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ON SELF-ADJOINTNESS OF 1–D SCHRÖDINGER OPERATORS WITH δ-INTERACTIONS

I. I. KARPENKO AND D. L. TYSHKEVICH

Abstract. In the present work we consider the Schrödinger operator $\mathrm{H}_{\mathrm{X},\alpha} = -\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n)$ acting in $L^2(\mathbb{R}_+)$. We investigate and complete the conditions of self-adjointness and nontriviality of deficiency indices for $\mathrm{H}_{\mathrm{X},\alpha}$ obtained in [13]. We generalize the conditions found earlier in the special case $d_n := x_n - x_{n-1} = 1/n$, $n \in \mathbb{N}$, to a wider class of sequences $\{x_n\}_{n=1}^{\infty}$. Namely, for $x_n = \frac{1}{n^\gamma \ln^\eta n}$ with $\langle \gamma, \eta \rangle \in (1/2,\, 1) \times (-\infty, +\infty) \cup \{1\} \times (-\infty, 1]$, the description of asymptotic behavior of the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is obtained for $\mathrm{H}_{\mathrm{X},\alpha}$ either to be self-adjoint or to have nontrivial deficiency indices.

1. Introduction

Let $X = \{x_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of nonnegative numbers, $x_0 = 0$, and $\lim_{n\to\infty} x_n = \infty$. Let also $\alpha = \{\alpha_n\}_1^{\infty}$ be a sequence of real numbers.

The differential expression

$$l_{X,\alpha} := -\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n) \tag{1}$$

on $L^2(0, +\infty)$ is connected with the symmetric differential operator

$$\mathrm{H}^0_{\mathrm{X},\alpha} := -\frac{\mathrm{d}^2}{\mathrm{dx}^2} \tag{2}$$

with domain

$$\begin{aligned} \mathrm{dom}(\mathrm{H}^0_{\mathrm{X},\alpha}) = \big\{ f \in W^{2,2}(\mathbb{R}_+ \setminus X) \cap L^2_{\mathrm{comp}}(\mathbb{R}_+) \mid f'(0) = 0,& \\ f'(x_{n+}) - f'(x_{n-}) = \alpha_n f(x_n) \big\}&. \end{aligned} \tag{3}$$

Denote by $\mathrm{H}_{\mathrm{X},\alpha}$ the closure of the operator $\mathrm{H}^0_{\mathrm{X},\alpha}$.

Schrödinger operators with distributional potentials have attracted considerable interest in the last decades, in particular, because they can be used as solvable models in many situations, see [2, 3, 4, 5, 7, 9, 14, 15]. For instance, the operator $\mathrm{H}^0_{\mathrm{X},\alpha}$ can be regarded as a Hamiltonian for a δ–interaction at points x_n with intensity α_n. In the general case, the operator $\mathrm{H}_{\mathrm{X},\alpha}$ does not need to be self-adjoint. One of the important problems in the spectral analysis of this operator is to find necessary and sufficient conditions for the operator $\mathrm{H}_{\mathrm{X},\alpha}$ to be self-adjoint. A thorough study of this problem was recently undertaken in the case of lower semi-bounded Hamiltonians. Namely, it is proved in [3] (see also [11]) that $\mathrm{H}_{\mathrm{X},\alpha}$ is always self-adjoint provided that it is lower semi-bounded.

Spectral properties of the operator $\mathrm{H}_{\mathrm{X},\alpha}$ depend on both the sequence α and the sequence X. In the latter case, the behavior of the sequence

$$d_n := x_n - x_{n-1}, \quad n \in \mathbb{N}, \tag{4}$$

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is an important characteristic. In particular, if $d_* := \inf_{n\in\mathbb{N}} d_n > 0$, then the operator $\mathrm{H}_{\mathrm{X},\alpha}$ is *always self-adjoint* [9].

This result is sharp in the sense that there is a sequence d_n satisfying $\lim_{n\to\infty} d_n = 0$ such that the Hamiltonian $\mathrm{H}_{\mathrm{X},\alpha}$ has nontrivial deficiency indices for some sequences $\alpha \subset \mathbb{R}$. Namely, C. Shubin Christ and G. Stolz showed in [15] that $n_\pm(\mathrm{H}_{\mathrm{X},\alpha}) = 1$ if $d_n = 1/n$ and $\alpha_n = -2n-1$, $n \in \mathbb{N}$. Thus the case $d_* = 0$ is fundamentally different from the case $d_* > 0$ since nontrivial deficiency indices can be realized there.

In [13], A. S. Kostenko and M. M. Malamud studied the Hamiltonian $\mathrm{H}_{\mathrm{X},\alpha}$ in the framework of boundary triplets and the corresponding Weyl function. Such an approach to the theory of extensions of symmetric operators was initiated about thirty years ago and still is being actively developed, see [6, 7, 8, 9, 10].

Using a corresponding boundary triple, the authors in [13] parameterized the set of Hamiltonians $\mathrm{H}_{\mathrm{X},\alpha}$ with certain classes of Jacobi matrices (three-diagonal matrices). It was also found there that spectral properties of the Hamiltonian $\mathrm{H}_{\mathrm{X},\alpha}$ are closely linked with the same properties of the corresponding Jacobi matrix,

$$B_{X,\alpha} = \begin{pmatrix} r_1^{-2}(\alpha_1 + \frac{1}{d_1} + \frac{1}{d_2}) & -r_1^{-1}r_2^{-1}d_2^{-1} & 0 & \dots \\ -r_1^{-1}r_2^{-1}d_2^{-1} & r_2^{-2}(\alpha_2 + \frac{1}{d_2} + \frac{1}{d_3}) & -r_2^{-1}r_3^{-1}d_3^{-1} & \dots \\ 0 & -r_2^{-1}r_3^{-1}d_3^{-1} & r_3^{-2}(\alpha_3 + \frac{1}{d_3} + \frac{1}{d_4}) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{5}$$

where

$$r_n = \sqrt{d_n + d_{n+1}}, \quad n \in \mathbb{N}. \tag{6}$$

As it turned out the deficiency indices for $\mathrm{H}_{\mathrm{X},\alpha}$ and $B_{X,\alpha}$ coincide, $n_\pm(\mathrm{H}_{\mathrm{X},\alpha}) = n_\pm(B_{X,\alpha})$ ([13, Theorem 5.4]) and, consequently, $n_\pm(\mathrm{H}_{\mathrm{X},\alpha}) \le 1$, see [7]. In particular, $\mathrm{H}_{\mathrm{X},\alpha}$ is self-adjoint if and only if the matrix $B_{X,\alpha}$ is self-adjoint.

Using the Carleman criterion A. S. Kostenko and M. M. Malamud obtained the following result [13, Proposition 5.7]:

if $\sum_{n=1}^\infty d_n^2 = \infty$, then the operator $\mathrm{H}_{\mathrm{X},\alpha}$ is self-adjoint for any sequence $\alpha \subset \mathbb{R}$.

In comparison with the mentioned result from [3], [11], this statement from [13] gives new information only for not lower semi-bounded Hamiltonians $\mathrm{H}_{\mathrm{X},\alpha}$.

Clearly, the result on self-adjointness of the operator $\mathrm{H}_{\mathrm{X},\alpha}$ for $d_* := \inf_{n\in\mathbb{N}} d_n > 0$ is a particular case of the latter statement.

Moreover, the example of Shubin Christ and Stolz in [15] was significantly specified in [13], namely,

if $d_n = 1/n$, then
1. $n_\pm(\mathrm{H}_{\mathrm{X},\alpha}) = 0$ *if* $\alpha_n \le -2(2n+1) + \frac{C_1}{n}$, $C_1 > 0$, *or* $\alpha_n \ge -\frac{C_2}{n}$, $C_2 > 0$;
2. $n_\pm(\mathrm{H}_{\mathrm{X},\alpha}) = 1$ *if* $\alpha_n = a(2n+1) + O(1/n)$, $a \in (-2, 0)$.

Note that the estimates in **1** follow from the estimates in **(ii)**, **(iii)** below that were obtained for a more general case of $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ in [13, Proposition 5.11] with the use of sufficient conditions for a Jacobi matrix to be self-adjoint.

Proposition 1. [13, Proposition 5.11]. *The operator $\mathrm{H}_{\mathrm{X},\alpha}$ is self-adjoint on $L^2(\mathbb{R}_+)$ if the sequence $\alpha = \{\alpha_n\}_1^\infty$ and the sequences $\{d_n\}_1^\infty$, $\{r_n\}_1^\infty$ defined by* (4) *and* (6), *correspondingly, satisfy one of the following conditions:*

(i) $\sum_{n=1}^\infty |\alpha_n| d_n d_{n+1} r_{n-1} r_{n+1} = \infty$;

(ii) *there exists a constant $C_1 > 0$ such that*

$$\alpha_n + \frac{1}{d_n}\Big(1 + \frac{r_n}{r_{n-1}}\Big) + \frac{1}{d_{n+1}}\Big(1 + \frac{r_n}{r_{n+1}}\Big) \le C_1(d_n + d_{n+1}), \quad n \in \mathbb{N};$$

(iii) *there exists a constant* $C_2 > 0$ *such that*

$$\alpha_n + \frac{1}{d_n}\Big(1 - \frac{r_n}{r_{n-1}}\Big) + \frac{1}{d_{n+1}}\Big(1 - \frac{r_n}{r_{n+1}}\Big) \geq -C_2(d_n + d_{n+1}), \quad n \in \mathbb{N}.$$

In this paper, we continue the study of the conditions obtained in [13] for the Schrödinger operator $\mathrm{H}_{\mathrm{X},\alpha}$ to be self-adjoint or to have nontrivial deficiency indices. It turned out that the conditions found for $d_n = 1/n$ can be generalized to a broader class of sequences, see Propositions 4, 5, 6. For example, for a class of the sequences $\Big\{\dfrac{1}{n^\gamma \ln^\eta n}\Big\}_2^\infty$ that belong to $\ell_2 \setminus \ell_1$ if $\langle\gamma, \eta\rangle \in (1/2, 1)\times(-\infty, +\infty) \cup \{1\}\times(-\infty, 1]$, we obtain a description for the asymptotic behavior of the sequence α such that the operator $\mathrm{H}_{\mathrm{X},\alpha}$ would either be self-adjoint or have nontrivial deficiency indices.

2. Sufficient conditions for self-adjointness of the operator $\mathrm{H}_{\mathrm{X},\alpha}$

Taking into account the above considerations, we will mention some properties of sequences $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ of positive numbers. The most important of them is the property

$$\liminf_{n\to\infty}\frac{d_{n+1}}{d_n} \leq 1 \leq \limsup_{n\to\infty}\frac{d_{n+1}}{d_n}, \tag{7}$$

which follows from the d'Alembert test for series. This immediately implies that if there exists the $\lim_{n\to\infty} \frac{d_{n+1}}{d_n}$ for a sequence $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ of positive numbers, then this limit equals 1. Using (7) and making certain restrictions on the sequence $\{d_n\}_1^\infty$ we can significantly simplify the sufficient conditions of self-adjointness in **(i)**–**(iii)**.

Proposition 2. *Let the sequence* $\{d_n\}_1^\infty$ *of positive numbers defined by* (4) *belongs to* $\ell_2 \setminus \ell_1$ *and satisfies the relation*

$$\liminf_{n\to\infty}\frac{d_{n+1}}{d_n} > 0. \tag{8}$$

Then $\mathrm{H}_{X,\alpha}$ *is self-adjoint provided that there holds the following condition:*

(I) $\sum_{n=1}^\infty |\alpha_n| d_n^3 = \infty$.

Proof. In fact, condition (8) implies that there exists a $C > 0$ such that

$$d_{n+1} > C d_n, \quad n \in \mathbb{N}.$$

It follows that

$$\begin{aligned} d_n d_{n+1} r_{n-1} r_{n+1} &= d_n d_{n+1}\sqrt{d_{n-1}+d_n}\sqrt{d_{n+1}+d_{n+2}} \\ &> d_n(Cd_n)\sqrt{d_n}\ \sqrt{Cd_n + C^2 d_n} > C\sqrt{C+C^2}d_n^3, \quad n \in \mathbb{N}. \end{aligned} \tag{9}$$

Thus, the divergence of series **(I)** implies the divergence of series **(i)** in Proposition 1, and there holds the sufficient condition for $\mathrm{H}_{X,\alpha}$ to be self-adjoint. □

Note that, for the class of sequences $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ satisfying the additional constraint

$$0 < \liminf_{n\to\infty}\frac{d_{n+1}}{d_n} \leq \limsup_{n\to\infty}\frac{d_{n+1}}{d_n} < \infty, \tag{10}$$

both series **(I)** and **(i)** converge and diverge simultaneously. Hence test **(i)** as well as Proposition 2 can be applied for such sequences.

In the next assertion, we present conditions sufficient for tests **(ii)**, **(iii)** of Proposition 1 to hold (for now, without any additional restrictions on the sequence $\{d_n\}_1^\infty$). These conditions will allow us to find simpler sufficient conditions for the Hamiltonian $\mathrm{H}_{X,\alpha}$ to be self-adjoint.

Proposition 3. *Let sequences $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ and $\{r_n\}_1^\infty$ be defined by (4), (6), and let the function*

$$F(n) = \frac{1}{d_n}\Big(\frac{r_n}{r_{n-1}} - 1\Big) + \frac{1}{d_{n+1}}\Big(\frac{r_n}{r_{n+1}} - 1\Big) \quad (n \in \mathbb{N}) \tag{11}$$

allow the representation of the form

$$F(n) = G(n) + O(d_n) \quad (n \in \mathbb{N}) \tag{12}$$

(for definiteness, we put $r_0 := 1$). Then the Hamiltonian $\mathrm{H}_{X,\alpha}$ is self-adjoint provided that one of the following conditions hold:

- **(II)** *there exists a constant $C_1 > 0$ such that*

$$\alpha_n \le -\Big(\frac{2}{d_n} + \frac{2}{d_{n+1}} + G(n)\Big) + C_1 d_n \quad (n \in \mathbb{N});$$

- **(III)** *there exists a constant $C_2 > 0$ such that $\alpha_n \ge G(n) - C_2 d_n \quad (n \in \mathbb{N})$.*

Proof. By condition **(II)**, we have

$$\alpha_n + \frac{2}{d_n} + \frac{2}{d_{n+1}} + G(n) \le C_1 d_n \quad (n \in \mathbb{N}).$$

Then

$$\alpha_n + \frac{2}{d_n} + \frac{2}{d_{n+1}} + F(n) - O(d_n) \le C_1 d_n \quad (n \in \mathbb{N}).$$

Since the sequence $\{d_n\}_1^\infty$ is positive, we conclude that

$$\alpha_n + \frac{1}{d_n}\Big(1 + \frac{r_n}{r_{n-1}}\Big) + \frac{1}{d_{n+1}}\Big(1 + \frac{r_n}{r_{n+1}}\Big) \le C d_n \le C(d_n + d_{n+1}) \quad (n \in \mathbb{N})$$

for some $C > 0$. Consequently, it follows from the sufficient condition **(ii)** of Proposition 1 that the Hamiltonian $\mathrm{H}_{X,\alpha}$ is self-adjoint.

Arguing similarly we can prove that condition **(III)** implies estimate **(iii)** of Proposition 1. □

It is important to note that, for sequences $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ satisfying (10), both estimates **(ii)**, **(II)** and **(iii)**,**(III)** are fulfilled or not fulfilled simultaneously. Hence Proposition 1 and Proposition 3 are equivalent for such sequences. Note also that tests **(II)** and **(III)** are of common use only in the case when the function G in decomposition (12) has simpler form than the function F. In this case, there are of great interest sequences $\{d_n\}_1^\infty$ such that G can be chosen as *zero* function[1]: due to test **(III)**, all the Hamiltonians $\mathrm{H}_{X,\alpha}$ with nonnegative sequences α are self-adjoint. In the propositions below, we present a number of requirements to properties of the sequence $\{d_n\}_1^\infty$ in order to provide the asymptotics $F(n) = O(d_n)$.

Proposition 4. *Let a sequence $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ satisfies the following asymptotic estimate:*

$$\frac{d_{n+1}}{d_n} = 1 + C d_n + O(d_n^2). \tag{13}$$

Then $F(n) = O(d_n)$.

Proof. Using (13) and carrying out direct calculations we can show that the following relations hold:

- **(a0)** $\frac{d_n}{d_{n+1}} = 1 - C d_n + O(d_n^2)$.

[1]See Proposition 5 and Example 2 below. Relation (12) implies directly that G is zero function if and only if $\sup_{n\in\mathbb{N}} \frac{F(n)}{d_n} < \infty$.

(a1) $\frac{r_n}{r_{n-1}} = \sqrt{\frac{d_n+d_{n+1}}{d_{n-1}+d_n}} = \sqrt{\frac{1+d_{n+1}/d_n}{1+d_{n-1}/d_n}} = \sqrt{\frac{2+Cd_n+O(d_n^2)}{2-Cd_n+O(d_n)}} = 1 + Cd_n + O(d_n^2)$;

(a2) Similarly, $\frac{r_n}{r_{n+1}} = 1 - Cd_n + O(d_n^2)$.

In this case, we obtain the following relations:

$$\begin{aligned} F(n) &= \tfrac{1}{d_n}\big(\tfrac{r_n}{r_{n-1}} - 1\big) + \tfrac{1}{d_{n+1}}\big(\tfrac{r_n}{r_{n+1}} - 1\big) = \tfrac{1}{d_n}\big(\tfrac{r_n}{r_{n-1}} - 1 + \tfrac{d_n}{d_{n+1}}\big(\tfrac{r_n}{r_{n+1}} - 1\big)\big) \\ &\overset{\mathbf{a0\text{–}a2}}{=} \tfrac{1}{d_n}\Big(Cd_n + O(d_n^2) + \Big(1 - Cd_n + O(d_n^2)\Big)\big(-Cd_n + O(d_n^2)\big)\Big) \\ &= \tfrac{1}{d_n}\big(Cd_n + O(d_n^2) - Cd_n + O(d_n^2)\big) = \tfrac{1}{d_n}O(d_n^2) = O(d_n). \end{aligned}$$

□

Example 1. *Suppose that the Hamiltonian* $\mathrm{H}_{\mathrm{X},\alpha}$ *is generated by the differential expression* (1) *as explained in Introduction, with* $X = \{x_n\}_0^\infty$ *defined by the relations*

$$x_0 = 0, \quad x_n = x_{n-1} + d_n \quad (n \in \mathbb{N}), \qquad d_n = \frac{1}{n^\gamma}, \quad \gamma \in (1/2,\, 1). \tag{14}$$

Then $\mathrm{H}_{X,\alpha}$ *is self-adjoint provided that there holds one of the following conditions:*

(s1) $\sum_{n=1}^\infty |\alpha_n| n^{-3\gamma} = \infty$.

(s2) *There exists a constant* $C_1 > 0$ *such that*

$$\alpha_n \le -2\big(n^\gamma + (n+1)^\gamma\big) + \frac{C_1}{n^\gamma} \quad (n \in \mathbb{N}).$$

(s3) *There exists a constant* $C_2 > 0$ *such that* $\alpha_n \ge -C_2 \frac{1}{n^\gamma}$ $(n \in \mathbb{N})$.

Indeed, in this case we have $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ and, by direct calculations, we conclude that $\frac{d_{n+1}}{d_n} = 1 - \frac{1}{n^\gamma} + O(\frac{1}{n^{2\gamma}})$. Then, in view of Proposition 2, condition **(s1)** provides the self-adjointness of the operator $\mathrm{H}_{X,\alpha}$. To prove that **(s2)** and **(s3)** can be applied, it suffices to note that the sequence $\{d_n\}_1^\infty$ satisfies the assumptions of Proposition 4 and, therefore, $G(n) = 0$. Further, we can use conditions **(II)**, **(III)** of Proposition 3 directly.

As was mentioned before, the particular case $\gamma = 1$ was considered in [13] (see **1** on p. 361).

In some cases (see Example 2) condition (13) is too strict. Nevertheless, if (13) does not hold, it is possible to carry out a more refined analysis of properties of the sequence $\{d_n\}_1^\infty$ leading to the asymptotics $F(n) = O(d_n)$.

Proposition 5. *Let* $p \in \mathbb{N}$, *and let the sequence* $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ *be generated by a function* d: $d_n := d(n)$ $(n \in \overline{p,\infty})$ *defined on the interval* (p,∞) *and twice continuously differentiable on it. Let also the function* d *satisfy the following conditions:*

(d0) $\frac{d_{n+1}}{d_n} = 1 + O(d_n)$.

(d1) $d'(n) \neq 0$ $(n \in \overline{m+1,\infty})$ *and* $\sup_{\substack{n\in\overline{m+1,\infty} \\ \zeta,\eta\in[-1,2]}} \frac{|d'(n+\zeta)|}{|d'(n+\eta)|} < \infty$ *for some number* $m \in \overline{p,\infty}$.

(d2) $d''(n) \neq 0$ $(n \in \overline{m+1,\infty})$ *and* $\sup_{\substack{n\in\overline{m+1,\infty} \\ \zeta,\eta\in[-1,2]}} \frac{|d''(n+\zeta)|}{|d''(n+\eta)|} < \infty$ *for some number* $m \in \overline{p,\infty}$.

(d3) $\frac{d''(n)}{d'(n)} = O(d_n)$.

Then $F(n) = O(d_n)$.

Proof. Since $d_n \to 0,\ n \to \infty$, for arbitrary positive numbers $u,\ v$ the following relations hold:

$$\text{(15)}\qquad \textbf{a)}\ \big(u + O(d_n)\big)^{-1} = u^{-1} + O(d_n); \quad \textbf{b)}\ \sqrt{\frac{u + O(d_n)}{v + O(d_n)}} = \sqrt{\frac{u}{v}} + O(d_n).$$

Then condition **(d0)** implies the preliminary asymptotic estimates

(b0) $\frac{d_n}{d_{n+1}} = 1 + O(d_n)$;

(b1) $\frac{r_n}{r_{n-1}} = 1 + O(d_n)$;

(b2) $\frac{r_n}{r_{n+1}} = 1 + O(d_n)$.

In this case, the following relations hold:

$$\text{(16)}\qquad \begin{aligned} F(n) &= \tfrac{1}{d_n}\big(\tfrac{r_n}{r_{n-1}} - 1\big) + \tfrac{1}{d_{n+1}}\big(\tfrac{r_n}{r_{n+1}} - 1\big) = \tfrac{1}{d_n}\big(\tfrac{r_n}{r_{n-1}} - 1\big)\big(1 + \tfrac{d_n}{d_{n+1}} \cdot \tfrac{r_{n-1}}{r_{n+1}} \cdot \tfrac{r_n - r_{n+1}}{r_n - r_{n-1}}\big) \\ &\overset{\textbf{(b1)}}{=} \tfrac{1}{d_n} O(d_n)\Big(1 + \tfrac{d_n}{d_{n+1}} \cdot \tfrac{r_{n-1}}{r_{n+1}} \cdot \tfrac{r_n + r_{n-1}}{r_n + r_{n+1}} \cdot \tfrac{r_n^2 - r_{n+1}^2}{r_n^2 - r_{n-1}^2}\Big) \\ &= \tfrac{1}{d_n} O(d_n)\Big(1 + \tfrac{d_n}{d_{n+1}} \cdot \big(\tfrac{r_{n-1}}{r_{n+1}}\big)^2 \cdot \tfrac{r_n/r_{n-1}+1}{r_n/r_{n+1}+1} \cdot \tfrac{r_n^2 - r_{n+1}^2}{r_n^2 - r_{n-1}^2}\Big) \\ &\overset{(\textbf{b0–b2},15,6)}{=} \tfrac{1}{d_n} O(d_n)\big(1 + \big(1 + O(d_n)\big)\big(1 + O(d_n)\big)^2 \cdot \tfrac{2+O(d_n)}{2+O(d_n)} \cdot \tfrac{d_n - d_{n+2}}{d_{n+1} - d_{n-1}}\big) \\ &\overset{(15)}{=} \tfrac{1}{d_n} O(d_n)\big(1 + \big(1 + O(d_n)\big)\tfrac{d_n - d_{n+2}}{d_{n+1} - d_{n-1}}\big) \\ &= \tfrac{1}{d_n} O(d_n)\big(\tfrac{(d_{n+1} - d_{n-1}) - (d_{n+2} - d_n)}{d_{n+1} - d_{n-1}} + O(d_n)\tfrac{d_n - d_{n+2}}{d_{n+1} - d_{n-1}}\big). \end{aligned}$$

Below we will need properties **(d1–d3)** of the sequence $\{d_n\}_1^\infty$. Namely, for a sufficiently large n and for some $C > 0$ we obtain

$$\begin{aligned} &\frac{(d_{n+1} - d_{n-1}) - (d_{n+2} - d_n)}{d_{n+1} - d_{n-1}} = 2\frac{d'(n + \zeta_n) - d'(n + \theta_n)}{d'(n + \zeta_n)} \\ &= 2(\zeta_n - \theta_n)\frac{d''(n + \xi_n)}{d'(n + \zeta_n)} = 2(\zeta_n - \theta_n)\frac{d''(n + \xi_n)}{d''(n)} \cdot \frac{d'(n)}{d'(n + \zeta_n)} \cdot \frac{d''(n)}{d'(n)} \overset{\textbf{(d1–d3)}}{=} O(d_n), \\ &\Big|\frac{d_n - d_{n+2}}{d_{n+1} - d_{n-1}}\Big| = \Big|\frac{d'(n + \theta_n)}{d'(n + \zeta_n)}\Big| \overset{\textbf{(d1)}}{<} C, \end{aligned}$$

where $\zeta_n \in [-1, 1],\ \theta_n \in [0, 2],\ \xi_n \in [-1, 2]$. Then (16) implies the following estimate of the function $F(n)$:

$$F(n) = \frac{1}{d_n} O(d_n) \cdot O(d_n) = O(d_n).$$

□

Assumptions of Proposition 5 can already be used for wider classes of sequences. Let us apply the above results to a two-parametric family of sequences including Example 1 as well.

Example 2. *Suppose that the Hamiltonian* $\mathrm{H}_{\mathrm{X},\alpha}$ *is generated by the differential expression* (1) *as explained in Introduction, with* $X = \{x_n\}_0^\infty$ *defined by the relations*

$$\text{(17)}\qquad \begin{aligned} &x_0 = 0, \quad x_n = x_{n-1} + d_n \quad (n \in \mathbb{N}), \\ &d_1 > 0, \quad d_n = \frac{1}{n^\gamma \ln^\eta n} \quad (n \in \overline{2, \infty}), \\ &\langle \gamma, \eta \rangle \in (1/2,\ 1) \times (-\infty, +\infty)\ \cup\ \{1\} \times (-\infty, 0]. \end{aligned}$$

Then $\mathrm{H}_{X,\alpha}$ *is self-adjoint provided that there holds one of the following conditions:*

(sa1) $\sum_{n=1}^\infty |\alpha_n| n^{-3\gamma} \ln^{-3\eta} n = \infty$.

(sa2) *There exists a constant* $C_1 > 0$ *such that*

$$\alpha_n \leq -2\big(n^\gamma \ln^\eta n + (n+1)^\gamma \ln^\eta (n+1)\big) + \frac{C_1}{n^\gamma \ln^\eta n} \quad (n \in \mathbb{N}).$$

(sa3) *There exists a constant* $C_2 > 0$ *such that* $\alpha_n \geq -\frac{C_2}{n^\gamma \ln^\eta n} \quad (n \in \mathbb{N})$.

Indeed, in this case we have $\{d_n\}_1^\infty \in \ell_2 \setminus \ell_1$ and, by direct calculations, we conclude that $\frac{d_{n+1}}{d_n} = 1 + O(d_n)$. It is important to note that the sequence $\frac{d_{n+1}}{d_n}$ does not satisfy estimate (13) if $\eta \neq 0$. To prove that **(sa1)** can be applied, we use Proposition 2. To prove that **(sa2)** and **(sa3)** are applicable, consider the function $d(x) = \dfrac{1}{x^\gamma \ln^\eta x}$ generating the sequence $\{d_n\}_1^\infty$. Derivatives of this function are of the form

$$d'(x) = -(\gamma \ln x + \eta) x^{-\gamma-1} \ln^{-\eta-1} x;$$
$$d''(x) = \big(\gamma(\gamma+1)\ln^2 x + (2\gamma+1)\eta \ln x + \eta(\eta+1)\big) x^{-\gamma-2} \ln^{-\eta-2} x.$$

This immediately implies conditions **(d1)**, **(d2)** of Proposition 5.

The relation

$$\frac{d''(n)}{d'(n) d_n} = -\frac{\big(\gamma(\gamma+1)\ln^2 n + (2\gamma+1)\eta \ln n + \eta(\eta+1)\big)}{(\gamma \ln^2 n + \eta \ln n)} \cdot n^{\gamma-1} \ln^\eta n$$

shows that condition **(d3)** is fulfilled either for $\frac{1}{2} < \gamma < 1$, $\eta \in (-\infty, +\infty)$, or for $\gamma = 1$, $\eta \leq 0$.

Note that in the two-parametric family of sequences of form (17) lying in $\ell_2 \setminus \ell_1$ there is a "gap" consisting of sequences of the form $\big\{\frac{1}{n \ln^\eta n}\big\}$, $\eta \in (0, 1]$. Condition **(d0)** is violated for sequences from this "gap" (despite the fact that $\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = 1$). To investigate these cases, we must know additional properties of the sequence $\{d_n\}_1^\infty$. Below, in Proposition 6, we present analytic conditions on the function d generating the sequence $\{d_n\}_1^\infty$ such that we can choose the function G of (12) to verify estimate **(II)** of Proposition 3.

Proposition 6. *Let* $p \in \mathbb{N}$*, and let a sequence* $\{d_n\}_1^\infty$ *be generated by a function* d: $d_n := d(n)$ $(n \in \overline{p, \infty})$ *that is defined on the interval* (p, ∞) *and is continuously differentiable on it. Assume also fulfillment of conditions **(d0)**, **(d1)** of Proposition 5 as well as the following condition:*

(d4) *There exists a* $k \in \mathbb{N}$ *such that* $\left|\frac{d'(n)}{d_n}\right|^k = O(d_n^2)$.

Then there exist numbers $\{C_i\}_{i \in \overline{0, k-1}}$ *such that, for* k *from **(d4)**, we have*

$$F(n) = \frac{1}{d_n} \sum_{i=1}^{k-1} C_i u(n)^i + \frac{1}{d_{n+1}} \sum_{i=1}^{k-1} C_i v(n)^i + O(d_n) \quad (n \in \overline{p, \infty}), \tag{18}$$

where $u(n) = \frac{d_{n+1} - d_{n-1}}{d_n + d_{n-1}}$, $v(n) = \frac{d_n - d_{n+2}}{d_{n+1} + d_{n+2}}$.

Proof. Taking into account (11) we obtain

$$F(n) = \frac{1}{d_n}\Big(\frac{r_n}{r_{n-1}} - 1\Big) + \frac{1}{d_{n+1}}\Big(\frac{r_n}{r_{n+1}} - 1\Big),$$

where

$$\frac{r_n}{r_{n-1}} = \sqrt{\frac{d_n + d_{n+1}}{d_n + d_{n-1}}} = \sqrt{1 + u(n)},$$
$$\frac{r_n}{r_{n+1}} = \sqrt{\frac{d_n + d_{n+1}}{d_{n+1} + d_{n+2}}} = \sqrt{1 + v(n)}.$$

Using Taylor's series expansion of the function $\sqrt{1+x} = 1 + \sum_{i=1}^{\infty} C_i x^i$ we have

$$F(n) = \frac{1}{d_n}\sum_{i=1}^{\infty} C_i u(n)^i + \frac{1}{d_{n+1}}\sum_{i=1}^{\infty} C_i v(n)^i. \tag{19}$$

Let us estimate the behavior of terms of these series at infinity. In view of **(d0)**, **(d1)**, for some $\zeta_n \in [-1,1]$, $\theta_n \in [0,2]$ we have

$$u(n) = 2\frac{d'(n+\zeta_n)}{d_n + d_{n-1}} = 2\frac{d'(n+\zeta_n)}{d'(n)} \cdot \frac{d_n}{d_n + d_{n-1}} \cdot \frac{d'(n)}{d_n} = O\Big(\frac{|d'(n)|}{d_n}\Big);$$
$$v(n) = -2\frac{d'(n+\theta_n)}{d_{n+1} + d_{n+2}} = -2\frac{d'(n+\theta_n)}{d'(n)} \cdot \frac{d_n}{d_{n+1} + d_{n+2}} \cdot \frac{d'(n)}{d_n} = O\Big(\frac{|d'(n)|}{d_n}\Big).$$

Then, for $k \in \mathbb{N}$ from **(d4)**, we obtain

$$\sum_{i=k}^{\infty} C_i u(n)^i = O\big(u(n)^k\big) = O\Big(\frac{|d'(n)|^k}{d_n^k}\Big) = O(d_n^2).$$

Similarly, we conclude that $\sum_{i=k}^{\infty} C_i v(n)^i = O(d_n^2)$. Hence estimate (18) holds. $\square$

In some cases, the right-hand side of (18) can be used to select from F "the best" estimator G satisfying the relation $F(n) = G(n) + O(d_n)$. Such a function G should include all the "parts" of sequences $\frac{d'(n)^i}{d_n^{i+1}}$ $(i \in \overline{1, k-1})$ that grow slower than d_n at infinity. Note that to obtain more effective estimates in assumptions of Proposition 6, the following argument is useful. Since

$$\frac{1}{d_n}u(n) - \frac{1}{d_{n+1}}v(n) = \frac{1}{d_n} \cdot \frac{d'(n+\zeta)}{d_n + d_{n-1}} - \frac{1}{d_{n+1}} \cdot \frac{d'(n+\theta)}{d_{n+1} + d_{n+2}}$$
$$= \frac{d_n}{d_n + d_{n+1}} \cdot \frac{d'(n+\zeta)}{d_n^2} - \frac{d_{n+1}}{d_{n+1} + d_{n+2}} \cdot \frac{d_n^2}{d_{n+1}^2} \cdot \frac{d'(n+\theta)}{d_n^2},$$

and since both expressions $\frac{d_n}{d_n + d_{n+1}}$ and $\frac{d_{n+1}}{d_{n+1}+d_{n+2}} \cdot \big(\frac{d_n}{d_{n+1}}\big)^2$ are close to 1 for sufficiently large n, we conclude that the behavior of the summand $C_1\big(\frac{1}{d_n}u(n) - \frac{1}{d_{n+1}}v(n)\big)$ at infinity is determined by the expression $\frac{d''(n)}{d_n^2}$. Trying to avoid general definitions here we will carry out the reasoning in the following example.

Example 3. Consider the sequences $\big\{\frac{1}{n \ln^\eta n}\big\}$, $\eta \in (0,1]$ (see the argument before Proposition 5, p. 366).

Put $d(x) = x^{-1}\ln^{-\eta} x$, $\eta \in (0,1]$, and $p = 3$. The function d derives the sequence $\big\{\frac{1}{n \ln^\eta n}\big\}$, is defined on the interval $(3,\infty)$ and is twice continuously differentiable on it. We have

$$d'(x) = -\tfrac{\ln x + \eta}{x^2 \ln^{\eta+1} x},$$
$$\tfrac{d'(x)}{d(x)} = -\tfrac{\ln x + \eta}{x \ln x},$$

from which we see that $\big(\frac{d\,'(x)}{d(x)}\big)^3 = O(d_n^2)$. Hence $k = 3$, and for estimating the function $F(n)$ we need to consider the expressions

$$d''(x) = \tfrac{1}{x^3}\big(\tfrac{2}{\ln^\eta x} + \tfrac{3\eta - 1}{\ln^{\eta+1} x} + \tfrac{\eta + \eta^2}{\ln^{\eta+2} x}\big);$$
$$\tfrac{d''(x)}{d^2(x)} = \tfrac{2\ln^\eta x}{x} + \tfrac{3\eta-1}{x \ln^{1-\eta} x} + \tfrac{\eta+\eta^2}{x \ln^{2-\eta} x};$$
$$\tfrac{d'(x)^2}{d^3(x)} = \tfrac{\ln^\eta x}{x} + \tfrac{2\eta}{x \ln^{1-\eta} x} + \tfrac{\eta^2}{x \ln^{2-\eta} x}.$$

This makes it possible to presuppose that F admits the asymptotic representation

$$F(n) = w_1 \tfrac{\ln^\eta n}{n} + w_2 \tfrac{1}{n \ln^{1-\eta} n} + O\big(\tfrac{1}{n \ln^\eta n}\big)$$

with some coefficients $w_1,\ w_2$. In fact, by direct calculations we can obtain that

$$\lim_{n\to\infty}\tfrac{n}{\ln^\eta n}F(n)=\tfrac14;$$
$$\lim_{n\to\infty} n\ln^{1-\eta} n\big(F(n)-\tfrac14\tfrac{\ln^\eta n}{n}\big)=\eta;$$
$$\lim_{n\to\infty} n\ln^{\eta} n\big(F(n)-\tfrac14\tfrac{\ln^\eta n}{n}-\tfrac{\eta}{n\ln^{1-\eta}n}\big)=\begin{cases}0, & \eta\in(0,1)\\ \tfrac14, & \eta=1\end{cases},$$

for $\eta\in(0,1]$, which gives the asymptotics

$$F(n)=\begin{cases}\tfrac14\tfrac{\ln^\eta n}{n}+O\big(\tfrac{1}{n\ln^\eta n}\big), & \eta\in(0,\tfrac12]\\ \tfrac14\tfrac{\ln^\eta n}{n}+\tfrac{\eta}{n\ln^{1-\eta}n}+O\big(\tfrac{1}{n\ln^\eta n}\big), & \eta\in(\tfrac12,1]\end{cases}. \tag{20}$$

Finally, we have $G(n)=\begin{cases}\tfrac14\tfrac{\ln^\eta n}{n}, & \eta\in(0,\tfrac12]\\ \tfrac14\tfrac{\ln^\eta n}{n}+\tfrac{\eta}{n\ln^{1-\eta}n}, & \eta\in(\tfrac12,1]\end{cases}$. Let us summarize our considerations in the form of sufficient conditions for the operator $\mathrm{H}_{\mathrm{X},\alpha}$ to be self-adjoint.

Suppose that the Hamiltonian $\mathrm{H}_{\mathrm{X},\alpha}$ is generated by the differential expression (1) *as explained in Introduction, with $X=\{x_n\}_0^\infty$ defined by the relations*

$$x_0=0,\quad x_n=x_{n-1}+d_n\quad (n\in\mathbb{N}),$$
$$d_1>0,\quad d_n=\tfrac{1}{n\ln^\eta n}\quad (n\in\overline{2,\infty}),\quad \eta\in(0,1].$$

Then $\mathrm{H}_{X,\alpha}$ is self-adjoint provided that there holds one of the following conditions:

- **(sa1)** $\sum_{n=1}^\infty|\alpha_n|n^{-3}\ln^{-3\eta}n=\infty$.
- **(sa2)** *There exists a constant $C_1>0$ such that*

$$\alpha_n\le -2\big(n\ln^\eta n+(n+1)\ln^\eta(n+1)\big)+\begin{cases}\tfrac14\tfrac{\ln^\eta n}{n}, & \eta\in(0,\tfrac12]\\ \tfrac14\tfrac{\ln^\eta n}{n}+\tfrac{\eta}{n\ln^{1-\eta}n}, & \eta\in(\tfrac12,1]\end{cases}+\frac{C_1}{n\ln^\eta n}.$$

- **(sa3)** *There exists a constant $C_2>0$ such that*

$$\alpha_n\ge\begin{cases}\tfrac14\tfrac{\ln^\eta n}{n}, & \eta\in(0,\tfrac12]\\ \tfrac14\tfrac{\ln^\eta n}{n}+\tfrac{\eta}{n\ln^{1-\eta}n}, & \eta\in(\tfrac12,1]\end{cases}-\frac{C_2}{n\ln^\eta n}.$$

3. Sufficient conditions for non-triviality of $n_\pm(\mathrm{H}_{\mathrm{X},\alpha})$

For a positive sequence $\{d_n\}_1^\infty$, define the sequence $\{\tilde r_n\}_1^\infty$ recursively

$$\tilde r_1:=1,\quad \tilde r_{n+1}:=-\frac{d_{n+1}}{\tilde r_n}\quad (n\in\mathbb{N}) \tag{21}$$

(here we generalize the arguments from [13, Proposition 5.13] regarding the case $d_n=1/n$).

It is easy to show by induction that

$$\tilde r_{n+1}:=(-1)^n\frac{d_{n+1}d_{n-1}\cdots}{d_nd_{n-2}\cdots}\quad (n\in\mathbb{N}). \tag{22}$$

We say that a sequence $\{d_n\}_1^\infty$ satisfies condition (A) if

$$\{r_n\tilde r_n\}_1^\infty\in\ell_2,$$

and we say that it satisfies condition (B) if

$$\left(\frac{1}{d_n}+\frac{1}{d_{n+1}}\right)\tilde r_n^2=u_n+O(r_n^2\tilde r_n^2),$$

where $\{u_n\}_1^\infty$ is a real *periodic* sequence.

It follows from the results of [13] that the sequence $d_n = 1/n$ satisfies both conditions (A) and (B), and the period of the sequence $\{u_n\}_1^\infty$ equals 2; $u_1 = 4/\pi,\ u_2 = \pi$. As it turns out, this result is general enough for sequences satisfying conditions (A) and (B). Namely, the following statement holds.

Lemma 1. *Suppose that a sequence $\{d_n\}_1^\infty \in \ell_2\backslash\ell_1$ such that $\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = 1$ satisfies also conditions (A) and (B). Then the sequence $\{u_n\}_1^\infty$ has the period equal to 2, with $u_1u_2 = 4$.*

Proof. Let the sequence $\{d_n\}_1^\infty$ satisfy conditions (A) and (B), and let N be the period of the sequence $\{u_n\}_1^\infty$. We denote

$$\rho_n := \Big(\frac{1}{d_n} + \frac{1}{d_{n+1}}\Big)\,\tilde{r}_n^2.$$

In view of condition (B), $\rho_{1+kN} = u_1 + O(r_{1+kN}^2\tilde{r}_{1+kN}^2)$. If $1 < s < N$ is an arbitrary odd number, we have

$$\rho_{s+kN} = \Big(\frac{1}{d_{s+kN}} + \frac{1}{d_{s+kN+1}}\Big)\,\Big(\frac{d_{s+kN}d_{s+kN-2}\dots}{d_{s+kN-1}d_{s+kN-3}\dots}\Big)^2 = \Theta(s,k)\rho_{1+kN}, \tag{23}$$

where

$$\Theta(s,k) = \Big(\frac{1}{d_{s+kN}} + \frac{1}{d_{s+kN+1}}\Big)\,\Big(\frac{1}{d_{1+kN}} + \frac{1}{d_{2+kN}}\Big)^{-1}\Big(\frac{d_{s+kN}d_{s+kN-2}\dots d_{3+kN}}{d_{s+kN-1}d_{s+kN-3}\dots d_{2+kN}}\Big)^2$$

and also $\lim_{k\to\infty}\Theta(s,k) = 1$. Since $\rho_{s+kN} = u_s + O(r_{s+kN}^2\tilde{r}_{s+kN}^2)$, by passing to the limit in relation (23) as $k\to\infty$, we obtain

$$u_s = u_1.$$

Thus, for an arbitrary odd $n = 2k+1$ we have

$$\rho_{2k+1} = u_1 + O(r_{2k+1}^2\tilde{r}_{2k+1}^2).$$

Arguing similarly we can show that for an arbitrary even $n = 2k$ we obtain

$$\rho_{2k} = u_2 + O(r_{2k}^2\tilde{r}_{2k}^2).$$

Since

$$\rho_{2k}\rho_{2k+1} = \Big(1 + \frac{d_{2k+1}}{d_{2k+2}}\Big)\Big(1 + \frac{d_{2k+1}}{d_{2k}}\Big),$$

we arrive at the relation

$$\big(u_1 + O(r_{2k+1}^2\tilde{r}_{2k+1}^2)\big)\big(u_2 + O(r_{2k}^2\tilde{r}_{2k}^2)\big) = \Big(1 + \frac{d_{2k+1}}{d_{2k+2}}\Big)\Big(1 + \frac{d_{2k+1}}{d_{2k}}\Big).$$

Finally, by passing to the limit in the latter as $k\to\infty$, we conclude that

$$u_1u_2 = 4.$$

□

Proposition 7. *Suppose that the Hamiltonian $\mathrm{H}_{X,\alpha}$ is defined by the sequence $\{d_n\}_1^\infty \in \ell_2\backslash\ell_1$ such that $\lim_{n\to\infty}\frac{d_{n+1}}{d_n} = 1$ and for which conditions (A) and (B) are satisfied.*

If

$$\alpha_n = a\Big(\frac{1}{d_n} + \frac{1}{d_{n+1}}\Big) + O(d_n),$$

where the parameter a satisfies the inequality $-2 < a < 0$, then the Hamiltonian $\mathrm{H}_{X,\alpha}$ is a symmetric operator with deficiency indices $n_\pm = 1$.

Proof. Due to Lemma 1, for the given sequence d_n, the real periodic sequence $\{u_n\}_1^\infty$ in condition (B) has the period equal to 2, with $u_1u_2 = 4$.

Consider the sequence

$$\alpha_n^0 := -\Big(\frac{1}{d_n} + \frac{1}{d_{n+1}}\Big) + (a+1)u_n\tilde{r}_n^{-2}.$$

It follows that

$$B_{X,\alpha^0} = \begin{pmatrix} r_1^{-2}\tilde{r}_1^{-2}(a+1)u_1 & -r_1^{-1}r_2^{-1}d_2^{-1} & 0 & \dots \\ -r_1^{-1}r_2^{-1}d_2^{-1} & r_2^{-2}\tilde{r}_2^{-2}(a+1)u_2 & -r_2^{-1}r_3^{-1}d_3^{-1} & \dots \\ 0 & -r_2^{-1}r_3^{-1}d_3^{-1} & r_3^{-2}\tilde{r}_3^{-2}(a+1)u_1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

If $R_X = \operatorname{diag}(r_n)$, $\widetilde{R}_1 = \operatorname{diag}(\widetilde{r}_n)$, we have

$$\widetilde{R}_1 R_X B_{X,\alpha^0} R_X \widetilde{R}_1 = \begin{pmatrix} (a+1)u_1 & -\tilde{r}_1\tilde{r}_2 d_2^{-1} & 0 & \dots \\ -\tilde{r}_1\tilde{r}_2 d_2^{-1} & (a+1)u_2 & -\tilde{r}_2\tilde{r}_3 d_3^{-1} & \dots \\ 0 & -\tilde{r}_2\tilde{r}_3 d_3^{-1} & (a+1)u_1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Since $-\tilde{r}_n\tilde{r}_{n+1}d_{n+1}^{-1} = -\tilde{r}_n \cdot \frac{-d_{n+1}}{\tilde{r}_n} \cdot \frac{1}{d_{n+1}} = 1$, then

$$\widetilde{R}_1 R_X B_{X,\alpha^0} R_X \widetilde{R}_1 = J_a, \tag{24}$$

where

$$J_a = \begin{pmatrix} (a+1)u_1 & 1 & 0 & \dots \\ 1 & (a+1)u_2 & 1 & \dots \\ 0 & 1 & (a+1)u_1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

is the periodic Jacobi matrix defined by both the sequence $\{u_n\}_1^\infty$ and the real parameter a.

In view of (24), the vector f^0 is the solution of the equation $J_a f = 0$ if and only if the vector $h^0 = R_X\widetilde{R}_1 f^0$ is the solution of the equation $B_{X,\alpha^0}h = 0$. If $f^0 = \{f_n\}$ is a bounded sequence, due to condition (A) we obtain that $h^0 = \{r_n\tilde{r}_n f_n\} \in \ell_2$.

As is known [16], solutions to the equation $J_a f = 0$ are bounded if there holds the inequality $|\Delta_a(0)| < 1$ for the Floquet discriminant. It follows from the above considerations that the Jacobi matrix determined by the sequence $\{u_n\}_1^\infty$ has the period equal to 2. Consequently,

$$\Delta_a(\lambda) = 1/2\big(-2 + (\lambda - (a+1)u_1)(\lambda - (a+1)u_2)\big).$$

This yields $\Delta_a(0) = 1/2(-2 + (a+1)^2u_1u_2) = 1/2(-2 + 4(a+1)^2) = 2(a+1)^2 - 1$. Hence, $|\Delta_a(0)| < 1$ if $-2 < a < 0$. Thus, under this condition, a solution to the equation $B_{X,\alpha^0}h = 0$ belongs to ℓ_2, and also B_{X,α^0} is symmetric operator with deficiency indices $n_\pm = 1$.

We can simplify the general form of the sequence α^0. Indeed, condition (B) implies that

$$(a+1)u_n\tilde{r}_n^{-2} = (a+1)\Big(\frac{1}{d_n} + \frac{1}{d_{n+1}}\Big) + (a+1)O(r_n^2\tilde{r}_n^2)\tilde{r}_n^{-2},$$

where $(a+1)O(r_n^2\tilde{r}_n^2)\tilde{r}_n^{-2} = O(r_n^2) = O(d_n)$. It follows that

$$(a+1)u_n\tilde{r}_n^{-2} = (a+1)\Big(\frac{1}{d_n} + \frac{1}{d_{n+1}}\Big) + O(d_n),$$

and the sequence α of the form

$$\alpha_n = a\Big(\frac{1}{d_n} + \frac{1}{d_{n+1}}\Big) + O(d_n)$$

derives a bounded self-adjoint perturbation of the operator B_{X,α^0}. Since deficiency indices do not change under such a perturbation, we conclude that $B_{X,\alpha}$ is a symmetric operator with deficiency indices $n_\pm = 1$ as well. $\square$

Example 4. Let $d_n = \frac{1}{n^\gamma}$ $(n \in \mathbb{N})$, $\gamma \in (\frac{1}{2}, 1]$. For the given sequence, we have

$$\tilde{r}_n = (-1)^{n-1}\big(\tfrac{(n-1)!!}{n!!}\big)^\gamma \quad (n \in \mathbb{N}),$$

which implies that

$$\tilde{r}_n^2 = \big(\tfrac{(n-1)!!}{n!!}\big)^{2\gamma} = \tfrac{1}{(2n+1)^\gamma}\Big((2n+1)\big(\tfrac{(n-1)!!}{n!!}\big)^2\Big)^\gamma \quad (n \in \mathbb{N}). \tag{25}$$

Let us use the asymptotics

$$(2n+1)\big(\tfrac{(n-1)!!}{n!!}\big)^2 = \begin{cases} \pi + O(n^{-2}), & \text{if } n \text{ is } odd \\ \frac{4}{\pi} + O(n^{-2}), & \text{if } n \text{ is } even \end{cases} \tag{26}$$

obtained in [13, Proposition 5.13], and also the asymptotics derived by the chain of relations

$$\tfrac{n^\gamma+(n+1)^\gamma}{(2n+1)^\gamma} = \tfrac{1+\left(1+\frac{1}{n}\right)^\gamma}{2^\gamma\left(1+\frac{1}{2n}\right)^\gamma} = \tfrac{1}{2^\gamma}\big(2+\tfrac{\gamma}{n}+O(n^{-2})\big)\big(1-\tfrac{\gamma}{2n}+O(n^{-2})\big) = 2^{1-\gamma}+O(n^{-2}). \tag{27}$$

We have several relations $(n \in \mathbb{N})$

$$d_n = n^{-\gamma} \sim r_n^2\,, \quad n^{-2} = O(n^{-2\gamma})\,, \quad \tilde{r}_n^2 \overset{(25,26)}{=} O(n^{-\gamma})\,.$$

They immediately yield that

$$\text{a) } n^{-2} = O(d_n\tilde{r}_n^2)\,, \quad \text{b) } r_n^2\tilde{r}_n^2 = O(n^{-2\gamma}) \tag{28}$$

(here $n \in \mathbb{N}$). In view of (28b), condition (A) holds for the sequence $\{d_n\}_1^\infty$. Moreover, putting $w := (\pi,\, \frac{4}{\pi},\, \pi,\, \frac{4}{\pi}, \ldots)$ and $u_n := 2^{1-\gamma}w_n^\gamma$ $(n \in \mathbb{N})$, we obtain the chain

$$\begin{aligned}\big(\tfrac{1}{d_n} + \tfrac{1}{d_{n+1}}\big)\tilde{r}_n^2 &\overset{(25)}{=} \tfrac{n^\gamma+(n+1)^\gamma}{(2n+1)^\gamma}\Big((2n+1)\big(\tfrac{(n-1)!!}{n!!}\big)^2\Big)^\gamma \\ &\overset{(26,27)}{=} \big(2^{1-\gamma} + O(n^{-2})\big)\big(w_n + O(n^{-2})\big)^\gamma \overset{(28a)}{=} u_n + O(d_n\tilde{r}_n^2) \quad (n \in \mathbb{N}).\end{aligned}$$

We combine the essence of these considerations with particular case **(sa2)**, **(sa3)** of Example 2 for $\eta = 0$ (see p. 366) in order to demonstrate the dependence of the Hamiltonian $\mathrm{H}_{X,\alpha}$ on an asymptotic behavior of the sequence α.

Suppose that the Hamiltonian $\mathrm{H}_{\mathrm{X},\alpha}$ *is generated by the differential expression* (1) *as explained in Introduction, with* $X = \{x_n\}_0^\infty$ *defined by the relations*

$$x_0 = 0, \quad x_n = x_{n-1} + \tfrac{1}{n^\gamma} \quad (n \in \mathbb{N}), \quad \gamma \in (\tfrac{1}{2}, 1].$$

Then we have

if $\alpha_n \le -2\big(n^\gamma + (n+1)^\gamma\big) + C_1 n^{-\gamma}$ $(n \in \mathbb{N})$ *for some* $C_1 > 0$, *then* $\mathrm{H}_{\mathrm{X},\alpha}$ *is self-adjoint;*

if $\alpha_n = a\big(n^\gamma + (n+1)^\gamma\big) + O(n^{-\gamma})$ $(n \in \mathbb{N})$ *for some* $a \in (-2, 0)$, *then* $n_\pm(\mathrm{H}_{\mathrm{X},\alpha}) = 1$*;*

if $\alpha_n \ge -C_2 n^{-\gamma}$ $(n \in \mathbb{N})$ *for some* $C_2 > 0$, *then* $\mathrm{H}_{\mathrm{X},\alpha}$ *is self-adjoint.*

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Tavrida National V. I. Vernadsky University, 4 Acad. Vernadsky Ave., Simferopol, 95007, Ukraine

E-mail address: i_karpenko@ukr.net

Tavrida National V. I. Vernadsky University, 4 Acad. Vernadsky Ave., Simferopol, 95007, Ukraine

E-mail address: dtyshk@inbox.ru

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