Methods of Functional Analysis and Topology
Vol. 19 (2013), no. 1, pp. 40–54

THE TWO-DIMENSIONAL MOMENT PROBLEM IN A STRIP

S. M. ZAGORODNYUK

To Professor M. L. Gorbachuk on the occasion of his 75th birthday

Abstract. In this paper we study the two-dimensional moment problem in a strip $\Pi(R) = \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_2| \leq R\}$, $R > 0$. We obtained an analytic parametrization of all solutions of this moment problem. Usually the problem is reduced to an extension problem for a pair of commuting symmetric operators but we have no possibility to construct such extensions in larger spaces in a direct way. It turns out that we can find solutions without knowing the corresponding extensions in larger spaces. We used the fundamental results of Shtraus on generalized resolvents and some results from the measure theory.

1. Introduction

In this paper we consider the following problem: to find a non-negative Borel measure μ in a strip

$$\Pi = \Pi(R) = \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_2| \leq R\}, \quad R > 0,$$

such that

$$\int_\Pi x_1^m x_2^n d\mu = s_{m,n}, \quad m, n \in \mathbb{Z}_+, \tag{1}$$

where $\{s_{m,n}\}_{m,n\in\mathbb{Z}_+}$ is a prescribed sequence of complex numbers. This problem is said to be *the two-dimensional moment problem in a strip*.

The two-dimensional moment problem and the complex moment problem have an extensive literature, see books [1], [2], [3], [4] surveys in [5], [6], [7] and [8]. The moment problem (1) has a solution if and only if for arbitrary complex numbers $\alpha_{m,n}$ (where all but finite numbers are zeros) the following relations hold:

$$\sum_{m,n,k,l=0}^{\infty} \alpha_{m,n} \overline{\alpha_{k,l}} s_{m+k,n+l} \geq 0, \tag{2}$$

$$\sum_{m,n,k,l=0}^{\infty} \alpha_{m,n} \overline{\alpha_{k,l}} (R^2 s_{m+k,n+l} - s_{m+k,n+l+2}) \geq 0. \tag{3}$$

This result is not new and it follows directly from more general results on the moment problems for semi-algebraic sets in [9] (see Theorem 5.1 and Proposition 1.3 therein) and in [10, Corollary 10] (for the case $R = 1$ see [11, Theorem 6]). Conditions of solvability and a description of all solutions for the two-dimensional moment problem on a semi-algebraic set (even for the multidimensional case) in terms of a dimensional extension were obtained in [12, Theorem 2.7]. Also we refer to [13, Theorem 4.2]. However, to the best of our knowledge, no analytic description for solutions of the two-dimensional moment problem in a strip was obtained. In general, only few steps are done to describe all solutions of the two-dimensional (or multidimensional) moment problem, see [14], [12],

2000 *Mathematics Subject Classification.* Primary 44A60; Secondary 30E05.
Key words and phrases. Moment problem, measure, generalized resolvent.

[8, Section 7], [15]. In [15] we proposed an algorithm for solving the two-dimensional moment problem. Roughly speaking, this algorithm reduces to the solving of linear systems of equations with an infinite number of variables by the Gauss algorithm. The two-dimensional moment problem has a solution iff the (countable) intersection of the sets of solutions of such systems and some other explicitly given sets is non-empty. Then the points of the limit set play a role of parameters in a description of all solutions (we refer to [15] for more details). Operator multidimensional moment problems were studied in [16], [17]. For a moment problems with an infinite number of variables we refer to the book of Berezansky and Kondratiev [18] and references therein.

The operator approach to the moment problems probably takes its origin in 1940–1943, in papers of Neumark [19], [20] for the case of the Hamburger moment problem. Neumark obtained the Nevanlinna formula using his results on the generalized resolvents of a symmetric operator with the deficiency index $(1,1)$. Then this approach was developed by Krein and Krasnoselskiy in [21], using the ideas of Krein of 1946–1948 [22], [23]. Various modifications appeared afterwords. Concerning the multidimensional moment problem we refer to the book of Berezansky [3], historical comments and the bibliography therein, to the papers of Fuglede [5], Putinar and Schmüdgen [8] and references therein.

The usual operator-theoretic core of our problem (1) is the following: in a Hilbert space H there are given a symmetric operator A and a bounded self-adjoint operator B of the norm $\|B\| \leq R$ and commuting with A. One has to find extensions $\widetilde{A} \supseteq A$, $\widetilde{B} \supseteq B$, in a Hilbert space $\widetilde{H} \supseteq H$, such that $\widetilde{A}$ and $\widetilde{B}$ are self-adjoint, commute and $\|\widetilde{B}\| \leq R$. However, this core does not help in our case: *we do not know how to construct such extensions in larger spaces.*

Suppose we know $(\widetilde{A}, \widetilde{B})$ satisfying the above properties. Let $\widetilde{E}, \widetilde{F}$ be their orthogonal resolutions of the identity, respectively. Let F be the orthogonal resolution of the identity of B. In our case we have

$$F = P_H^{\widetilde{H}} \widetilde{F}.$$

Set

$$\mathbf{E} = P_H^{\widetilde{H}} \widetilde{E}. \tag{4}$$

The pair $(\mathbf{E}, F)$ commute and generate a solution by the formula (13) below.

Question (A). Given is an arbitrary spectral function $\mathbf{E}$ of the operator A which commutes with the orthogonal spectral function F of B. Does there exists a pair $(\widetilde{A}, \widetilde{B})$ with the above-mentioned properties such that (4) holds? In other words, is the mapping

$$(\widetilde{A}, \widetilde{B}) \mapsto (\mathbf{E}, F)$$

surjective?

We do not see a direct way to answer Question (A) in the usual operator-theoretic interpretation. Of course, for an arbitrary spectral function $\mathbf{E}$ of the operator A there exists a self-adjoint operator $\widehat{A} \supseteq A$ in a Hilbert space $\widehat{H} \supseteq H$, such that $\mathbf{E} = P_H^{\widehat{H}} \widehat{E}$, where $\widehat{E}$ is the orthogonal resolution of the identity of the operator $\widehat{A}$. However it is not known that it commutes with some (at least one) extension $\widehat{B}$ of B in $\widehat{H}$ (even without the property $\|\widehat{B}\| \leq R$).

Thus, we do not see a way to solve the problem (1) in this general operator-theoretic formulation.

The choice of a concrete version of the operator approach depends on the concrete problem under consideration and the taste of the researcher. For our problem (1) we shall use the "pure operator" approach of Szökefalvi-Nagy and Koranyi ([24], [25], [2, p. 217]). The modern exposition of this approach and related results are presented in the book of Nikolski [26]. This approach allows to hide details which are not important for us.

We shall use the fundamental results of Shtraus on generalized resolvents [27], as well as some facts from the measure theory and properties of decomposable operators to obtain an analytic description of all solutions of the moment problem (1). Also we shall use our results in [28].

The problem of a continuation of a positive definite function in a strip to the whole plane was studied by Berezansky and Gorbachuk [29], by Levin and Ovcharenko [30]. In [29] the problem was reduced to the problem of a continuation of a positive definite matrix-valued function (of one variable) on an interval with additional properties. The results of Gorbachuk on a continuation of positive-definite operator-valued functions of one variable (with an additional condition of "maximality" of defect numbers) [31] were used. Since the matrix moment problem on an interval is always determinate, it is clear that such a reduction is impossible in the case of the two-dimensional moment problem in a strip.

On the other hand the approach from [30] was used by Ovcharenko to study the two-dimensional moment problem in [32]. However, no description of solutions was obtained there.

The reason of the above situation, probably, is that the "strip" appears in different circumstances in the above-mentioned problems: while the positive definite function is defined in a strip, the positive definite sequence (1) is defined for $m,n\in\mathbb{Z}_+$.

The paper is organized as follows. In Section 2 we describe solutions of the moment problem (1) in terms of spectral functions of the corresponding symmetric operators. As a first application of this description, we describe canonical solutions of the moment problem (see the precise definition below).

In Section 3 we obtain the main result of this paper: an analytic parametrization of all solutions of the two-dimensional moment problem in a strip. In a consequence, in Section 4 we derive conditions of the solvability and describe all solutions of the complex moment problem with the support in a strip.

In this paper we followed notions and notations from books [33], [34].

Notations. As usual, we denote by $\mathbb{R},\mathbb{C},\mathbb{N},\mathbb{Z},\mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. For a subset S of the complex plane we denote by $\mathfrak{B}(S)$ the set of all Borel subsets of S. Everywhere in this paper, all Hilbert spaces are assumed to be separable. By $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ we denote the scalar product and the norm in a Hilbert space H, respectively. The indices may be omitted in obvious cases. For a set M in H, by $\overline{M}$ we mean the closure of M in the norm $\|\cdot\|_H$. For $\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}$, $x_{m,n}\in H$, we write $\mathrm{Lin}\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}$ for the span of vectors $\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}$ and $\mathrm{span}\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}=\overline{\mathrm{Lin}\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}}$. The identity operator in H is denoted by E_H. For an arbitrary linear operator A in H, the operators A^*,$\overline{A}$,A^{-1} mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator A. By $\sigma(A)$, $\rho(A)$ we denote the spectrum of A and the resolvent set of A, respectively. We denote by $R_z(A)$ the resolvent function of A, $z\in\rho(A)$; $\Delta_A(z)=(A-zE_H)D(A)$, $z\in\mathbb{C}$. The norm of a bounded operator A is denoted by $\|A\|$. By $P^H_{H_1}=P_{H_1}$ we mean the operator of orthogonal projection in H on a subspace H_1 in H. By $\mathbf{B}(H)$ we denote the set of all bounded operators in H.

2. A description of solutions of the two-dimensional moment problem in a strip via spectral functions. Canonical solutions

Let the moment problem (1) be given and conditions (2) and (3) hold. Relations (2) mean that the kernel $K((m,n),(k,l))=s_{m+k,n+l}$, $m,n,k,l\in\mathbb{Z}_+$, is non-negative definite and by Lemma in [25, p. 177] (see also [35, pp. 361–363]) there exists a Hilbert space H

and a sequence $\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}$, $x_{m,n}\in H$, such that

$$(x_{m,n},x_{k,l})_H = s_{m+k,n+l},\quad m,n,k,l\in\mathbb{Z}_+. \tag{5}$$

Such a Hilbert space and a sequence of elements in it are not uniquely defined. We choose an arbitrary such a space H and $\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}$, and fix them for the rest of this paper.

Set $L=\mathrm{Lin}\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}$. Introduce the following operators:

$$A_0x=\sum_{m,n\in\mathbb{Z}_+}\alpha_{m,n}x_{m+1,n},\quad B_0x=\sum_{m,n\in\mathbb{Z}_+}\alpha_{m,n}x_{m,n+1}, \tag{6}$$

where

$$x=\sum_{m,n\in\mathbb{Z}_+}\alpha_{m,n}x_{m,n}\in L,\quad \alpha_{m,n}\in\mathbb{C}. \tag{7}$$

The values of A_0 and B_0 does not depend on the choice of the representation of x. To see that scalar multiply the corresponding values by an arbitrary element of L and use the properties of K. It is not hard to see that operators A_0 and B_0 are symmetric and therefore they are closable. Moreover, condition (3) implies that the operator B_0 is bounded. Set

$$A=\overline{A_0},\quad B=\overline{B_0}. \tag{8}$$

Observe that B is a bounded self-adjoint operator in H. We shall also need the following operator:

$$J_0x=\sum_{m,n\in\mathbb{Z}_+}\overline{\alpha_{m,n}}x_{m,n}, \tag{9}$$

where $x=\sum_{m,n\in\mathbb{Z}_+}\alpha_{m,n}x_{m,n}\in L$. The value J_0x does not depend on the choice of the representation for x. To verify that, calculate the norm of the difference of the corresponding values, using the properties of K. We can easily check that $(J_0x,J_0y)_H=(y,x)$, $x,y\in L$. By continuity we extend J_0 to a a conjugation J in H. Notice that J_0 commutes with A_0 and B_0.

Let $\widetilde{A}\supseteq A$ be a self-adjoint extension of A in a Hilbert space $\widetilde{H}\supseteq H$ and $E_{\widetilde{A}}$ be the spectral measure of $\widetilde{A}$. Recall that the function

$$\mathbf{R}_z(A):=P^{\widetilde{H}}_H R_z(\widetilde{A}),\quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{10}$$

is said to be a *generalized resolvent* of A. The function

$$\mathbf{E}_A(\delta):=P^{\widetilde{H}}_H E_{\widetilde{A}}(\delta),\quad \delta\in\mathfrak{B}(\mathbb{R}), \tag{11}$$

is said to be a *spectral measure* of A. There exists a one-to-one correspondence between generalized resolvents and (left-continuous) spectral measures according to the following relation [35]:

$$(\mathbf{R}_z(A)x,y)_H=\int_{\mathbb{R}}\frac{1}{t-z}d(\mathbf{E}_Ax,y)_H,\quad x,y\in H. \tag{12}$$

The spectral measure $\mathbf{E}_A$ is defined by the Stieltjes-Perron inversion formula. The generalized resolvent $\mathbf{R}_z(A)$ and the spectral function $\mathbf{E}_A$, related by (12), are said *to belong to each other* [35, p. 378].

Theorem 2.1. *Let the moment problem (1) be given and conditions (2),(3) hold. Choose a Hilbert space H and a sequence $\{x_{m,n}\}_{m,n\in\mathbb{Z}_+}$, $x_{m,n}\in H$, such that relation (5) holds, and fix them. Consider operators A_0,B_0,A,B defined by (6) and (8). Then the following statements hold:*

1) *All solutions of the moment problem (6) have the following form:*

$$\mu(\delta)=((\mathbf{E}\times F)(\delta)x_{0,0},x_{0,0})_H,\quad \delta\in\mathfrak{B}(\Pi), \tag{13}$$

where F is the orthogonal spectral measure of B, $\mathbf{E}$ is a spectral measure of A which commutes with F. Here by $((\mathbf{E}\times F)(\delta)x_{0,0},x_{0,0})_H$ we denote the unique *non-negative Borel measure on $\mathbb{R}$ which is obtained by the Lebesgue continuation procedure from the following non-negative measure on rectangles*

$$((\mathbf{E}\times F)(I_{x_1}\times I_{x_2})x_{0,0},x_{0,0})_H := (\mathbf{E}(I_{x_1})F(I_{x_2})x_{0,0},x_{0,0})_H, \tag{14}$$

where $I_{x_1}\subset\mathbb{R}$, $I_{x_2}\subseteq[-R,R]$ are arbitrary intervals.

2) *Given an arbitrary spectral measure $\mathbf{E}$ of A which commutes with the spectral measure F of B, by relation (13) we can define a solution μ of the moment problem (1).*
3) *The correspondence between the spectral measures of A which commute with the spectral measure of B and solutions of the moment problem, which is established by (13), is bijective.*

Remark. It is straightforward to check that the measure in (14) is non-negative and additive. Moreover, repeating the standard arguments [36, Chapter 5, Theorem 2, p. 254–255] we conclude that the measure in (14) is σ-additive. Consequently, it has the (unique) Lebesgue continuation to a (finite) non-negative Borel measure on Π.

Proof. Let us check the first statement of the Theorem. Let μ be an arbitrary solution of the moment problem (1). We need to verify that it admits a representation of the form (13).

Consider the space L^2_μ of complex functions on Π which are square integrable with respect to the measure μ. The scalar product and the norm are given by

$$(f,g)_\mu = \int_\Pi f(x_1,x_2)\overline{g(x_1,x_2)}\,d\mu,\quad \|f\|_\mu = ((f,f)_\mu)^{\frac{1}{2}},\quad f,g\in L^2_\mu.$$

Consider the following operators:

$$A_\mu f(x_1,x_2) = x_1 f(x_1,x_2),\quad D(A_\mu)=\{f\in L^2_\mu:\ x_1 f(x_1,x_2)\in L^2_\mu\}, \tag{15}$$

$$B_\mu f(x_1,x_2) = x_2 f(x_1,x_2),\quad D(B_\mu)=L^2_\mu. \tag{16}$$

The operator A_μ is self-adjoint and the operator B_μ is self-adjoint and bounded. These operators commute and therefore the spectral measure E_μ of A_μ and the spectral measure F_μ of B_μ commute, as well.

Let $p(x_1,x_2)$ be a polynomial of the form (1) and $q(x_1,x_2)$ be a polynomial of the form (1) with $\beta_{m,n}\in\mathbb{C}$ instead of $\alpha_{m,n}$. Then

$$\begin{aligned}(p,q)_\mu &= \sum_{m,n,k,l\in\mathbb{Z}_+}\alpha_{m,n}\overline{\beta_{k,l}}\int_\Pi x_1^{m+k}x_2^{n+l}d\mu\\ &= \sum_{m,n,k,l\in\mathbb{Z}_+}\alpha_{m,n}\overline{\beta_{k,l}}s_{m+k,n+l}.\end{aligned}$$

On the other hand, we may write

$$\begin{aligned}\left(\sum_{m,n\in\mathbb{Z}_+}\alpha_{m,n}x_{m,n},\sum_{k,l\in\mathbb{Z}_+}\beta_{k,l}x_{k,l}\right)_H &= \sum_{m,n,k,l\in\mathbb{Z}_+}\alpha_{m,n}\overline{\beta_{k,l}}(x_{m,n},x_{k,l})_H\\ = \sum_{m,n,k,l\in\mathbb{Z}_+}\alpha_{m,n}\overline{\beta_{k,l}}K((m,n),(k,l)) &= \sum_{m,n,k,l\in\mathbb{Z}_+}\alpha_{m,n}\overline{\beta_{k,l}}s_{m+k,n+l}.\end{aligned}$$

Therefore

$$(p,q)_\mu = \left(\sum_{m,n\in\mathbb{Z}_+}\alpha_{m,n}x_{m,n},\sum_{k,l\in\mathbb{Z}_+}\beta_{k,l}x_{k,l}\right)_H. \tag{17}$$

Consider the following operator:

$$V[p] = \sum_{m,n\in\mathbb{Z}_+}\alpha_{m,n}x_{m,n},\quad p=\sum_{m,n\in\mathbb{Z}_+}\alpha_{m,n}x_1^m x_2^n. \tag{18}$$

Here by $[p]$ we mean the class of equivalence in L^2_μ defined by p. If two different polynomials p and q belong to the same class of equivalence then by (17) we get

$$0 = \|p-q\|^2_\mu = (p-q,p-q)_\mu = \left(\sum_{m,n\in\mathbb{Z}_+} (\alpha_{m,n} - \beta_{m,n}) x_{m,n}, \sum_{k,l\in\mathbb{Z}_+} (\alpha_{k,l} - \beta_{k,l}) x_{k,l} \right)_H$$
$$= \left\| \sum_{m,n\in\mathbb{Z}_+} \alpha_{m,n} x_{m,n} - \sum_{m,n\in\mathbb{Z}_+} \beta_{m,n} x_{m,n} \right\|^2_H .$$

Thus, the definition of V is correct. The operator V maps the set of all polynomials $P^2_{0,\mu}$ in L^2_μ on L. By continuity we extend V to an isometric transformation from the closure of polynomials $P^2_\mu = \overline{P^2_{0,\mu}}$ onto H.

Set $H_0 := L^2_\mu \ominus P^2_\mu$. Introduce the following operator:

$$U := V \oplus E_{H_0}, \tag{19}$$

which maps isometrically L^2_μ onto $\widetilde{H} := H \oplus H_0$. Set

$$\widetilde{A} := U A_\mu U^{-1}, \quad \widetilde{B} := U B_\mu U^{-1}. \tag{20}$$

Notice that

$$\widetilde{A} x_{m,n} = U A_\mu U^{-1} x_{m,n} = U A_\mu x_1^m x_2^n = U x_1^{m+1} x_2^n = x_{m+1,n},$$
$$\widetilde{B} x_{m,n} = U B_\mu U^{-1} x_{m,n} = U B_\mu x_1^m x_2^n = U x_1^m x_2^{n+1} = x_{m,n+1}.$$

Therefore $\widetilde{A} \supseteq A$ and $\widetilde{B} \supseteq B$. Let

$$\widetilde{A} = \int_{\mathbb{R}} x_1 d\widetilde{E}(x_1), \quad \widetilde{B} = \int_{[-R,R]} x_2 d\widetilde{F}(x_2), \tag{21}$$

where $\widetilde{E}$ and $\widetilde{F}$ are the spectral measures of $\widetilde{A}$ and $\widetilde{B}$, respectively.

By the induction argument we get

$$x_{m,n} = A^m x_{0,n}, \quad m,n \in \mathbb{Z}_+,$$

and

$$x_{0,n} = B^n x_{0,0}, \quad n \in \mathbb{Z}_+.$$

Therefore we obtain

$$x_{m,n} = A^m B^n x_{0,0}, \quad m,n \in \mathbb{Z}_+. \tag{22}$$

We may write

$$x_{m,n} = \int_{\mathbb{R}} x_1^m d\widetilde{E}(x_1) \int_{[-R,R]} x_2^n d\widetilde{F}(x_2) x_{0,0} = \int_\Pi x_1^m x_2^n d(\widetilde{E} \times \widetilde{F})(x_1,x_2) x_{0,0},$$

where $(\widetilde{E} \times \widetilde{F})$ is the product measure of $\widetilde{E}$ and $\widetilde{F}$. Then

$$s_{m,n} = (x_{m,n}, x_{0,0})_H = \int_\Pi x_1^m x_2^n d((\widetilde{E} \times \widetilde{F}) x_{0,0}, x_{0,0})_H, \quad m,n \in \mathbb{Z}_+. \tag{23}$$

Thus, the measure $\widetilde{\mu} := ((\widetilde{E} \times \widetilde{F}) x_{0,0}, x_{0,0})_{\widetilde{H}}$ is a solution of the moment problem. Let $I_{x_1} \subset \mathbb{R}$, $I_{x_2} \subseteq [-R,R]$ be arbitrary intervals. Observe that

$$P^{\widetilde{H}}_H \widetilde{E}(I_{x_1}) \widetilde{F}(I_{x_2}) P^{\widetilde{H}}_H = P^{\widetilde{H}}_H \widetilde{E}(I_{x_1}) P^{\widetilde{H}}_H \widetilde{F}(I_{x_2}) P^{\widetilde{H}}_H = \mathbf{E}(I_{x_1}) F(I_{x_2});$$
$$P^{\widetilde{H}}_H \widetilde{E}(I_{x_1}) \widetilde{F}(I_{x_2}) P^{\widetilde{H}}_H = P^{\widetilde{H}}_H \widetilde{F}(I_{x_2}) \widetilde{E}(I_{x_1}) P^{\widetilde{H}}_H = P^{\widetilde{H}}_H \widetilde{F}(I_{x_2}) P^{\widetilde{H}}_H \widetilde{E}(I_{x_1}) P^{\widetilde{H}}_H$$
$$= F(I_{x_2}) \mathbf{E}(I_{x_1}),$$

and therefore

$$\mathbf{E}(I_{x_1}) F(I_{x_2}) = F(I_{x_2}) \mathbf{E}(I_{x_1}), \tag{24}$$

where $\mathbf{E}$ is the corresponding spectral function of A and F is the spectral function of B. Then

$$\begin{aligned}\widetilde{\mu}(I_{x_1}\times I_{x_2}) &= ((\widetilde{E}\times\widetilde{F})(I_{x_1}\times I_{x_2})x_{0,0},x_{0,0})_{\widetilde{H}}\\ &= (\widetilde{E}(I_{x_1})\widetilde{F}(I_{x_2})x_{0,0},x_{0,0})_{\widetilde{H}} = (P_H^{\widetilde{H}}\widetilde{F}(I_{x_2})\widetilde{E}(I_{x_1})x_{0,0},x_{0,0})_{\widetilde{H}}\\ &= (P_H^{\widetilde{H}}\widetilde{F}(I_{x_2})P_H^{\widetilde{H}}\widetilde{E}(I_{x_1})x_{0,0},x_{0,0})_{\widetilde{H}} = (F(I_{x_2})\mathbf{E}(I_{x_1})x_{0,0},x_{0,0})_H\\ &= (\mathbf{E}(I_{x_1})F(I_{x_2})x_{0,0},x_{0,0})_H,\end{aligned}$$

where $\mathbf{E}$ is the corresponding spectral function of A and F is the spectral function of B. Thus, the measure $\widetilde{\mu}$ admits the representation (13) since the Lebesgue continuation is unique.

Let us show that $\widetilde{\mu}=\mu$. Consider the following transformation:

$$S:\ (x_1,x_2)\in\Pi\mapsto\left(\operatorname{Arg}\frac{x_1-i}{x_1+i},x_2\right)\in\Pi_0, \tag{25}$$

where $\Pi_0=[-\pi,\pi)\times[-R,R]$ and $\operatorname{Arg}e^{iy}=y\in[-\pi,\pi)$. By virtue of V we define the following measures:

$$\mu_0(VG):=\mu(G),\quad \widetilde{\mu}_0(VG):=\widetilde{\mu}(G),\quad G\in\mathfrak{B}(\Pi). \tag{26}$$

It is not hard to see that μ_0 and $\widetilde{\mu}_0$ are non-negative measures on $\mathfrak{B}(\Pi_0)$. Then

$$\int_\Pi\left(\frac{x_1-i}{x_1+i}\right)^m x_2^n d\mu=\int_{\Pi_0}e^{im\psi}x_2^n d\mu_0, \tag{27}$$

$$\int_\Pi\left(\frac{x_1-i}{x_1+i}\right)^m x_2^n d\widetilde{\mu}=\int_{\Pi_0}e^{im\psi}x_2^n d\widetilde{\mu}_0,\quad m\in\mathbb{Z},\quad n\in\mathbb{Z}_+; \tag{28}$$

and

$$\begin{aligned}\int_\Pi\left(\frac{x_1-i}{x_1+i}\right)^m x_2^n d\widetilde{\mu} &= \int_\Pi\left(\frac{x_1-i}{x_1+i}\right)^m x_2^n d((\widetilde{E}\times\widetilde{F})x_{0,0},x_{0,0})_{\widetilde{H}}\\ &= \left(\int_\Pi\left(\frac{x_1-i}{x_1+i}\right)^m x_2^n d(\widetilde{E}\times\widetilde{F})x_{0,0},x_{0,0}\right)_{\widetilde{H}}\\ &= \left(\int_{\mathbb{R}}\left(\frac{x_1-i}{x_1+i}\right)^m d\widetilde{E}\int_{[-R,R]}x_2^n d\widetilde{F}x_{0,0},x_{0,0}\right)_{\widetilde{H}}\\ &= \left(\left((\widetilde{A}-iE_{\widetilde{H}})(\widetilde{A}+iE_{\widetilde{H}})^{-1}\right)^m\widetilde{B}^n x_{0,0},x_{0,0}\right)_{\widetilde{H}}\\ &= \left(U^{-1}\left((\widetilde{A}-iE_{\widetilde{H}})(\widetilde{A}+iE_{\widetilde{H}})^{-1}\right)^m\widetilde{B}^n U1,U1\right)_\mu\\ &= \left(\left((A_\mu-iE_{L^2_\mu})(A_\mu+iE_{L^2_\mu})^{-1}\right)^m B_\mu^n 1,1\right)_\mu\\ &= \int_\Pi\left(\frac{x_1-i}{x_1+i}\right)^m x_2^n d\mu,\quad m\in\mathbb{Z},\quad n\in\mathbb{Z}_+.\end{aligned} \tag{29}$$

By virtue of relations (27), (28) and (29) we get

$$\int_{\Pi_0}e^{im\psi}x_2^n d\mu_0=\int_{\Pi_0}e^{im\psi}x_2^n d\widetilde{\mu}_0,\quad m\in\mathbb{Z},\quad n\in\mathbb{Z}_+. \tag{30}$$

By the Weierstrass theorem we can approximate any continuous function by exponentials and therefore

$$\int_{\Pi_0}f(\psi)x_2^n d\mu_0=\int_{\Pi_0}f(\psi)x_2^n d\widetilde{\mu}_0,\quad n\in\mathbb{Z}_+, \tag{31}$$

for arbitrary continuous functions on Π_0. In particular, we have

$$\int_{\Pi_0}x_1^m x_2^n d\mu_0=\int_{\Pi_0}x_1^m x_2^n d\widetilde{\mu}_0,\quad n,m\in\mathbb{Z}_+. \tag{32}$$

However, the two-dimensional Hausdorff moment problem is determinate ([1]) and therefore we get $\mu_0 = \widetilde{\mu}_0$ and $\mu = \widetilde{\mu}$. Thus, we have proved that an arbitrary solution μ of the moment problem (1) can be represented in the form (13).

Let us check the second assertion of the Theorem. Let an arbitrary spectral measure $\mathbf{E}$ of A which commutes with the spectral measure F of B be given. By relation (13) we define a non-negative Borel measure μ on Π. Let us show that the measure μ is a solution of the moment problem (1).

Let $\widehat{A}$ be a self-adjoint extension of the operator A in a Hilbert space $\widehat{H} \supseteq H$, such that

$$\mathbf{E} = P_H^{\widehat{H}} \widehat{E},$$

where $\widehat{E}$ is the spectral measure of $\widehat{A}$. By (22) we get

$$\begin{aligned} x_{m,n} &= A^m B^n x_{0,0} = \widehat{A}^m B^n x_{0,0} = P_H^{\widehat{H}} \widehat{A}^m B^n x_{0,0} \\ &= P_H^{\widehat{H}} \lim_{a\to+\infty} \int_{[-a,a)} x_1^m d\widehat{E}(x_1) B^n x_{0,0} = \lim_{a\to+\infty} P_H^{\widehat{H}} \int_{[-a,a)} x_1^m d\widehat{E}(x_1) B^n x_{0,0} \\ &= \lim_{a\to+\infty} \int_{[-a,a)} x_1^m d\mathbf{E}(x_1) B^n x_{0,0}, \\ & \qquad m,n \in \mathbb{Z}_+, \end{aligned} \tag{33}$$

where the integrals are understood as strong limits of the Stieltjes operator sums. Let a be a positive integer. We choose arbitrary points

$$\begin{aligned} & -a = x_{1,0} < x_{1,1} < \cdots < x_{1,N} = a, \\ & \max_{1\leq i\leq N} |x_{1,i} - x_{1,i-1}| =: d, \quad N \in \mathbb{N}; \end{aligned} \tag{34}$$

$$\begin{aligned} & -R = x_{2,0} < x_{2,1} < \cdots < x_{2,M} = R, \\ & \max_{1\leq j\leq M} |x_{2,j} - x_{2,j-1}| =: r, \quad M \in \mathbb{N}. \end{aligned} \tag{35}$$

Set $I_{2,j} = [x_{2,j-1}, x_{2,j})$, if $1 \leq j < M$, and $I_{2,M} = [x_{2,M-1}, x_{2,M}]$. Then

$$C_a := \int_{[-a,a)} x_1^m d\mathbf{E} \int_{[-R,R]} x_2^n dF = \lim_{d\to0} \sum_{i=1}^N x_{1,i-1}^m \mathbf{E}([x_{1,i-1}, x_{1,i})) \lim_{r\to0} \sum_{j=1}^M x_{2,j-1}^n F(I_{2,j}),$$

where the integral sums converge in the strong operator topology. Then

$$\begin{aligned} C_a &= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N x_{1,i-1}^m \mathbf{E}([x_{1,i-1}, x_{1,i})) \sum_{j=1}^M x_{2,j-1}^n F(I_{2,j}) \\ &= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n \mathbf{E}([x_{1,i-1}, x_{1,i})) F(I_{2,j}), \end{aligned}$$

where the limits are understood in the strong operator topology. Then

$$\begin{aligned} (C_a x_{0,0}, x_{0,0})_H &= \left(\lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n \mathbf{E}([x_{1,i-1}, x_{1,i})) F(I_{2,j}) x_{0,0}, x_{0,0} \right)_H \\ &= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n \left(\mathbf{E}([x_{1,i-1}, x_{1,i})) F(I_{2,j}) x_{0,0}, x_{0,0} \right)_H \\ &= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n \left((\mathbf{E} \times F)([x_{1,i-1}, x_{1,i}) \times I_{2,j}) x_{0,0}, x_{0,0} \right)_H \\ &= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{1,i-1}^m x_{2,j-1}^n \left(\mu([x_{1,i-1}, x_{1,i}) \times I_{2,j}) x_{0,0}, x_{0,0} \right)_H . \end{aligned}$$

Therefore

$$(C_a x_{0,0}, x_{0,0})_H = \lim_{d\to 0}\lim_{r\to 0}\int_{[-a,a)\times[-R,R]} f_{d,r}(x_1,x_2)\,d\mu,$$

where $f_{d,r}$ is equal to $x_{1,i-1}^m x_{2,j-1}^n$ on the rectangular $[x_{1,i-1},x_{1,i})\times I_{2,j}$, $1\le i\le N$, $1\le j\le M$.

If $r\to 0$, then the function $f_{d,r}(x_1,x_2)$ converges pointwise to a function $f_d(x_1,x_2)$ which is equal to $x_{1,i-1}^m x_2^n$ on the rectangular $[x_{1,i-1},x_{1,i})\times[-R,R]$, $1\le i\le N$. Moreover, the function $f_{d,r}(x_1,x_2)$ is uniformly bounded. By the Lebesgue theorem we obtain

$$(C_a x_{0,0}, x_{0,0})_H = \lim_{d\to 0}\int_{[-a,a)\times[-R,R]} f_d(x_1,x_2)\,d\mu.$$

If $d\to 0$, then the function f_d converges pointwise to a function $x_1^m x_2^n$. Since $|f_d|\le a^m R^n$, by the Lebesgue theorem we get

$$(C_a x_{0,0}, x_{0,0})_H = \int_{[-a,a)\times[-R,R]} x_1^m x_2^n d\mu. \tag{36}$$

By virtue of relations (33) and (36) we get

$$\begin{aligned} s_{m,n} = (x_{m,n}, x_{0,0})_H &= \lim_{a\to+\infty}(C_a x_{0,0}, x_{0,0})_H \\ &= \lim_{a\to+\infty}\int_{[-a,a)\times[-R,R]} x_1^m x_2^n d\mu = \int_\Pi x_1^m x_2^n d\mu. \end{aligned} \tag{37}$$

Thus, the measure μ is a solution of the moment problem (1).

Let us prove the last assertion of the Theorem. Suppose to the contrary that two different spectral measures $\mathbf{E}_1$ and $\mathbf{E}_1$ of A commute with the spectral measure F of B and produce by relation (13) the same solution μ of the moment problem. Choose an arbitrary $z\in\mathbb{C}\backslash\mathbb{R}$. In what follows a will be a positive integer. Let us check that

$$\begin{aligned} \int_\Pi \frac{x_1^m}{x_1-z}x_2^n d\mu &= \int_\Pi \frac{x_1^m}{x_1-z}x_2^n d((\mathbf{E}_k\times F)(\delta)x_{0,0},x_{0,0})_H \\ &= \lim_{a\to+\infty} D_a(k),\quad k=1,2, \end{aligned} \tag{38}$$

where

$$D_a = D_a(k) := \int_{[-a,a)\times[-R,R]} \frac{x_1^m}{x_1-z}x_2^n d((\mathbf{E}_k\times F)(\delta)x_{0,0},x_{0,0})_H,\quad k=1,2.$$

In fact, set

$$\Pi_a = [-a,a)\times[-R,R],\quad a\in\mathbb{N},$$
$$D_1=\Pi_1,\quad D_a=\Pi_a\backslash\Pi_{a-1},\quad a=2,3,\dots$$

Notice that the sets D_a are pairwise disjoint and

$$\Pi_a=\bigcup_{j=1}^a D_a,\quad \Pi=\bigcup_{j=1}^\infty D_a.$$

By the known property of the Lebesgue integral ([36, Theorem 3, p. 298]) we may write

$$\begin{aligned} \int_\Pi \frac{x_1^m}{x_1-z}x_2^n d((\mathbf{E}_k\times F)(\delta)x_{0,0},x_{0,0})_H &= \sum_{j=1}^\infty \int_{D_j}\frac{x_1^m}{x_1-z}x_2^n d((\mathbf{E}_k\times F)(\delta)x_{0,0},x_{0,0})_H \\ = \lim_{a\to\infty}\sum_{j=1}^a\int_{D_j}\frac{x_1^m}{x_1-z}x_2^n d((\mathbf{E}_k\times F)(\delta)x_{0,0},x_{0,0})_H &= \lim_{a\to\infty} D_a(k),\quad k=1,2. \end{aligned}$$

Consider arbitrary partitions of the type (34), (35). Then

$$D_a = \lim_{d\to 0}\lim_{r\to 0}\int_{[-a,a)\times[-R,R]} g_{z;d,r}(x,\varphi)d((\mathbf{E}_k\times F)(\delta)x_{0,0},x_{0,0})_H.$$

Here the function $g_{z;d,r}(x_1,x_2)$ is equal to $\frac{x_{1,i-1}^m}{x_{1,i-1}-z}x_{2,j-1}^n$ on the rectangular $[x_{i-1},x_i)\times I_{2,j-1}$, $1\leq i\leq N$, $1\leq j\leq M$. Then

$$\begin{aligned} D_a &= \lim_{d\to 0}\lim_{r\to 0}\sum_{i=1}^N\sum_{j=1}^M \frac{x_{1,i-1}^m}{x_{1,i-1}-z}x_{2,j-1}^n\left(\mathbf{E}_k([x_{1,i-1},x_{1,i}))F(I_{2,j})x_{0,0},x_{0,0}\right)_H \\ &= \lim_{d\to 0}\lim_{r\to 0}\left(\sum_{i=1}^N \frac{x_{1,i-1}^m}{x_{1,i-1}-z}\mathbf{E}_k([x_{i-1},x_i))\sum_{j=1}^M x_{2,j}^n F(I_{2,j})x_{0,0},x_{0,0}\right)_H \\ &= \left(\int_{[-a,a)}\frac{x_1^m}{x_1-z}d\mathbf{E}_k\int_{[-R,R]}x_2^n dFx_{0,0},x_{0,0}\right)_H. \end{aligned}$$

Let $n=n_1+n_2$, $n_1,n_2\in\mathbb{Z}_+$. Then we may write

$$\begin{aligned} D_a &= \left(B^{n_1}\int_{[-a,a)}\frac{x_1^m}{x_1-z}d\mathbf{E}_k B^{n_2}x_{0,0},x_{0,0}\right)_H \\ &= \left(\int_{[-a,a)}\frac{x_1^m}{x_1-z}d\mathbf{E}_k x_{0,n_2},x_{0,n_1}\right)_H. \end{aligned}$$

By substitution of the last expression for D_a into relation (38) we get

$$\begin{aligned} \int_\Pi \frac{x_1^m}{x_1-z}x_2^n d\mu &= \lim_{a\to+\infty}D_a = \lim_{a\to+\infty}\left(\int_{[-a,a)}\frac{x_1^m}{x_1-z}d\widehat{E}_k x_{0,n_2},x_{0,n_1}\right)_{\widehat{H}_k} \\ &= \left(\int_{\mathbb{R}}\frac{x_1^m}{x_1-z}d\widehat{E}_k x_{0,n_2},x_{0,n_1}\right)_{\widehat{H}_k} = \left(\widehat{A}_k^{m_2}R_z(\widehat{A}_k)\widehat{A}_k^{m_1}x_{0,n_2},x_{0,n_1}\right)_{\widehat{H}_k} \\ &= \left(R_z(\widehat{A}_k)x_{m_1,n_2},x_{m_2,n_1}\right)_H, \end{aligned} \tag{39}$$

where $m_1,m_2\in\mathbb{Z}_+:\ m_1+m_2=m$, and $\widehat{A}_k$ is a self-adjoint extension of A in a Hilbert space $\widehat{H}_k\supseteq H$ such that its spectral measure $\widehat{E}_k$ generates $\mathbf{E}_k$: $\mathbf{E}_k=P_H^{\widehat{H}_k}\widehat{E}_k$; $k=1,2$.

Relation (39) shows that the generalized resolvents corresponding to $\mathbf{E}_k$, $k=1,2$, coincide. This means that the spectral measures $\mathbf{E}_1$ and $\mathbf{E}_2$ coincide. We obtained a contradiction. This completes the proof. □

Definition 2.1. *A solution μ of the moment problem (1) is said to be* **canonical** *if it is generated by relation (13) where* $\mathbf{E}$ *is an* **orthogonal** *spectral measure of A which commutes with the spectral measure of B. Orthogonal spectral measures are those measures which are the spectral measures of self-adjoint extensions of A inside H.*

Let the moment problem (1) be given and conditions (2), (3) hold. Let us describe canonical solutions of the two-dimensional moment problem in a strip. Let μ be an arbitrary canonical solution and $\mathbf{E}$ be the corresponding orthogonal spectral measure of A. Let $\widetilde{A}$ be the self-adjoint operator in H which corresponds to $\mathbf{E}$. Consider the Cayley transformation of $\widetilde{A}$

$$U_{\widetilde{A}}=(\widetilde{A}+iE_H)(\widetilde{A}-iE_H)^{-1}\supseteq V_A, \tag{40}$$

where $V_A:=(A+iE_H)(A-iE_H)^{-1}$.

Since $\mathbf{E}$ commutes with the spectral measure F of B, then $U_{\widetilde{A}}$ commutes with B and with U_B. By the considerations in the proof of Theorem 2.1 in [28], the operator $U_{\widetilde{A}}$ have the following form:

$$U_{\widetilde{A}}=V_A\oplus\widetilde{U}_{2,4}, \tag{41}$$

where $\widetilde{U}_{2,4}$ is an isometric operator which maps H_2 onto H_4, and commutes with U_B. Let the operator $U_{2,4}$ be defined as in the proof of Theorem 2.1 in [28]. Then the following operator:

$$U_2 = U_{2,4}^{-1} \widetilde{U}_{2,4} \tag{42}$$

is a unitary operator in H_2 which commutes with $U_{B;H_2}$.

Denote by $\mathbf{S}(U_B; H_2)$ a set of all unitary operators in H_2 which commute with $U_{B;H_2}$. Choose an arbitrary operator $\widehat{U}_2 \in \mathbf{S}(U_B; H_2)$. Define $\widehat{U}_{2,4}$ by the following relation:

$$\widehat{U}_{2,4} = U_{2,4} \widehat{U}_2. \tag{43}$$

Notice that $\widehat{U}_{2,4} U_B h = \widehat{U}_{2,4} U_B h$, $h \in H_2$. Then we define a unitary operator $U = V_A \oplus \widehat{U}_{2,4}$ and its Cayley transformation $\widehat{A}$ which commute with the operator B. Repeating arguments before (23) we get a canonical solution of the moment problem.

Thus, all canonical solutions of the Devinatz moment problem are generated by operators $\widehat{U}_2 \in \mathbf{S}(U_B; H_2)$. Notice that different operators $U', U'' \in \mathbf{S}(U_B; H_2)$ produce different orthogonal spectral measures $\mathbf{E}', \mathbf{E}$. By Theorem 2.1, these spectral measures produce different solutions of the moment problem.

Recall some well-known definitions. A pair $(Y, \mathfrak{A})$, where Y is an arbitrary set and $\mathfrak{A}$ is a fixed σ-algebra of subsets of A is said to be a *measurable space*. A triple $(Y, \mathfrak{A}, \mu)$, where $(Y, \mathfrak{A})$ is a measurable space and μ is a measure on $\mathfrak{A}$ is said to be a *space with a measure*.

Let $(Y, \mathfrak{A})$ be a measurable space, $\mathbf{H}$ be a Hilbert space and $\mathcal{P} = \mathcal{P}(\mathbf{H})$ be a set of all orthogonal projectors in $\mathbf{H}$. A countably additive mapping $E: \mathfrak{A} \to \mathcal{P}$, $E(Y) = E_{\mathbf{H}}$, is said to be a *spectral measure* in $\mathbf{H}$. A set $(Y, \mathfrak{A}, H, E)$ is said to be a *space with a spectral measure*. By $S(Y, E)$ one means a set of all E-measurable E-a.e. finite complex-valued functions on Y.

Let $(Y, \mathfrak{A}, \mu)$ be a separable space with a σ-finite measure and to μ-everyone $y \in Y$ it corresponds a Hilbert space $G(y)$. A function $N(y) = \dim G(y)$ is called the *dimension function*. It is supposed to be μ-measurable. Let Ω be a set of vector-valued functions $g(y)$ with values in $G(y)$ which are defined μ-everywhere and are measurable with respect to some base of measurability. A set of (classes of equivalence) of such functions with the finite norm

$$\|g\|^2_{\mathcal{H}} = \int |g(y)|^2_{G(y)} d\mu(y) < \infty \tag{44}$$

form a Hilbert space $\mathcal{H}$ with the scalar product given by

$$(g_1, g_2)_{\mathcal{H}} = \int (g_1, g_2)_{G(y)} d\mu(y). \tag{45}$$

The space $\mathcal{H} = \mathcal{H}_{\mu,N} = \int_Y \oplus G(y) d\mu(y)$ is said to be a *direct integral of Hilbert spaces*. Consider the following operator:

$$\mathbf{X}(\delta) g = \chi_\delta g, \quad g \in \mathcal{H}, \quad \delta \in \mathfrak{A}, \tag{46}$$

where χ_δ is the characteristic function of the set δ. The projectors $\mathbf{X}(\delta)$, $\delta \in \mathfrak{A}$, define a spectral measure in $\mathcal{H}$.

Let $t(y)$ be a measurable operator-valued function with values in $\mathbf{B}(G(y))$ which is μ-a.e. defined and $\mu - \sup \|t(y)\|_{G(y)} < \infty$. The operator

$$T: \ g(y) \mapsto t(y) g(y), \tag{47}$$

is said to be *decomposable*. It is a bounded operator in $\mathcal{H}$ which commutes with $\mathbf{X}(\delta)$, $\forall \delta \in \mathfrak{A}$. Moreover, every bounded operator in $\mathcal{H}$ which commutes with $\mathbf{X}(\delta)$, $\forall \delta \in \mathfrak{A}$, is decomposable. In the case $t(y) = \varphi(y) E_{G(y)}$, where $\varphi \in S(Y, \mu)$, we set $T =: Q_\varphi$. The decomposable operator is unitary if and only if μ-a.e. the operator $t(y)$ is unitary.

Return to the investigation of canonical solutions. Consider the spectral measure F_2 of the operator $U_{B;H_2}$ in H_2. There exists an element $h \in H_2$ of the maximal type, i.e.

the non-negative Borel measure

$$\mu(\delta) := (F_2(\delta)h, h), \quad \delta \in \mathfrak{B}([-\pi, \pi)), \tag{48}$$

has the maximal type between all such measures (generated by other elements of H_2). This type is said to be the *spectral type* of the measure F_2. Let N_2 be the multiplicity function of the measure F_2. Then there exists a unitary transformation W of the space H_2 on $\mathcal{H} = \mathcal{H}_{\mu,N_2}$ such that

$$WU_{B;H_2}W^{-1} = Q_{e^{iy}}, \quad WF_2(\delta)W^{-1} = \mathbf{X}(\delta). \tag{49}$$

Notice that $\widehat{U}_2 \in \mathbf{S}(U_B; H_2)$ if and only if the operator

$$V_2 := W\widehat{U}_2W^{-1} \tag{50}$$

is unitary and commutes with $\mathbf{X}(\delta)$, $\forall\delta \in [-\pi, \pi)$. The latter is equivalent to the condition that V_2 is decomposable and the values of the corresponding operator-valued function $t(y)$ are μ-a.e. unitary operators. A set of all decomposable operators in $\mathcal{H}$ such that the values of the corresponding operator-valued function $t(y)$ are μ-a.e. unitary operators we denote by $\mathbf{D}(U_B; H_2)$.

Theorem 2.2. *Let the moment problem (1) be given. In the conditions of Theorem 2.1 all canonical solutions of the moment problem have the form (13) where the spectral measures $\mathbf{E}$ of the operator A are constructed by operators from $\mathbf{D}(U_B; H_2)$. Namely, for an arbitrary $V_2 \in \mathbf{D}(U_B; H_2)$ we set $\widehat{U}_2 = W^{-1}V_2W$, $\widehat{U}_{2,4} = U_{2,4}\widehat{U}_2$, $U = V_A \oplus \widehat{U}_{2,4}$, $\widehat{A} = i(U + E_H)(U - E_H)^{-1}$, and then $\mathbf{E}$ is the spectral measure of $\widehat{A}$.*
Moreover, the correspondence between $\mathbf{D}(U_B; H_2)$ and a set of all canonical solutions of the moment problem is bijective.

Proof. The proof follows from the previous considerations. $\square$

3. An analytic description of all solutions of the two-dimensional moment problem in a strip

Consider the moment problem (1) and suppose that conditions (2), (3) hold. Let us turn to a parametrization of all solutions of the moment problem. We shall use Theorem 2.1. Consider relation (13). The spectral measure $\mathbf{E}$ commutes with the operator U_B. Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$. By virtue of relation (12) we may write

$$\begin{aligned}(U_B\mathbf{R}_z(A)x, y)_H &= (\mathbf{R}_z(A)x, U_B^*y)_H = \int_{\mathbb{R}} \frac{1}{t-z} d(\mathbf{E}(t)x, U_B^*y)_H \\ &= \int_{\mathbb{R}} \frac{1}{t-z} d(U_B\mathbf{E}(t)x, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} d(\mathbf{E}(t)U_Bx, y)_H, \quad x, y \in H;\end{aligned} \tag{51}$$

$$(\mathbf{R}_z(A)U_Bx, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} d(\mathbf{E}(t)U_Bx, y)_H, \quad x, y \in H, \tag{52}$$

where $\mathbf{R}_z(A)$ is the generalized resolvent which corresponds to $\mathbf{E}$. Therefore we get

$$\mathbf{R}_z(A)U_B = U_B\mathbf{R}_z(A), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{53}$$

On the other hand, if relation (53) holds, then

$$\int_{\mathbb{R}} \frac{1}{t-z} d(\mathbf{E}U_Bx, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} d(U_B\mathbf{E}x, y)_H, \quad x, y \in H, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{54}$$

By the Stieltjes inversion formula [1], we obtain that $\mathbf{E}$ commutes with U_B.

We denote by $\mathbf{M}(A, B)$ a set of all generalized resolvents $\mathbf{R}_z(A)$ of A which satisfy relation (53).

Recall some known facts from [27] which we shall need here. Let K be a closed symmetric operator in a Hilbert space $\mathbf{H}$, with the domain $D(K)$, $\overline{D(K)} = \mathbf{H}$. Set $N_\lambda = N_\lambda(K) = \mathbf{H} \ominus \Delta_K(\lambda)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Consider an arbitrary bounded linear operator C, which maps N_i into N_{-i}. For

$$g = f + C\psi - \psi, \quad f \in D(K), \quad \psi \in N_i, \tag{55}$$

we set

$$K_C g = Kf + iC\psi + i\psi. \tag{56}$$

The operator K_C is said to be a *quasiself-adjoint extension of the operator K, defined by the operator K.*

The following theorem can be found in [27, Theorem 7]:

Theorem 3.1. *Let K be a closed symmetric operator in a Hilbert space $\mathbf{H}$ with the domain $D(K)$, $\overline{D(K)} = \mathbf{H}$. All generalized resolvents of the operator K have the following form:*

$$\mathbf{R}_\lambda(K) = \begin{cases} (K_{F(\lambda)} - \lambda E_{\mathbf{H}})^{-1}, & \operatorname{Im}\lambda > 0 \\ (K_{F^*(\overline{\lambda})} - \lambda E_{\mathbf{H}})^{-1}, & \operatorname{Im}\lambda < 0 \end{cases}, \tag{57}$$

where $F(\lambda)$ is an analytic in $\mathbb{C}_+$ operator-valued function, which values are contractions which map $N_i(A) = H_2$ into $N_{-i}(A) = H_4$ ($\|F(\lambda)\| \leq 1$), and $K_{F(\lambda)}$ is the quasiself-adjoint extension of K defined by $F(\lambda)$.

On the other hand, for any operator function $F(\lambda)$ having the above properties there corresponds by relation (57) a generalized resolvent of K.

Observe that the correspondence between all generalized resolvents and functions $F(\lambda)$ in Theorem 3.1 is bijective [27].

Return to the study of the moment problem (1). Let us describe the set $\mathbf{M}(A,B)$. Choose an arbitrary $\mathbf{R}_\lambda \in \mathbf{M}(A,B)$. By (57) we get

$$\mathbf{R}_\lambda = (A_{F(\lambda)} - \lambda E_H)^{-1}, \quad \operatorname{Im}\lambda > 0, \tag{58}$$

where $F(\lambda)$ is an analytic in $\mathbb{C}_+$ operator-valued function, which values are contractions which map H_2 into H_4, and $A_{F(\lambda)}$ is the quasiself-adjoint extension of A defined by $F(\lambda)$. Then

$$A_{F(\lambda)} = \mathbf{R}_\lambda^{-1} + \lambda E_H, \quad \operatorname{Im}\lambda > 0.$$

By virtue of relation (53) we obtain

$$U_B A_{F(\lambda)} h = A_{F(\lambda)} U_B h, \quad h \in D(A_{F(\lambda)}), \quad \lambda \in \mathbb{C}_+. \tag{59}$$

Consider the following operators:

$$W_\lambda := (A_{F(\lambda)} + iE_H)(A_{F(\lambda)} - iE_H)^{-1} = E_H + 2i(A_{F(\lambda)} - iE_H)^{-1}, \tag{60}$$

$$V_A = (A + iE_H)(A - iE_H)^{-1} = E_H + 2i(A - iE_H)^{-1}, \tag{61}$$

where $\lambda \in \mathbb{C}_+$. Notice that ([27])

$$W_\lambda = V_A \oplus F(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{62}$$

The operator $(A_{F(\lambda)} - iE_H)^{-1}$ is defined on the whole H, see [27, p. 79]. By relation (59) we obtain

$$U_B (A_{F(\lambda)} - iE_H)^{-1} h = (A_{F(\lambda)} - iE_H)^{-1} U_B h, \quad h \in H, \quad \lambda \in \mathbb{C}_+. \tag{63}$$

Then

$$U_B W_\lambda = W_\lambda U_B, \quad \lambda \in \mathbb{C}_+. \tag{64}$$

Recall that the operator U_B reduces the subspaces H_j, $1 \leq j \leq 4$, and $U_B V_A = V_A U_B$ (see the proof of Theorem 2.1 in [28]). If we choose an arbitrary $h \in H_2$ and apply relations (64), (62), we get

$$U_B F(\lambda) = F(\lambda) U_B, \quad \lambda \in \mathbb{C}_+. \tag{65}$$

Denote by $\mathbf{F}(A,B)$ a set of all analytic in $\mathbb{C}_+$ operator-valued functions which values are contractions which map H_2 into H_4 and which satisfy relation (65). Thus, for an arbitrary $\mathbf{R}_\lambda \in \mathbf{M}(A,B)$ the corresponding function $F(\lambda)$ belongs to $\mathbf{F}(A,B)$.

On the other hand, choose an arbitrary $F(\lambda) \in \mathbf{F}(A,B)$. Then we derive (64) with W_λ defined by (60). Then we get (63), (59) and therefore

$$U_B \mathbf{R}_\lambda = \mathbf{R}_\lambda U_B, \quad \lambda \in \mathbb{C}_+. \tag{66}$$

Calculating the conjugate operators for the both sides of the last equality we conclude that this relation holds for all $\lambda \in \mathbb{C}$.

Consider the spectral measure F_2 of the operator $U_{B;H_2}$ in H_2. We shall use relation (49). Observe that $F(\lambda) \in \mathbf{F}(A,B)$ if and only if the operator-valued function

$$G(\lambda) := W U_{2,4}^{-1} F(\lambda) W^{-1}, \quad \lambda \in \mathbb{C}_+, \tag{67}$$

is analytic in $\mathbb{C}_+$ and has values which are contractions in $\mathcal{H}$ which commute with $\mathbf{X}(\delta)$, $\forall \delta \in [-\pi, \pi)$.

This means that for an arbitrary $\lambda \in \mathbb{C}_+$ the operator $G(\lambda)$ is decomposable and the values of the corresponding operator-valued function $t(y)$ are μ-a.e. contractions. A set of all decomposable operators in $\mathcal{H}$ such that the values of the corresponding operator-valued function $t(y)$ are μ-a.e. contractions we denote by $\mathrm{T}(B; H_2)$. A set of all analytic in $\mathbb{C}_+$ operator-valued functions $G(\lambda)$ with values in $\mathrm{T}(B; H_2)$ we denote by $\mathbf{G}(A,B)$.

Theorem 3.2. *Let the two-dimensional moment problem in a strip (1) be given. In the conditions of Theorem 2.1 all solutions of the moment problem have the form (13) where the spectral measures* $\mathbf{E}$ *of the operator* A *are defined by the corresponding generalized resolvents* $\mathbf{R}_\lambda$ *which are constructed by the following relation:*

$$\mathbf{R}_\lambda = (A_{F(\lambda)} - \lambda E_H)^{-1}, \quad \operatorname{Im} \lambda > 0, \tag{68}$$

where $F(\lambda) = U_{2,4} W^{-1} G(\lambda) W$, $G(\lambda) \in \mathbf{G}(A,B)$.
Moreover, the correspondence between $\mathbf{G}(A,B)$ *and a set of all solutions of the moment problem is bijective.*

Proof. The proof follows from the previous considerations. □

References

1. J. A. Shohat, J. D. Tamarkin, *The Problem of Moments*, Amer. Math. Soc., New York City, 1943.
2. N. I. Akhiezer, *Classical Moment Problem*, Fizmatlit, Moscow, 1961. (Russian)
3. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
4. C. Berg, J. Christensen, P. Ressel, *Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions*, Springer Verlag, New York, 1984.
5. B. Fuglede, *The multidimensional moment problem*, Expo. Math. **1** (1983), 47–65.
6. Yu. M. Berezansky, *Spectral theory of the infinite block Jacobi type normal matrices, orthogonal polynomials on the complex domain, and the complex moment problem*, Oper. Theory Adv. Appl. **191** (2009), 37–50.
7. J. Stochel, F. H. Szafraniec, *The complex moment problem and subnormality: a polar decomposition approach*, J. Funct. Anal. **159** (1998), 432–491.
8. M. Putinar, K. Schmüdgen, *Multivariate determinateness*, Indiana Univ. Math. J. **57** (2008), no. 6, 2931–2968.
9. S. Kuhlmann, M. Marshall, *Positivity, sums of squares and the multi-dimensional moment problem*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4285–4301.
10. K. Schmüdgen, *On the moment problem of closed semi-algebraic sets*, J. Reine Angew. Math. (Crelle) **558** (2003), 225–234.
11. J. L. McGregor, *Solvability criteria for certain N-dimensional moment problems*, J. Approx. Theory **30** (1980), 315–333.
12. M. Putinar, F.-H. Vasilescu, *Solving moment problems by dimensional extension*, Ann. Math. **149** (1999), 1087–1107.
13. F.-H. Vasilescu, *Spaces of fractions and positive functionals*, Math. Scand. **96** (2005), 257–279.
14. F.-H. Vasilescu, *Hamburger and Stieltjes moment problems in several variables*, Trans. Amer. Math. Soc. **354** (2005), no. 3, 1265–1278.

15. S. Zagorodnyuk, *On the two-dimensional moment problem*, Ann. Funct. Anal. **1** (2010), no. 1, 80–104.
16. F.-H. Vasilescu, *Operator moment problems in unbounded sets*, Oper. Theory Adv. Appl. **127** (2001), 613–638.
17. E. Albrecht and F.-H. Vasilescu, *Unbounded extensions and operator moment problems*, Preprint, 2004.
18. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis,* Vols. 1, 2, Kluwer Acad. Publ., Dordrecht—Boston—London, 1995. (Russian edition: Naukova Dumka, Kiev, 1988)
19. M. A. Neumark, *Spectral functions of a symmetric operator*, Izv. Akad. Nauk SSSR, Ser. Mat. **4** (1940), no. 3, 277–318. (Russian)
20. M. A. Neumark, *On spectral functions of a symmetric operator*, Izv. Akad. Nauk SSSR, Ser. Mat. **7** (1943), no. 6, 285–296. (Russian)
21. M. G. Krein, M. A. Krasnoselskiy, *Basic theorems on an extension of Hermitian operators and some their applications to the theory of orthogonal polynomials and the moment problem*, Uspekhi Mat. Nauk, **3(19)** (1947), 60–106. (Russian)
22. M. G. Krein, *On a general method of decomposing Hermite-positive nuclei into elementary products*, Doklady Acad. Sci. URSR **53** (1946), 3–6. (Russian)
23. M. G. Krein, *On Hermitian operators with directed functionals*, Akad. Nauk Ukrain. RSR, Zbirnyk Prac' Inst. Mat. (1948), no. 10, 83–106. (Russian)
24. B. Szökefalvi-Nagy, A. Koranyi, *Relations d'un probléme de Nevanlinna et Pick avec la théorie de l'espace hilbertien*, Acta Math. Acad. Sci. Hungar. **7** (1956), 295–303.
25. B. Szökefalvi-Nagy, A. Koranyi, *Operatortheoretische Behandlung und Verallgemeinerung eines Problemkreises in der komplexen Funktionentheorie*, Acta Math. **100** (1958), 171–202.
26. N. K. Nikolski, *Operators, Functions, and Systems: an Easy Reading.* Vol. I*: Hardy, Hankel, and Toeplitz*, Math. Surveys and Monographs, Vol. 92, Amer. Math. Soc., Providence, RI, 2002.
27. A. V. Shtraus, *Generalized resolvents of symmetric operators*, Izv. Akad. Nauk SSSR, Ser. Mat. **18** (1954), 51–86. (Russian)
28. S. M. Zagorodnyuk, *On commuting symmetric operators*, Methods Funct. Anal. Topology **18** (2012), no. 2, 198–200.
29. Yu. M. Berezansky, M. L. Gorbachuk, *On a continuation of positive definite functions of two variables*, Ukrain. Mat. Zh. **17** (1965), no. 5, 96–102. (Russian)
30. B. Ya. Levin, I. E. Ovcharenko, *A continuation of Hermitian positive functions defined on a strip*, Teor. Funkts., Funkts. Analiz Prilozh. **1** (1967), no. 5, 68–83. (Russian)
31. M. L. Gorbachuk, *On a description of continuations of a positive definite operator function*, Ukrain. Mat. Zh. **17** (1965), no. 5, 102–110. (Russian)
32. I. E. Ovcharenko, *On two-dimensional power moment sequences*, Ukrain. Mat. Zh. **36** (1984), no. 1, 51–56. (Russian)
33. M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, Amer. Math. Soc., Colloquium Publications, Providence, RI, 1932.
34. M. Sh. Birman, M. Z. Solomyak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Izdat. Leningrad. Univ., Leningrad, 1980. (Russian)
35. N. I. Akhiezer, I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Gostekhizdat, Moscow—Leningrad, 1950. (Russian)
36. A. N. Kolmogorov, S. V. Fomin, *Elements of the Theory of Functions and the Functional Analysis*, Nauka, Moscow, 1981. (Russian)

School of Mathematics and Mechanics, Karazin Kharkiv National University, 4 Svobody sq., Kharkiv, 61077, Ukraine

E-mail address: Sergey.M.Zagorodnyuk@univer.kharkov.ua

Received 09/04/2012; Revised 27/10/2012