > 0$ such that $||(A - rI)\, f|| \geq c\, ||f||$ for all $f \in \mathcal{D}(A)$ [7].

3. Condition of the reality of the spectrum

Let S satisfy (2.2) and let S have deficiency indices $\langle 2,2\rangle$. Then the Weyl function $M(\mu)$ of S associated with an admissible boundary triplet has the form $M(\cdot) = m(\cdot)\, I$, where $m(\cdot)$ is a scalar-valued function [2].

In [6], a criterion for the existence of non-real eigenvalues of a $\mathcal{PT}$-symmetric extension A_T was obtained (see Theorem 3.3 in [6]). Rewriting it in terms of det T and Tr T we obtain the following theorem.

Theorem 3.1. *Let a symmetric operator S with deficiency indices $\langle 2,2\rangle$ have a real point of regular type and satisfy (2.2), let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be an admissible boundary triplet of S^*, and let $M(\mu) = m(\mu)\, I$ be the Weyl function of S associated with $(\mathcal{H}, \Gamma_0, \Gamma_1)$. Then the $\mathcal{PT}$-symmetric extension A_T determined by (2.4) has an eigenvalue $\mu \in \mathbb{C}\backslash\mathbb{R}$ if and only if*

$$\left\{ \begin{array}{l} \mathrm{Re}\; m(\mu) = \frac{1}{2}\,\mathrm{Tr}\; T, \\ (\mathrm{Im}\; m(\mu))^2 = \det T - \left(\frac{1}{2}\,\mathrm{Tr}\; T\right)^2 . \end{array} \right. \tag{3.1}$$

Theorem 3.1 implies the following sufficient condition of the reality of the spectrum.

Corollary 3.1. *If the inequality*

$$4 \det T < (\mathrm{Tr}\; T)^2 \tag{3.2}$$

is satisfied then the $\mathcal{PT}$-symmetric extension A_T has real spectrum.

4. A necessary and sufficient condition for existence of a stable $\mathcal{C}$-symmetry

Definition 4.1. *A $\mathcal{PT}$-symmetric operator A has the property of $\mathcal{C}$-symmetry (or, briefly, has the $\mathcal{C}$-symmetry) if there exists a bounded linear operator $\mathcal{C}$ ($\mathcal{C} \neq \pm I$) in $\mathfrak{H}$ such that*

$$\mathcal{C}^2 = I, \quad \mathcal{CPT} = \mathcal{PTC}, \quad \mathcal{C}A = A\mathcal{C}. \tag{4.1}$$

Lemma 4.1. ([9, Lemma 2.7]). *The operator $\mathcal{C} \in \mathcal{C}l_2(\mathcal{P}, \mathcal{R})$ ($\mathcal{C} \neq \pm I$) satisfies the first two equalities in (4.1) if and only if there exist $\chi \in \mathbb{R}$ and $\xi \in [0, 2\pi)$ such that*

$$\mathcal{C} = e^{\chi i \mathcal{R}\mathcal{P}_\xi}\mathcal{P}_\xi = (\mathrm{ch}\,\chi \cdot I + \mathrm{sh}\,\chi \cdot i\mathcal{R}\mathcal{P}_\xi)\,\mathcal{P}_\xi, \tag{4.2}$$

where the unitary involution $\mathcal{P}_\xi$ is defined as follows:

$$\mathcal{P}_\xi = e^{i\xi\mathcal{R}}\mathcal{P} = (\cos\,\xi \cdot I + i \sin\,\xi \cdot \mathcal{R})\mathcal{P}. \tag{4.3}$$

$\mathcal{C}$-symmetries are not always uniquely defined for $\mathcal{PT}$-symmetric operators. In particular, if we consider a symmetric operator S that satisfies (2.2), then every operator $\mathcal{C}$ of the form (4.2) turns out to be the $\mathcal{C}$-symmetry for S. In the case of deficiency indices $\langle 2,2\rangle$ of S, this assertion can be strengthened. Namely, some additional analysis shows that the set of all $\mathcal{C}$-symmetries for S is exhausted by the operators of the form (4.2), i.e. $\mathcal{C}$ is the $\mathcal{C}$-symmetry for S if and only if $\mathcal{C} = e^{\chi i \mathcal{R}\mathcal{P}_\xi}\mathcal{P}_\xi$ for some $\chi \in \mathbb{R}$ and $\xi \in [0, 2\pi)$.

Definition 4.2. *Let a symmetric operator S have deficiency indices $\langle 2,2\rangle$ and let S satisfy (2.2). A $\mathcal{PT}$-symmetric extension A of S has the property of stable $\mathcal{C}$-symmetry if A has the $\mathcal{C}$-symmetry realized by an operator $\mathcal{C} = e^{\chi i \mathcal{R}\mathcal{P}_\xi}\mathcal{P}_\xi$ for some choice of $\chi \in \mathbb{R}$ and $\xi \in [0, 2\pi)$.*

Theorem 4.1. *Let a symmetric operator S satisfy the conditions of Theorem 3.1 and let A_T be a $\mathcal{PT}$-symmetric non-self-adjoint extension of S. Then A_T has the stable $\mathcal{C}$-symmetry if and only if the inequality (3.2) is satisfied.*

Proof. First, we prove the necessary condition. Let A_T have the stable $\mathcal{C}$-symmetry. Then $A_T\mathcal{C} = \mathcal{C}A_T$, where $\mathcal{C} = e^{\chi i\mathcal{R}\mathcal{P}_\xi}\mathcal{P}_\xi$.

Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be an admissible boundary triplet. In that case, relations (2.3) hold for the boundary operators Γ_j and, hence,

$$\mathcal{C}_\mathcal{H}\Gamma_0 = \Gamma_0\mathcal{C}, \quad \mathcal{C}_\mathcal{H}\Gamma_1 = \Gamma_1\mathcal{C}, \tag{4.4}$$

where the operator $\mathcal{C}_\mathcal{H} = e^{\chi i\mathcal{R}_\mathcal{H}\mathcal{P}_{\xi_\mathcal{H}}}\mathcal{P}_{\xi_\mathcal{H}}$ acts in $\mathcal{H}$.

Using (4.4) and reasoning by analogy with the proof of part 1 of Proposition 3.1 in [6], we conclude that the equality $A_T\mathcal{C} = \mathcal{C}A_T$ is equivalent to the relation

$$T\mathcal{C}_\mathcal{H} = \mathcal{C}_\mathcal{H}T. \tag{4.5}$$

It follows from [6, Theorem 3.1] that the $\mathcal{PT}$-symmetric operator A_T can be interpreted as a self-adjoint operator in the Krein space $(\mathfrak{H}, [\cdot,\cdot]_{\mathcal{P}_\xi})$ for some choice of $\xi \in [0, 2\pi)$. Then, according to [6, Lemma 3.3],

$$T = \beta_0 I + \beta_1\mathcal{P}_{\xi_\mathcal{H}} + \beta_2\mathcal{R}_\mathcal{H}, \quad \beta_0, \beta_1 \in \mathbb{R}, \quad \beta_2 \in i\mathbb{R}. \tag{4.6}$$

Note that $\beta_2 \neq 0$ in (4.6). Indeed, if $\beta_2 = 0$, then $T^* = T$ and hence, $A_T^* = A_T$ that contradicts to the assumption about nonself-adjointness of A_T.

Substituting (4.6) in (4.5) and taking into account that

$$\mathcal{C}_\mathcal{H} = e^{\chi i\mathcal{R}_\mathcal{H}\mathcal{P}_{\xi_\mathcal{H}}}\mathcal{P}_{\xi_\mathcal{H}} = (\operatorname{ch}\chi \cdot I + \operatorname{sh}\chi \cdot i\mathcal{R}_\mathcal{H}\mathcal{P}_{\xi_\mathcal{H}})\,\mathcal{P}_{\xi_\mathcal{H}}, \tag{4.7}$$

we conclude that the commutation relation (4.5) holds if and only if

$$\operatorname{ch}\chi \cdot \beta_2 = i\operatorname{sh}\chi \cdot \beta_1. \tag{4.8}$$

Using (4.6) we can express the quantities $\det T$ and $\operatorname{Tr} T$ in terms of β_j. To this end we choose an orthonormal basis of $\mathcal{H}$ in such a way that the action of $\mathcal{P}_\mathcal{H}$ and $\mathcal{R}_\mathcal{H}$ is determined, respectively, by the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $-i\mathcal{R}_\mathcal{H}\mathcal{P}_\mathcal{H}$ corresponds to $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and hence, the operator $\mathcal{P}_{\xi_\mathcal{H}}$ is determined by $\cos\xi \cdot \sigma_1 - \sin\xi \cdot \sigma_2$.

Summing up the obtained relations and using (4.6), we arrive at the conclusion that the operator T corresponds to the matrix

$$\mathbf{T} = \begin{pmatrix} \beta_0 + \beta_2 & \beta_1 e^{i\xi} \\ \beta_1 e^{-i\xi} & \beta_0 - \beta_2 \end{pmatrix}.$$

Therefore,

$$\det T = \beta_0^2 - \beta_1^2 + |\beta_2|^2, \quad \operatorname{Tr} T = 2\,\beta_0. \tag{4.9}$$

It follows from (4.8) that $|\beta_2|^2 < \beta_1^2$. Then, $4\det T < 4\,\beta_0^2 = (\operatorname{Tr} T)^2$ that completes the proof of the necessary condition.

Now we turn to the proof of the sufficient condition. Suppose that inequality (3.2) is satisfied. Taking (4.9) into account we deduce that $|\beta_2|^2 < \beta_1^2$. Hence, we can choose $\chi \in \mathbb{R}$ such that (4.8) takes place. Then the operator $\mathcal{C}_\mathcal{H}$ defined by (4.7) satisfies (4.5) and the operator $\mathcal{C} = e^{\chi i\mathcal{R}\mathcal{P}_\xi}\mathcal{P}_\xi$. commutes with A_T. Thus, A_T has the stable $\mathcal{C}$-symmetry. The theorem is proved. □

5. Example

To illustrate the theorems, which were formulated in the previous sections, we consider the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V(x) \tag{5.1}$$

with singular potential

$$V(x) = a\langle\delta, \cdot\rangle\delta(x) + b\langle\delta', \cdot\rangle\delta(x) + c\langle\delta, \cdot\rangle\delta'(x) + d\langle\delta', \cdot\rangle\delta'(x), \tag{5.2}$$

where δ and δ' are the delta-function and its derivative, respectively, and $a, b, c, d \in \mathbb{C}$.

The differential expression (5.1) with the potential (5.2) is formal in the Hilbert space $L_2(\mathbb{R})$, but it can be regarded as a mapping of the Sobolev space $W_2^2(\mathbb{R})$ in the negative Sobolev space $W_2^{-2}(\mathbb{R})$. In order to determine the operator realization of (5.1) in the space $L_2(\mathbb{R})$, we use the well-known regularization method [10].

First of all, note that the minimal operator [11, c. 173] generated by (5.1) is defined as

$$S = -\frac{d^2}{dx^2}, \quad D(S) = \{f(x) \in W_2^2(\mathbb{R}) \mid f(0) = f'(0) = 0\}. \tag{5.3}$$

In the Hilbert space $L_2(\mathbb{R})$, the operator being adjoint to S is

$$S^* = -\frac{d^2}{dx^2}, \quad D(S^*) = W_2^2(\mathbb{R}\backslash\{0\}) := W_2^2(\mathbb{R}_-) \oplus W_2^2(\mathbb{R}_+). \tag{5.4}$$

Next, let us expand the action of δ-function and its derivative on the domain of the adjoint operator $D(S^*)$ as follows:

$$\langle\delta_{\text{ex}}, f\rangle = \frac{f(+0) + f(-0)}{2}, \quad \langle\delta'_{\text{ex}}, f\rangle = -\frac{f'(+0) + f'(-0)}{2}, \quad \forall f \in D(S^*).$$

Then the regularization $A_{\text{reg}} : D(S^*) \to W_2^{-2}(\mathbb{R})$ of (5.1) has the form

$$A_{\text{reg}} = -\frac{d^2}{dx^2} + a\langle\delta_{\text{ex}}, \cdot\rangle\delta(x) + b\langle\delta'_{\text{ex}}, \cdot\rangle\delta(x) + c\langle\delta_{\text{ex}}, \cdot\rangle\delta'(x) + d\langle\delta'_{\text{ex}}, \cdot\rangle\delta'(x),$$

where the action of $-\frac{d^2}{dx^2}$ on the function of $D(S^*) = W_2^2(\mathbb{R}\backslash\{0\})$ must be understood in the distribution sense [12].

Definition 5.1. *The operator realization of expression (5.1) in the space $L_2(\mathbb{R})$ is given by the formula*

$$A = A_{\text{reg}}\restriction_{D(A)}, \quad D(A) = \{f \in D(S^*) \mid A_{\text{reg}} f \in L_2(\mathbb{R})\}. \tag{5.5}$$

Definition 5.1 establishes a direct connection between the operator realization A and the coefficients a, b, c, d of the potential V. Repeating the arguments of Theorem 1 in [13], we conclude that the operator A given by (5.5) is the extension of the symmetric operator S and the restriction of the operator S^*, i.e., A is a proper extension of S.

We define the unitary involutions $\mathcal{P}$ and $\mathcal{R}$, and the conjugation operator $\mathcal{T}$ as

$$\mathcal{P}f(x) = f(-x), \quad \mathcal{R}f(x) = \operatorname{sign}(x)f(x), \quad \mathcal{T}f(x) = \overline{f(x)}, \quad \forall f \in L_2(\mathbb{R}). \tag{5.6}$$

It is easy to see that the operators $\mathcal{P}$, $\mathcal{R}$, and $\mathcal{T}$ satisfy conditions (2.1). Furthermore, the operator S defined by formula (5.3) satisfies conditions (2.2). Therefore, the operator S is $\mathcal{PT}$-symmetric and it commutes with all elements of the Clifford algebra $\mathcal{C}l_2(\mathcal{P}, \mathcal{R})$.

Let us now describe the extensions of the operator S in terms of boundary triplets. To that end, we will present the operator S in the form $S = S_+ + S_-$, where S_+ acts in the space $L_2(\mathbb{R}_+)$, and S_- acts in the space $L_2(\mathbb{R}_-)$. Since the operator S has deficiency indices $\langle 2, 2\rangle$, then the operators S_+ and S_- have deficiency indices $\langle 1, 1\rangle$, and hence, one can construct the boundary triplets of S_+^* and S_-^*. Then the proof of Lemma 2.11 in [14] implies that the admissible boundary triplet of S^* can be defined as

$$\mathcal{H} = \mathbb{C}^2, \quad \Gamma_0 f = \begin{pmatrix} \Gamma_0^+ f_+ \\ \Gamma_0^+ \mathcal{P} f_- \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} \Gamma_1^+ f_+ \\ \Gamma_1^+ \mathcal{P} f_- \end{pmatrix}, \tag{5.7}$$

where $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ is an *arbitrary* boundary triplet of S_+^*,

$$f(x) = f_+(x) + f_-(x), \quad f_+(x) \in L_2(\mathbb{R}_+), \quad f_-(x) \in L_2(\mathbb{R}_-).$$

Define the boundary triplet of S_+^* as follows:

$$\mathcal{H} = \mathbb{C}, \quad \Gamma_0^+ f = f(+0), \quad \Gamma_1^+ f = f'(+0). \tag{5.8}$$

Then the admissible boundary triplet of S^* constructed by (5.7) takes the form

$$\mathcal{H} = \mathbb{C}^2, \quad \Gamma_0 f = \begin{pmatrix} f(+0) \\ f(-0) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} f'(+0) \\ -f'(-0) \end{pmatrix}. \tag{5.9}$$

The corresponding images $\mathcal{P}_\mathcal{H}$ and $\mathcal{R}_\mathcal{H}$ of operators $\mathcal{P}$ and $\mathcal{R}$ (see (2.3)) coincide with the Pauli matrices σ_1 and σ_3, respectively. Furthermore, the operator $\mathcal{T}_\mathcal{H}$ acts as the componentwise complex conjugation operator τ

$$\tau \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}, \quad x, y \in \mathbb{C}. \tag{5.10}$$

Repeating the arguments of Theorem 1 in [13], we can prove the following proposition.

Proposition 5.1. *Assume that* $d \neq 0$ *in (5.2). Then the operator realization* A *defined by (5.5) coincides with the restriction of* $S^* = -\frac{d^2}{dx^2}$ *onto the set*

$$D(A_T) = \{f \in D(S^*) \mid \mathbf{T}\Gamma_0 f = \Gamma_1 f\}, \tag{5.11}$$

where the operators Γ_0 *and* Γ_1 *are defined by (5.9), and the matrix* $\mathbf{T}$ *has the form*

$$\mathbf{T} = \frac{1}{4d} \begin{pmatrix} \Delta - 4 + 2(b+c) & \Delta + 4 - 2(b-c) \\ \Delta + 4 + 2(b-c) & \Delta - 4 - 2(b+c) \end{pmatrix}, \quad \Delta = ad - bc. \tag{5.12}$$

Remark 5.1. *The condition* $d \neq 0$ *in Proposition 5.1 has a technical character and it deals with the choice of admissible boundary triplet.*

Now we can give description of the $\mathcal{PT}$-symmetric extensions in term of coefficients a, b, c, d.

Theorem 5.1. ([15, Theorem 2]). *The extension* A_T *given by formula (5.11) is* $\mathcal{PT}$*-symmetric if and only if*

$$a, d \in \mathbb{R}, \quad b, c \in i\mathbb{R}. \tag{5.13}$$

Proof. According to Lemma 2.1, the operator A_T is $\mathcal{PT}$-symmetric if and only if the operator T is $\mathcal{P}_\mathcal{H}\mathcal{T}_\mathcal{H}$-symmetric in $\mathcal{H} = \mathbb{C}^2$. The latter condition is reduced to the equality $\sigma_1 \tau \mathbf{T} h = \mathbf{T} \sigma_1 \tau h, \ \forall\, h \in \mathbb{C}^2$. Using (5.10) and (5.12), we rewrite this equality in the following form:

$$\begin{cases} d(\overline{\Delta} + 4 + 2(\overline{b} - \overline{c})) = \overline{d}(\Delta + 4 - 2(b-c)), \\ d(\overline{\Delta} - 4 - 2(\overline{b} + \overline{c})) = \overline{d}(\Delta - 4 + 2(b+c)), \\ d(\overline{\Delta} - 4 + 2(\overline{b} + \overline{c})) = \overline{d}(\Delta - 4 - 2(b+c)), \\ d(\overline{\Delta} + 4 - 2(\overline{b} - \overline{c})) = \overline{d}(\Delta + 4 + 2(b-c)). \end{cases}$$

It is easy to verify that the obtained system is equivalent to conditions (5.13). The theorem is proved. □

The subset of $\mathcal{PT}$-symmetric *nonself-adjoint* extensions A_T is distinguished by the condition $|b| + |c| \neq 0$. For these operators, we can describe the case of stable $\mathcal{C}$-symmetry.

Proposition 5.2. *The* $\mathcal{PT}$*-symmetric nonself-adjoint extension* A_T *given by formula (5.11) has the stable* $\mathcal{C}$*-symmetry if and only if*

$$(\Delta - 4)^2 + 16ad > 0. \tag{5.14}$$

Proof. Theorem 4.1 implies that a $\mathcal{PT}$-symmetric nonself-adjoint extension A_T has the stable $\mathcal{C}$-symmetry if and only if inequality (3.2) is satisfied. Using (5.12), we conclude that $4 \det T = -\dfrac{4ad}{d^2}$ and $(\mathrm{Tr}\, T)^2 = \dfrac{(\Delta - 4)^2}{4d^2}$. Substituting these quantities into (3.2), we obtain(5.14). □

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Received 21/11/2012; Revised 01/12/2012