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INTERTWINING PROPERTIES OF BOUNDED LINEAR OPERATORS ON THE BERGMAN SPACE

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Abstract. In this paper we find conditions on $\phi, \psi \in L^{\infty}(\mathbb{D})$ that are necessary and sufficient for the existence of bounded linear operators S, T from the Bergman space $L^2_a(\mathbb{D})$ into itself such that for all $z \in \mathbb{D}$, $\phi(z) = \langle Sk_z, k_z, \rangle, \psi(z) = \langle Tk_z, k_z\rangle$ and $C_aS = TC_a$ for all $a \in \mathbb{D}$ where $C_af = f \circ \phi_a$ for all $f \in L^2_a(\mathbb{D})$ and $\phi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$. Applications of the results are also discussed.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of the disk is 1. Let $L^2(\mathbb{D}, dA)$ be the Hilbert space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_2 = \Big[\int_{\mathbb{D}} |f(z)|^2 dA(z)\Big]^{\frac{1}{2}} < \infty.$$

The inner product is defined as

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z)$$

for $f, g \in L^2(\mathbb{D}, dA)$. The Bergman space $L^2_a(\mathbb{D})$ is the set of those functions in $L^2(\mathbb{D}, dA)$ that are analytic on $\mathbb{D}$. The Bergman space $L^2_a(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$, and so there is an orthogonal projection P from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. Let $K(z, \bar{w})$ be the function on $\mathbb{D}\times\mathbb{D}$ defined by $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, \bar{w})$ is called the Bergman kernel of $\mathbb{D}$ or the reproducing kernel of $L^2_a(\mathbb{D})$ because the formula

$$f(z) = \int_{\mathbb{D}} f(w)K(z, \bar{w})\, dA(w)$$

reproduces each f in L^2_a. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. Then $\{e_n\}$ forms an orthonormal basis for $L^2_a(\mathbb{D})$ and $K(z, \bar{w}) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)} = \frac{1}{(1-z\bar{w})^2}$. Let $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of L^2_a; it is clear that they are unit vectors in L^2_a. For any $a \in \mathbb{D}$, let ϕ_a be the analytic mapping on $\mathbb{D}$ defined by $\phi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$. An easy calculation shows [12] that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is

$$J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4}.$$

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Let Aut($\mathbb{D}$) be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. Let $L^\infty(\mathbb{D}, dA)$ be the Banach space of all essentially bounded measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \text{ ess sup } \{|f(z)| : z \in \mathbb{D}\}$$

and $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$.

For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ from $L^2_a(\mathbb{D})$ into itself is the operator defined by $T_\phi f = P(\phi f)$. For $\phi \in L^\infty(\mathbb{D})$, the multiplication operator M_ϕ with symbol ϕ from $L^2(\mathbb{D}, dA)$ into itself is defined by $M_\phi f = \phi f$. By a harmonic function we mean a complex valued function on $\mathbb{D}$ whose Laplacian is identically 0.

Let $h^\infty(\mathbb{D})$ be the space of bounded harmonic functions on $\mathbb{D}$. Then $h^\infty(\mathbb{D}) \subset L^\infty(\mathbb{D})$. It is well known that every harmonic function on $\mathbb{D}$ is the sum of an analytic function and conjugate of another analytic function. Hence if $f \in h^\infty(\mathbb{D})$ then $f(z) = \sum_{n=0}^\infty a_n z^n + \sum_{n=0}^\infty b_n \overline{z}^n$.

Let $\mathcal{L}(L^2_a(\mathbb{D}))$ be the of all bounded linear operators from $L^2_a(\mathbb{D})$ into itself and $\mathcal{LC}(L^2_a(\mathbb{D}))$ be the subspace of $\mathcal{L}(L^2_a(\mathbb{D}))$ consisting of all compact operators from $L^2_a(\mathbb{D})$ into itself. For linear operators $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ define the Berezin transform by the formula

$$\widetilde{T}(z) = \sigma(T)(z) = \langle Tk_z, k_z \rangle, \quad z \in \mathbb{D}.$$

If $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ then $|\sigma(T)(z)| = |\langle Tk_z, k_z\rangle| \leq \|T\|$ for all $z \in \mathbb{D}$. Hence $\sigma(T) \in L^\infty(\mathbb{D})$ and $\|\sigma(T)\|_\infty \leq \|T\|$.

2. The unitary operator U_λ and the Berezin transform

Given $\lambda \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_\lambda f$ on $\mathbb{D}$ by $U_\lambda f(z) = k_\lambda(z) f(\phi_\lambda(z))$. Notice that U_λ is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $L^2_a(\mathbb{D})$ for all $\lambda \in \mathbb{D}$. Further, it can be checked that $U_\lambda^2 = I$, the identity operator, $U_\lambda^* = U_\lambda, U_\lambda(L^2_a) \subset (L^2_a)$ and $U_\lambda((L^2_a)^\perp) \subset (L^2_a)^\perp$ for all $\lambda \in \mathbb{D}$. Thus $U_\lambda P = PU_\lambda$ for all $\lambda \in \mathbb{D}$ where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define the function $C_a f$ by $C_a f(z) = f(\phi_a(z))$, where $\phi_a \in Aut(\mathbb{D})$. The map C_a is a composition operator on $L^2_a(\mathbb{D})$.

Lemma 2.1. For $z, \omega \in \mathbb{D}$, $U_z k_\omega = \alpha k_{\phi_z(\omega)}$ for some complex constant α such that $|\alpha| = 1$.

Proof. Suppose $z, \omega \in \mathbb{D}$. If $f \in L^2_a(\mathbb{D})$, then

$$\langle f, U_z K_\omega \rangle = \langle U_z f, K_\omega\rangle = (U_z f)(\omega) = -(f \circ \phi_z)(\omega)\phi_z^{'}(\omega) = \langle f, (-\overline{\phi_z'(\omega)})K_{\phi_z(\omega)}\rangle.$$

Thus $U_z K_\omega = -\overline{\phi_z'(\omega)} K_{\phi_z(\omega)}$. Rewriting this in terms of the normalized reproducing kernels, we have

$$U_z k_\omega = \alpha k_{\phi_z(\omega)}$$

for some complex constant α. Since U_z is unitary and $\|k_\omega\|_2 = \|k_{\phi_z(\omega)}\|_2 = 1$, we obtain that $|\alpha| = 1$. □

Lemma 2.2. For all $a \in \mathbb{D}$, $U_a k_a = 1$.

Proof. If $a \in \mathbb{D}$, then first observe that $\phi_a^{'}(z) = -k_a(z)$. Since $(\phi_a \circ \phi_a)(z) = z$ for all $z \in \mathbb{D}$, taking derivatives with respect to z both the sides we obtain

$$(U_a k_a)(z) = k_a(\phi_a(z))k_a(z) = 1.$$

□

Notice that for all $a \in \mathbb{D}$, since $U_a k_a = 1$, hence $k_a \circ \phi_a = \dfrac{1}{k_a}$ and $k_a^{-1} \in H^\infty(\mathbb{D})$, the space of bounded analytic functions on $\mathbb{D}$.

Lemma 2.3. If $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ and for all $z \in \mathbb{D}$, $\widetilde{S}(z) = \widetilde{T}(z)$, then $S = T$.

Proof. If $\widetilde{S}(z) = \widetilde{T}(z)$ for all $z \in \mathbb{D}$, then

$$\langle (S-T)k_z, k_z\rangle = 0$$

for all $z \in \mathbb{D}$. This implies

$$\langle (S-T)K_z, K_z\rangle = K(z,z)\langle (S-T)k_z, k_z\rangle = K(z,z)\cdot 0 = 0.$$

Let $L = S - T$ and define

$$F(x,y) = \langle LK_{\bar{x}}, K_y\rangle.$$

The function F is holomorphic in x and y and $F(x,y) = 0$ if $x = \bar{y}$. It can now be verified that such functions must vanish identically. Let $x = u + iv,\ y = u - iv$. Let $G(u,v) = F(x,y)$. The function G is holomorphic and vanishes if u and v are real. Hence by the uniqueness theorem (see [3],[8]), $F(x,y) = G(u,v) \equiv 0$. Thus even $\langle LK_x, K_y\rangle = 0$ for any x, y. Since linear combinations of $K_x, x \in \mathbb{D}$ are dense in $L^2_a(\mathbb{D})$, it follows that $L = 0$. That is, $S = T$. □

3. Intertwining properties of operators

In this section we find conditions on $\phi, \psi \in L^\infty(\mathbb{D})$ that are necessary and sufficient for the existence of bounded linear operators S, T from the Bergman space $L^2_a(\mathbb{D})$ into itself such that for all $z \in \mathbb{D}$, $\phi(z) = \langle Sk_z, k_z, \rangle, \psi(z) = \langle Tk_z, k_z\rangle$ and $C_aS = TC_a$ for all $a \in \mathbb{D}$ where $C_af = f \circ \phi_a$ for all $f \in L^2_a(\mathbb{D})$ and $\phi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$.

Definition 3.1. A function $g(x,\bar{y})$ on $\mathbb{D}\times\mathbb{D}$ is called of positive type (or positive definite), written $g \gg 0$, if

$$\sum_{j,k=1}^{n} c_j\bar{c}_k g(x_j, \bar{x}_k) \geq 0 \tag{1}$$

for any $n-$ tuple of complex numbers $c_1, \ldots, c_n$ and points $x_1, \ldots, x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$.

We say $\phi \in \mathcal{A}$ if $\phi \in L^\infty(\mathbb{D})$ and is such that

$$\phi(z) = \Omega(z, \bar{z}), \tag{2}$$

where $\Omega(x,\bar{y})$ is a function on $\mathbb{D}\times\mathbb{D}$ meromorphic in x and conjugate meromorphic in y. It is a fact that (see [5],[7]) Ω as in (2), if it exists, is uniquely determined by ϕ.

We say the function Ω satisfies the condition (*) if there exists a constant $C > 0$ such that

$$CK(x,\bar{y}) \gg \Omega(x,\bar{y})K(x,\bar{y}) \gg -CK(x,\bar{y}).$$

For $\phi \in L^\infty(\mathbb{D}, dA)$, let

$$\widehat{\phi}(z) = \int_{\mathbb{D}} \phi(\phi_a(z))\, dA(a)$$

and

$$\widetilde{\phi}(z) = \int_{\mathbb{D}} \phi(\phi_z(w))\, dA(w).$$

Notice that

$$\widetilde{\phi}(z) = \langle \phi k_z, k_z\rangle.$$

If $\phi \in L^\infty(\mathbb{D})$ then ϕ is said to satisfy the condition (**) if $\phi \in \mathcal{A}$ and $\phi(z) = \Omega(z,\bar{z})$ as in (2) and

$$\Omega_1(x,\bar{y}) = \Omega(x,\bar{y}) + \overline{\Omega(y,\bar{x})}$$

and

$$\Omega_2(x,\bar{y}) = i(\Omega(x,\bar{y}) - \overline{\Omega(y,\bar{x})})$$

satisfy the condition (*).

Theorem 3.2. The functions $\phi, \theta \in L^\infty(\mathbb{D})$ satisfy the condition (**) and $\widehat{\phi} = \theta$ if and only if there exist $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that for all $z \in \mathbb{D}$, $\phi(z) = \langle Sk_z, k_z\rangle$ and $\theta(z) = \langle Tk_z, k_z\rangle$ and $C_a S = TC_a$ for all $a \in \mathbb{D}$.

Proof. Let $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ and

$$\Omega(x, \bar{y}) = \frac{\langle SK_y, K_x\rangle}{\langle K_y, K_x\rangle}, \tag{3}$$

where $K_x = K(., \bar{x})$ is the unnormalized reproducing kernel at x. Then $\Omega(x, \bar{y})$ is a function on $\mathbb{D}\times\mathbb{D}$ meromorphic in x and conjugate meromorphic in y. Let $\phi(z) = \Omega(z, \bar{z})$. Then $\phi(z) = \langle Sk_z, k_z\rangle$ for all $z \in \mathbb{D}$ and $\phi \in L^\infty(\mathbb{D})$ as S is bounded. Thus $\phi \in \mathcal{A}$.

Now let $\phi \in \mathcal{A}$ and $\phi(z) = \Omega(z, \bar{z})$ where $\Omega(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. We shall prove the existence of some S (possibly unbounded) such that $\langle Sk_z, k_z\rangle = \phi(z)$. Let

$$Sf(x) = \int_{\mathbb{D}} f(z)\Omega(x, \bar{z})K(x, \bar{z})\, dA(z). \tag{4}$$

Indeed,

$$\begin{aligned} Sf(x) = \langle Sf, K_x\rangle = \langle f, S^*K_x\rangle &= \int_{\mathbb{D}} f(z)\overline{\langle S^*K_x, K_z\rangle}\, dA(z) \\ = \int_{\mathbb{D}} f(z)\langle SK_z, K_x\rangle\, dA(z) &= \int_{\mathbb{D}} f(z)\Omega(x, \bar{z})K(x, \bar{z})\, dA(z). \end{aligned}$$

Then

$$\begin{aligned} \langle SK_y, K_x\rangle &= \int_{\mathbb{D}} K_y(z)\Omega(x, \bar{z})K(x, \bar{z})\, dA(z) = \int_{\mathbb{D}} K_y(z)\Omega(x, \bar{z})\overline{K_x(z)}\, dA(z) \\ &= \overline{\langle \overline{\Omega(x, \bar{z})}K_x, K_y\rangle} = \overline{\overline{\Omega(x, \bar{y})}\langle K_x, K_y\rangle} = \Omega(x, \bar{y})\langle K_y, K_x\rangle. \end{aligned}$$

Hence $\Omega(x, \bar{y}) = \frac{\langle SK_y, K_x\rangle}{\langle K_y, K_x\rangle}$ and $\phi(z) = \Omega(z, \bar{z}) = \langle Sk_z, k_z\rangle$. Notice however that the operator S given by (4) may well be unbounded. We shall now prove a necessary and sufficient condition for S to be bounded and positive is that there exists $C > 0$ such that

$$CK(x, \bar{y}) \gg \Omega(x, \bar{y})K(x, \bar{y}) \gg 0. \tag{5}$$

Suppose there exists a constant $C > 0$ such that for all $x, y \in \mathbb{D}$, (5) holds. We shall show that S is bounded and positive. Let $f = \sum_{j=1}^n c_j K_{x_j}$ where c_j are constants, $x_j \in \mathbb{D}$ for $j = 1, 2, \ldots, n$. Then

$$\begin{aligned} \langle Sf, f\rangle &= \Big\langle S\Big(\sum_{j=1}^n c_j K_{x_j}\Big), \sum_{j=1}^n c_j K_{x_j}\Big\rangle \\ &= \sum_{j,k=1}^n c_j\overline{c_k}\langle SK_{x_j}, K_{x_k}\rangle = \sum_{j,k=1}^n c_j\overline{c_k}\Omega(x_k, \bar{x_j})K(x_k, \bar{x_j}) \geq 0 \end{aligned}$$

and

$$\begin{aligned} \langle Sf, f\rangle &= \sum_{j,k=1}^n c_j\overline{c_k}\langle SK_{x_j}, K_{x_k}\rangle = \sum_{j,k=1}^n c_j\overline{c_k}\Omega(x_k, \bar{x_j})K(x_k, \bar{x_j}) \\ &\leq C\sum_{j,k=1}^n c_j\overline{c_k}K(x_k, \bar{x_j}) = C\|f\|^2. \end{aligned}$$

Since the set of vectors $\{\sum_{j=1}^n c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, 2, \ldots, n\}$ is dense in $L^2_a(\mathbb{D})$, hence $0 \leq \langle Sf, f\rangle \leq C\|f\|^2$ for all $f \in L^2_a(\mathbb{D})$ and S is bounded and positive.

Conversely, suppose S is bounded and positive. Then there exists a constant $C > 0$ such that $0 \leq \langle Sf, f\rangle \leq C\|f\|^2$ for all $f \in L^2_a(\mathbb{D})$. That is, if $f = \sum_{j=1}^n c_j K_{x_j}$, then

$$0 \leq \langle Sf, f\rangle = \sum_{j,k=1}^n c_j\overline{c_k}\langle SK_{x_j}, K_{x_k}\rangle = \sum_{j,k=1}^n c_j\overline{c_k}\Omega(x_k, \bar{x_j})K(x_k, \bar{x_j})$$
$$\leq C\|f\|^2 = C\sum_{j,k=1}^n c_j\overline{c_k}K(x_k, \bar{x_j}).$$

Thus $CK(x, \bar{y}) \gg \Omega(x, \bar{y})K(x, \bar{y}) \gg 0$.

Now suppose $CK(x, \bar{y}) \gg \Omega(x, \bar{y})K(x, \bar{y}) \gg -CK(x, \bar{y})$ for all $x, y \in \mathbb{D}$. Let $f = \sum_{j=1}^n c_j K_{x_j}$. Then

$$\langle Sf, f\rangle = \sum_{j,k=1}^n c_j\overline{c_k}\Omega(x_k, \bar{x_j})K(x_k, \bar{x_j}) \leq C\sum_{j,k=1}^n c_j\overline{c_k}K(x_k, \bar{x_j}) = C\|f\|^2$$

and

$$\langle Sf, f\rangle = \sum_{j,k=1}^n c_j\overline{c_k}\Omega(x_k, \bar{x_j})K(x_k, \bar{x_j}) \geq -C\sum_{j,k=1}^n c_j\overline{c_k}K(x_k, \bar{x_j}) = -C\|f\|^2.$$

Hence S is bounded and self-adjoint. Conversely, if S is bounded and self-adjoint then there exists a constant $C > 0$ such that $-C\|f\|^2 \leq \langle Sf, f\rangle \leq C\|f\|^2$. That is, $CK(x, \bar{y}) \gg \Omega(x, \bar{y})K(x, \bar{y}) \gg -CK(x, \bar{y})$ and thus Ω satisfies the condition (*). Suppose S is bounded. Then $S = \frac{S+S^*}{2} + i\frac{S-S^*}{2i} = S_1 + iS_2$ where S_1 and S_2 are bounded and self-adjoint.

Let $\Psi_1(x, \bar{y}) = \frac{\langle S_1K_y, K_x\rangle}{\langle K_y, K_x\rangle}$ and $\Psi_2(x, \bar{y}) = \frac{\langle S_2K_y, K_x\rangle}{\langle K_y, K_x\rangle}$. Since S_1 and S_2 are bounded and self-adjoint, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1K(x, \bar{y}) \gg \Psi_1(x, \bar{y})K(x, \bar{y}) \gg -c_1K(x, \bar{y})$$

and

$$c_2K(x, \bar{y}) \gg \Psi_2(x, \bar{y})K(x, \bar{y}) \gg -c_2K(x, \bar{y}).$$

Further

$$\Psi_1(x, \bar{y}) = \frac{\langle S_1K_y, K_x\rangle}{\langle K_y, K_x\rangle} = \frac{1}{2}\{\Omega(x, \bar{y}) + \overline{\Omega(y, \bar{x})}\}$$

and

$$\Psi_2(x, \bar{y}) = \frac{\langle S_2K_y, K_x\rangle}{\langle K_y, K_x\rangle} = -\frac{1}{2}[(i)\{\Omega(x, \bar{y}) - \overline{\Omega(y, \bar{x})}\}].$$

Thus $\Omega_1(x, \bar{y}) = \Omega(x, \bar{y}) + \overline{\Omega(y, \bar{x})}$ and $\Omega_2(x, \bar{y}) = (i)\{\Omega(x, \bar{y}) - \overline{\Omega(y, \bar{x})}\}$ satisfy the condition (*). Conversely, suppose $\Omega_1(x, \bar{y})$ and $\Omega_2(x, \bar{y})$ satisfy condition (*). Then there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1K(x, \bar{y}) \gg \Psi_1(x, \bar{y})K(x, \bar{y}) \gg -c_1K(x, \bar{y})$$

and

$$c_2K(x, \bar{y}) \gg \Psi_2(x, \bar{y})K(x, \bar{y}) \gg -c_2K(x, \bar{y}),$$

where

$$\Psi_1(x, \bar{y}) = \frac{1}{2}\Omega_1(x, \bar{y}) = \frac{1}{2}\{\Omega(x, \bar{y}) + \overline{\Omega(y, \bar{x})}\} = \frac{\langle(\frac{S+S^*}{2})K_y, K_x\rangle}{\langle K_y, K_x\rangle}$$

and

$$\Psi_2(x, \bar{y}) = -\frac{1}{2}\Omega_2(x, \bar{y}) = \frac{1}{2i}\{\Omega(x, \bar{y}) - \overline{\Omega(y, \bar{x})}\} = \frac{\langle(\frac{S-S^*}{2i})K_y, K_x\rangle}{\langle K_y, K_x\rangle}.$$

Thus $\frac{S+S^*}{2}$ and $\frac{S-S^*}{2i}$ are bounded and self-adjoint and hence S is bounded. Thus we have shown that $\phi \in L^\infty(\mathbb{D})$ satisfy the condition (**) if and only if there exist $S \in \mathcal{L}(L^2_a(\mathbb{D}))$

such that $\phi(z) = \langle Sk_z, k_z\rangle$ for all $z \in \mathbb{D}$. Similarly one can show that $\theta \in L^\infty(\mathbb{D})$ satisfy the condition (**) if and only if there exist $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\theta(z) = \langle Tk_z, k_z\rangle$ for all $z \in \mathbb{D}$. To establish the theorem we have to show that $\widehat{\phi} = \theta$ if and only if $C_aS = TC_a$ for all $a \in \mathbb{D}$.

Suppose $\widehat{\phi} = \theta$. That is,

$$\int_{\mathbb{D}} \phi(\phi_a(z))\, dA(a) = \theta(z)$$

for all $z \in \mathbb{D}$. This implies

$$\int_{\mathbb{D}} \widetilde{S}(\phi_a(z))\, dA(a) = \widetilde{T}(z)$$

for all $z \in \mathbb{D}$.

Then by Lemma 2.1, there exists a constant $\alpha, |\alpha| = 1$ such that for all $z \in \mathbb{D}$

$$\begin{aligned}\langle Tk_z, k_z\rangle &= \int_{\mathbb{D}} \langle Sk_{\phi_a(z)}, k_{\phi_a(z)}\rangle\, dA(a) = \int_{\mathbb{D}} \langle \alpha SU_ak_z, \alpha U_ak_z\rangle\, dA(a)\\ &= \int_{\mathbb{D}} \langle U_aSU_ak_z, k_z\rangle\, dA(a) = \Big\langle \Big(\int_{\mathbb{D}} U_aSU_a\, dA(a)\Big) k_z, k_z \Big\rangle = \langle \widehat{S}k_z, k_z\rangle,\end{aligned}$$

where $\widehat{S} = \int_{\mathbb{D}} U_aSU_a\, dA(a)$.

Thus by Lemma 2.3, $T = \widehat{S}$. Hence for all $f, g \in L^2_a(\mathbb{D}), \langle Tf, g\rangle = \langle \widehat{S}f, g\rangle$. That is,

$$\int_{\mathbb{D}} \langle SU_af, U_ag\rangle\, dA(a) = \int_{\mathbb{D}} Tf(z)\overline{g(z)}\, dA(z).$$

The boundedness of T and the anti-analyticity of $K(z, \bar{a})$ in $\bar{a}$ imply that for each $z \in \mathbb{D}$, the function

$$T\Big(\frac{f}{K(\cdot, \bar{a})}\Big)(z)K(z, \bar{a})$$

is anti-analytic in $\bar{a}$. Therefore, by the mean value property of harmonic functions, we have

$$\int_{\mathbb{D}} T\Big(\frac{f}{K(\cdot, \bar{a})}\Big)(z)K(z, \bar{a})\, dA(a) = T\Big(\frac{f}{K(\cdot, 0)}\Big)(z)K(z, 0) = Tf(z). \tag{6}$$

Thus, from (6), it follows that

$$\langle Tf, g\rangle = \int_{\mathbb{D}} \overline{g(z)}\, dA(z) \int_{\mathbb{D}} T\Big(\frac{f}{K(\cdot, \bar{a})}\Big)(z)K(z, \bar{a})\, dA(a).$$

Using Fubini's theorem, we obtain

$$\langle Tf, g\rangle = \int_{\mathbb{D}} dA(a) \int_{\mathbb{D}} T\Big(\frac{f}{K(\cdot, \bar{a})}\Big)(z)\overline{g(z)}K(z, \bar{a})\, dA(z). \tag{7}$$

Now since $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}}$ and $(k_a \circ \phi_a)(z)k_a(z) = 1$ for all $z, a \in \mathbb{D}$, the right hand side of (7) is equal to

$$\begin{aligned}&\int_{\mathbb{D}} dA(a) \int_{\mathbb{D}} T\Big(\frac{f}{k_a}\Big)(z)\overline{g(z)}k_a(z)\, dA(z)\\ &\quad = \int_{\mathbb{D}} dA(a) \int_{\mathbb{D}} T\Big(\frac{f}{k_a}\Big)(z)\overline{g(z)}\ \overline{k_a(\phi_a(z))}|k_a(z)|^2 dA(z).\end{aligned}$$

Finally, as $(\phi_a \circ \phi_a)(z) \equiv z$ and $\mathbf{J}_{\phi_a(z)} = |k_a(z)|^2$ we obtain

$$\langle Tf, g\rangle = \int_{\mathbb{D}} dA(a) \int_{\mathbb{D}} T\Big(\frac{f}{k_a}\Big)(\phi_a(z))\overline{k_a(z)}\ \overline{g(\phi_a(z))}\, dA(z).$$

By hypothesis, $\langle Tf, g\rangle = \int_{\mathbb{D}} \langle SU_a f, U_a g\rangle \, dA(a)$ and by using Lemma 2.2 we have

$$\begin{aligned}\langle SU_a f, U_a g\rangle &= \Big\langle S\Big(\frac{f\circ\phi_a}{k_a\circ\phi_a}\Big), (g\circ\phi_a)k_a\Big\rangle = \Big\langle S\Big(\frac{f}{k_a}\circ\phi_a\Big), (g\circ\phi_a)k_a\Big\rangle \\ &= \int_{\mathbb{D}} S\Big(\frac{f}{k_a}\circ\phi_a\Big)(z)\overline{g(\phi_a(z))}\;\overline{k_a(z)}\, dA(z).\end{aligned}$$

Thus we obtain for all $f, g \in L^2_a(\mathbb{D})$,

$$\int_{\mathbb{D}} S\Big(\frac{f}{k_a}\circ\phi_a\Big)(z)\overline{g(\phi_a(z))}\;\overline{k_a(z)}\, dA(z) = \int_{\mathbb{D}} T\Big(\frac{f}{k_a}\Big)(\phi_a(z))\overline{k_a(z)}\;\overline{g(\phi_a(z))}\, dA(z).$$

Hence for all $f, g \in L^2_a(\mathbb{D})$, $a \in \mathbb{D}$,

$$\Big\langle S\Big(\frac{f}{k_a}\circ\phi_a\Big), U_a g\Big\rangle = \Big\langle T\Big(\frac{f}{k_a}\Big)\circ\phi_a, U_a g\Big\rangle.$$

Since U_a is unitary, $U_a \in \mathcal{L}(L^2_a(\mathbb{D}))$, we get

$$S\Big(\frac{f}{k_a}\circ\phi_a\Big) = T\Big(\frac{f}{k_a}\Big)\circ\phi_a$$

for all $f \in L^2_a(\mathbb{D}), a \in \mathbb{D}$.
That is, for all $f \in L^2_a(\mathbb{D}), a \in \mathbb{D}$,

$$SC_a\Big(\frac{f}{k_a}\Big) = C_a T\Big(\frac{f}{k_a}\Big).$$

Since $k_a^{-1} \in H^\infty$, hence $SC_a = C_a T$ for all $a \in \mathbb{D}$. That is, $C_a S = TC_a$ for all $a \in \mathbb{D}$ as $C_a^2 = I$, the identity operator in $\mathcal{L}(L^2_a(\mathbb{D}))$. Now we shall prove the converse. Suppose $C_a Sf = TC_a f$ for all $a \in \mathbb{D}, f \in L^2_a(\mathbb{D})$. That is, for all $f \in L^2_a(\mathbb{D}), a \in \mathbb{D}$,

$$(Sf)\circ\phi_a = T(f\circ\phi_a) \text{ and } (Tf)\circ\phi_a = S(f\circ\phi_a).$$

By Lemma 2.2, $(k_a\circ\phi_a)k_a = 1$ for all $a \in \mathbb{D}$. Hence

$$SU_a f = S(k_a(f\circ\phi_a)) = S\Big(\frac{f\circ\phi_a}{k_a\circ\phi_a}\Big) = S\Big(\Big(\frac{f}{k_a}\Big)\circ\phi_a\Big) = \Big(T\frac{f}{k_a}\Big)\circ\phi_a.$$

Thus for $f, g \in L^2_a(\mathbb{D})$, since $\overline{k_a(\phi_a(z))}\;\overline{k_a(z)} = 1, \mathbf{J}_{\phi_a(z)} = |k_a(z)|^2$ and $k_a(z) = \frac{K(z,\bar a)}{\sqrt{K(a,\bar a)}}$ for all $z, a \in \mathbb{D}$, we obtain

$$\begin{aligned}\langle SU_a f, U_a g\rangle &= \int_{\mathbb{D}} \Big(T\frac{f}{k_a}\Big)(\phi_a(z))\overline{(g\circ\phi_a)(z)}\;\overline{k_a(z)}\, dA(z) \\ &= \int_{\mathbb{D}} T\Big(\frac{f}{k_a}\Big)(z)\overline{g(z)}\;\overline{(k_a\circ\phi_a)(z)}|k_a(z)|^2 dA(z) \\ &= \int_{\mathbb{D}} T\Big(\frac{f}{k_a}\Big)(z)\overline{g(z)}k_a(z)\, dA(z) \\ &= \int_{\mathbb{D}} T\Big(\frac{f}{K(\cdot,\bar a)}\Big)(z)\overline{g(z)}K(z,\bar a)\, dA(z).\end{aligned}$$

Hence by using Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \langle SU_a f, U_a g\rangle\, dA(a) &= \int_{\mathbb{D}}\int_{\mathbb{D}} T\Big(\frac{f}{K(\cdot,\bar a)}\Big)(z)\overline{g(z)}K(z,\bar a)\, dA(z)\, dA(a) \\ &= \int_{\mathbb{D}} \overline{g(z)}\, dA(z) \int_{\mathbb{D}} T\Big(\frac{f}{K(\cdot,\bar a)}\Big)(z)K(z,\bar a)\, dA(a).\end{aligned}$$

We have already checked in the first part of the proof that for all $z \in \mathbb{D}$,

$$\int_{\mathbb{D}} T\Big(\frac{f}{K(\cdot,\bar a)}\Big)(z)K(z,\bar a)\, dA(a) = T\Big(\frac{f}{K(\cdot,0)}\Big)(z)K(z,0) = Tf(z).$$

Thus

$$\int_{\mathbb{D}} \langle SU_a f, U_a g\rangle \, dA(a) = \int_{\mathbb{D}} Tf(z)\overline{g(z)}\, dA(z) = \langle Tf, g\rangle.$$

When $f = g = k_z, z \in \mathbb{D}$, we obtain by Lemma 2.1 that

$$\langle Tk_z, k_z\rangle = \int_{\mathbb{D}} \langle SU_a k_z, U_a k_z\rangle \, dA(a) = \int_{\mathbb{D}} \langle Sk_{\phi_a(z)}, k_{\phi_a(z)}\rangle \, dA(a) = \int_{\mathbb{D}} \widetilde{S}(\phi_a(z))\, dA(a)$$

and this completes the proof. □

We shall now discuss about some of the applications of Theorem 3.2. An operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is hyponormal (respectively cohyponormal) if $T^*T \geq TT^*$ (respectively, $TT^* \geq T^*T$). The operator T is paranormal if $\|Tf\|^2 \leq \|T^2 f\|\|f\|$ for all $f \in L^2_a(\mathbb{D})$. The operator T is a coisometry if T^* is an isometry.An operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is said to be algebraically hyponormal if there exists a nonconstant complex polynomial p such that $p(T)$ is hyponormal. The operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is called cyclic with cyclic vectors $f \in L^2_a(\mathbb{D})$ if the finite linear combinations of the vectors $f, Tf, T^2 f, \ldots$ are dense in $L^2_a(\mathbb{D})$. An operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is said to be power bounded if there exists a constant $K > 0$ such that $\|T^n\| \leq K$ for all $n \in \mathbb{N}$.

Corollary 3.1. Let $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ are such that T^* is a hyponormal operator and S is an isometry. If $SC_a = C_a T$ for some $a \in \mathbb{D}$ then T is unitary.

Proof. Suppose $SC_a = C_a T$ for some $a \in \mathbb{D}$. Then from [11] and [9] it follows that $S^* C_a = C_a T^*$ and since S is an isometry we obtain $C_a = S^* C_a T$. Thus $C_a = C_a T^* T$. That is, $I - T^*T = 0$ as $C_a^2 = I$. Since T^* is hyponormal and $T^*T = I$, it follows that T is normal and hence, unitary. □

Corollary 3.2. Let $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ are such that $T^n \to 0$ in the strong operator topology and S is an isometry. Then there does not exist $a \in \mathbb{D}$ such that $C_a T = SC_a$.

Proof. Suppose S is an isometry. Then $\|S^n f\| = \|f\|$ for all $n \in \mathbb{Z}_+$ and $f \in L^2_a(\mathbb{D})$. If $T^n \to 0$ in the strong operator topology and $C_a T = SC_a$ for some $a \in \mathbb{D}$, then $0 \leq \|C_a f\| = \|S^n C_a f\| = \|C_a T^n f\| \leq \|C_a\|\|T^n f\| \to 0$ for all $f \in L^2_a(\mathbb{D})$. That is, $C_a f = 0$ for all $f \in L^2_a(\mathbb{D})$ which is impossible. □

If $f \in L^1(\mathbb{D}, dA)$, the Berezin transform of f is, by definition,

$$(Bf)(w) = \widetilde{f}(w) = \langle fk_w, k_w\rangle = \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\bar{w}z|^4} f(z)\, dA(z), \quad w \in \mathbb{D},$$

where k_w is the normalized reproducing kernel at $w \in \mathbb{D}$ given by $k_w(z) = \frac{1-|w|^2}{(1-\bar{w}z)^2}$.

Notice that $k_w \in L^\infty(\mathbb{D})$ for all $w \in \mathbb{D}$, so the definition makes sense. On $\mathbb{D}$, the only measure left invariant by all Mobius transformations $z \mapsto e^{i\theta}\frac{z-w}{1-\bar{z}w} := e^{i\theta}\phi_w(z), w \in \mathbb{D}, \theta \in \mathbb{R}$ is the pseudo-hyperbolic measure $d\eta(z) = \frac{dA(z)}{(1-|z|^2)^2}$.

The invariance may be verified by direct computation. It turns out that the Berezin transform behaves well with respect to the invariant measures. The mapping $B : f \to \widetilde{f}$ is a contractive linear operator on each of the spaces $L^p(\mathbb{D}, d\eta(z)), 1 \leq p \leq \infty$ and $L^1(\mathbb{D}, d\eta) \subset L^1(\mathbb{D}, dA)$.

Corollary 3.3. Let B be the Berezin transform defined on $L^2(\mathbb{D}, d\eta)$ and $S \in \mathcal{L}(L^2(\mathbb{D}, d\eta))$ is an isometry. Then there exists no $a \in \mathbb{D}$ such that $C_a B = SC_a$.

Proof. The map B is a contraction on $L^2(\mathbb{D}, d\eta)$. This can be verified as follows:

$$|\widetilde{f}(w)| = \Big| \int_{\mathbb{D}} f(z) \frac{(1-|w|^2)^2}{|1-\bar{w}z|^4}\, dA(z) \Big| \leq B(|f|)(w).$$

Hence

$$\begin{aligned}\int_{\mathbb{D}} |\widetilde{f}(w)| \frac{dA(w)}{(1-|w|^2)^2} &\leq \int_{\mathbb{D}} \Big(\int_{\mathbb{D}} |f(z)| \frac{(1-|w|^2)^2}{|1-\bar{w}z|^4} \, dA(z) \Big) \frac{dA(w)}{(1-|w|^2)^2} \\ &= \int_{\mathbb{D}} |f(z)| \int_{\mathbb{D}} \frac{dA(w)}{|1-\bar{w}z|^4} \, dA(z) \\ &= \int_{\mathbb{D}} |f(z)| \langle K_z, K_z \rangle \, dA(z) = \int_{\mathbb{D}} |f(z)| \frac{dA(z)}{(1-|z|^2)^2},\end{aligned}$$

the change of the order of integration being justified by the positivity of the integrand. If $f \in L^2(\mathbb{D}, d\eta)$ and $\widetilde{f} = f$, then f is harmonic but the only harmonic function in $L^2(\mathbb{D}, d\eta)$ is constant zero. To see this, let

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt.$$

This is a nonnegative and nondecreasing function of r. Further,

$$\|f\|^2_{L^2(\mathbb{D}, d\eta)} = \int_0^1 M(r) \frac{2r}{(1-r^2)^2} \, dr < \infty.$$

So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$, whence $f = 0$. Thus there is no nonzero fixed point of B in $L^2(\mathbb{D}, d\eta)$. Since B is a contraction and B is positive [4] on $L^2(\mathbb{D}, d\eta)$, its spectrum must be contained in $[0, 1]$. Let $E(\lambda)$ be the resolution of identity for the self-adjoint operator B. Then

$$\|B^n f\|^2 = \int_{[0,1]} |\lambda^n|^2 d\langle E(\lambda) f, f \rangle.$$

According to the Lebesgue monotone convergence theorem, this tends to $\|(I - E(1-))f\|^2 = \|P_{\ker(B-I)} f\|^2$. But from the above discussion it follows that $\ker(B - I) = \{0\}$. Hence $\|B^n f\| \to 0$ as $n \to \infty$. Further, it is well known [6] that $BC_a = C_a B$ for all $a \in \mathbb{D}$. If now $C_a B = SC_a$ for some $a \in \mathbb{D}$ then this implies $BC_a = SC_a$. That is, $B = S$ as C_a is invertible. Since S is an isometry and $B^n \to 0$ in strong operator topology this is not possible. $\square$

Corollary 3.4. Suppose $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ are power bounded operators and $C_a T = SC_a$ for some $a \in \mathbb{D}$. Then

- **(i):** (i) $T^n \to 0$ in the weak operator topology if and only if $S^n \to 0$ in the weak operator topology.
- **(ii):** $\{T^n h\}$ is weakly convergent for each $h \in L^2_a(\mathbb{D})$ if and only if $\{S^n g\}$ is weakly convergent for each $g \in L^2_a(\mathbb{D})$.
- **(iii):** If for each $h \in L^2_a(\mathbb{D})$ and every increasing sequence $\{n_j\}$ of positive integers, the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^N T^{n_j} h$ exists in the norm topology then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^N S^{n_j} g$$

exists in the norm topology for all $g \in L^2_a(\mathbb{D})$.

Proof. (i) Suppose $\langle T^n h, h' \rangle \to 0$ for all $h, h' \in L^2_a(\mathbb{D})$. Then $\langle S^n C_a h, g \rangle = \langle C_a T^n h, g \rangle = \langle T^n h, C_a^* g \rangle \to 0$ as $n \to \infty$. Hence $S^n f \to 0$ weakly for all $f \in L^2_a(\mathbb{D})$ as $C_a L^2_a(\mathbb{D}) = L^2_a(\mathbb{D})$. Since S is power bounded, we have $S^n \to 0$ in weak operator topology.

Conversely, suppose that $S^n \to 0$ in weak operator topology. Then $S^{*n} \to 0$ in weak operator topology and $T^* C_a^* = C_a^* S^*$. Hence $T^{*n} \to 0$ in weak operator topology and so $T^n \to 0$ in weak operator topology.

(ii) Let $h \in L_a^2(\mathbb{D})$. Notice that the sequence $\{T^n h\}$ converges weakly if and only if $\langle T^n h, h'\rangle$ is convergent for each $h' \in L_a^2(\mathbb{D})$. Suppose that this condition is satisfied and define $\Phi(h') = \lim_{n\to\infty}\langle T^n h, h'\rangle$. Then Φ is a bounded conjugate linear functional, and so there is an $\bar{h} \in L_a^2(\mathbb{D})$ such that $\langle \bar{h}, h'\rangle = \Phi(h')$ for all $h' \in L_a^2(\mathbb{D})$. Hence $T^n h \to \bar{h}$ weakly. From this it follows easily that $\{T^n h\}$ is weakly convergent for each $h \in L_a^2(\mathbb{D})$ if and only if $\{T^{*n} h\}$ is weakly convergent for each $h \in L_a^2(\mathbb{D})$. Furthermore, $\{h \in L_a^2(\mathbb{D}) : \{T^n h\}$ converges weakly $\}$ is a closed subspace of $L_a^2(\mathbb{D})$. Suppose now that $\{T^n h\}$ converges weakly for each $h \in L_a^2(\mathbb{D})$ and suppose $C_a T = S C_a$ for some $a \in \mathbb{D}$. Let $h \in L_a^2(\mathbb{D})$ and $T^n h \to \bar{h}$ weakly. Then $T\bar{h} = \bar{h}$ and $T^n(h - \bar{h}) \to 0$ weakly. Thus $L_a^2(\mathbb{D}) = \ker(I - T) + (L_a^2)_0$ where $(L_a^2)_0 = \{h \in L_a^2(\mathbb{D}) : T^n h \to 0$ weakly $\}$. It is easy to see that $C_a \ker(I - T) \subset \ker(I - S)$ and $C_a(L_a^2)_0 \subset \{g \in L_a^2(\mathbb{D}) : S^n g \to 0$ weakly$\}$. Thus $\{S^n g\}$ converges weakly for each $g \in L_a^2(\mathbb{D})$.

Conversely, suppose that $\{S^n g\}$ converges weakly for each $g \in L_a^2(\mathbb{D})$. Then $T^* C_a^* = C_a^* S^*$ and $\{S^{*n} g\}$ converges weakly for each $g \in L_a^2(\mathbb{D})$. As in the previous case, one can show that $\{T^{*n} h\}$ converges weakly for each $h \in L_a^2(\mathbb{D})$, and so $\{T^n h\}$ converges weakly for each $h \in L_a^2(\mathbb{D})$.

(iii) Suppose for each increasing subsequence of positive integer $\{n_j\}$ and every $h \in L_a^2(\mathbb{D})$ the limit $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} T^{n_j} h$ exists in the norm topology. Then $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} S^{n_j} C_a h = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} C_a T^{n_j} h$ exists for each $h \in L_a^2(\mathbb{D})$. Since Range$C_a = L_a^2$ and the sequence $\frac{1}{N}\sum_{j=1}^{N} T^{n_j}$ is bounded, the limit $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} S^{n_j} g$ exists for all $g \in L_a^2(\mathbb{D})$. $\square$

Corollary 3.5. Let $S, T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Suppose $TC_a = C_a S$ and $T^* C_a = C_a S^*$ for some $a \in \mathbb{D}$. Then the operators S and T are unitarily equivalent and the following hold:

- **(i):** If S is hyponormal (cohyponormal) then T is also hyponormal (cohyponormal).
- **(ii):** If S is an isometry (coisometry) then T is also an isometry (coisometry).
- **(iii):** If S is normal then T is also normal.

Proof. Let $C_a^* = W^* Q$ be the polar decomposition of C_a^* such that $\ker W^* = \ker Q$. Since $C_a^2 = I$ we obtain ${C_a^*}^2 = I$. Hence $C_a^* f = 0$ implies ${C_a^*}^2 f = 0$. Therefore $f = 0$. Thus C_a^* is injective and $\ker C_a^* = \{0\}$. The operator W^* is a partial isometry. That is, $\|W^* f\| = \|f\|$ for all $f \in (\ker W^*)^\perp$. Now $f \in \ker Q$ implies $f \in \ker C_a^*$. Hence $\ker Q \subseteq \ker C_a^* = \{0\}$. Thus $\ker Q = \{0\}$. But $\ker W^* = \ker Q$. Hence $\ker W^* = \{0\}$ and W^* is injective. Thus $\|W^* f\| = \|f\|$ for all $f \in (\ker W^*)^\perp = \{0\}^\perp = L_a^2(\mathbb{D})$. Thus W^* is an isometry.

Now $C_a^* = W^* Q$ implies $C_a = QW$. If $f \in \ker W$ then $f \in \ker C_a$. But $\ker C_a = \{0\}$. Hence $\ker W = \{0\}$ and W is injective. Further, W^* is a partial isometry implies [2] the operator W is a partial isometry. That is, $\|Wf\| = \|f\|$ for all $f \in (\ker W)^\perp = \{0\}^\perp = L_a^2(\mathbb{D})$. Hence W is an isometry. Thus W is unitary and $Q^2 = C_a C_a^*$ is injective. From equations $TC_a = C_a S$ and $T^* C_a = C_a S^*$, we have

$$TC_a C_a^* = C_a S C_a^*, \quad C_a C_a^* T = C_a S C_a^*.$$

Thus, $Q^2 = C_a C_a^*$ commutes with T, and [2] so $QT = TQ$. Hence we obtain

$$QTW = TQW = TC_a = C_a S = QWS,$$

which implies that $TW = WS$ because Q is injective.

Since W is a coisometry, we have

$$T = TWW^* = WSW^*. \tag{8}$$

Hence from (8) it follows that S and T are unitarily equivalent. From the equations $C_a^*T = SC_a^*$ and $TQ = QT$, we have

$$W^*TQ = W^*QT = C_a^*T = SC_a^* = SW^*Q,$$

which implies that $W^*T = SW^*$. This is so as $\overline{\text{Range}Q} = (\ker Q)^\perp = (\ker W^*)^\perp = \{0\}^\perp = L_a^2(\mathbb{D})$. Hence

$$W^*WS = W^*TW = SW^*W.$$

Suppose now that S is normal. Since $S^*S = SS^*$, we have

$$\begin{aligned} T^*T &= (WSW^*)^*(WSW^*) = WS^*W^*WSW^* = WS^*SW^*WW^* \\ &= WS^*SW^* = WSS^*W^* = WW^*WSS^*W^* = (WSW^*)(WSW^*)^* = TT^*. \end{aligned}$$

Thus T is normal. This proves (iii).

To prove (i), assume that S is hyponormal (respectively, cohyponormal). Since $S^*S \geq SS^*$ (respectively, $SS^* \geq S^*S$), from the above arguments it follows that $T^*T = WS^*SW^* \geq WSS^*W^* = TT^*$ (respectively, $TT^* = WSS^*W^* \geq WS^*SW^* = T^*T$) and the result follows.

To prove (ii), assume that S is an isometry (respectively, coisometry). Again, by the above computation, $T^*T = WS^*SW^* = WW^* = I$ (respectively, $TT^* = WSS^*W^* = WW^* = I$). Thus T is an isometry (respectively, coisometry). So (ii) is established. □

Corollary 3.6. Let $S, T \in \mathcal{L}(L_a^2(\mathbb{D}))$ are such that T is a paranormal contraction, S^* is an isometry and $TC_a = C_aS$ for some $a \in \mathbb{D}$. Then T and S are unitary operators.

Proof. Let f be a nonzero vector in $L_a^2(\mathbb{D})$. Then $C_af \neq 0$. Let $g_n = C_aS^{*n}f, n = 0, 1, 2, \cdots$. Then $Tg_{n+1} = TC_aS^{*n+1}f = C_aSS^{*n+1}f = C_aS^{*n}f = g_n$. Since T is a contraction,

$$\|g_n\| = \|Tg_{n+1}\| \leq \|g_{n+1}\| = \|C_aS^{*n+1}f\| \leq \|C_a\|\|f\|$$

and hence $\{\|g_{n+1}\|\}$ is a monotone increasing convergent sequence. Since T is paranormal, we obtain

$$\|g_n\|^2 = \|Tg_{n+1}\|^2 \leq \|T^2g_{n+1}\|\|g_{n+1}\| = \|g_{n+1}\|\|g_{n+1}\|$$

and

$$1 \geq \frac{\|g_0\|}{\|g_1\|} \geq \frac{\|g_1\|}{\|g_2\|} \geq \cdots \geq \frac{\|g_{n-1}\|}{\|g_n\|} \to 1$$

as $n \to \infty$. In particular, $\|g_0\| = \|g_1\|$, that is, $\|C_af\| = \|C_aS^*f\|$. Thus

$$\|C_aS^*f\| = \|C_af\| = \|C_aSS^*f\| = \|TC_aS^*f\| \leq \|C_aS^*f\|,$$

and so

$$\|C_aS^*f\| = \|C_af\| = \|TC_aS^*f\|.$$

If $f = 0$ then

$$\|C_aS^*f\| = \|C_af\| = \|TC_aS^*f\| = 0.$$

Hence

$$\begin{aligned} &\|T^*C_af - C_aS^*f\|^2 \\ &= \|T^*C_af\|^2 + \|C_aS^*f\|^2 - \langle T^*C_af, C_aS^*f\rangle - \langle C_aS^*f, T^*C_af\rangle \\ &\leq 2\|C_af\|^2 - \langle C_af, TC_aS^*f\rangle - \langle TC_aS^*f, C_af\rangle \\ &= 2\|C_af\|^2 - \langle C_af, C_aSS^*f\rangle - \langle C_aSS^*f, C_af\rangle \\ &= 2\|C_af\|^2 - 2\|C_af\|^2 = 0 \end{aligned}$$

for all $f \in L_a^2(\mathbb{D})$ and $T^*C_a = C_aS^*$. It follows from Corollary 3.5 that T is a coisometry. That is, $TT^* = I$. Since T is a paranormal contraction we have $\langle T^*Tf, f\rangle = \langle Tf, Tf\rangle =$

$\|Tf\|^2 \leq \|T^2 f\|\|f\| \leq \|T\|^2\|f\|^2 \leq \|f\|^2$ for all $f \in L^2_a(\mathbb{D})$. Hence $T^*T \leq I = TT^*$. Thus T^* is hyponormal.

From [10] and [9] it follows that the operator T is unitary. Let $C_a^* = W^*Q$ be the polar decomposition of C_a^* such that $\ker W^* = \ker Q$. Then proceeding as in Corollary 3.5 one can show that W is unitary and $T = WSW^*$. Since T is an unitary operator, hence S is an unitary operator. □

Recall that for $T \in \mathcal{L}(L^2_a(\mathbb{D}))$, we have defined $\widehat{T} = \int_{\mathbb{D}} U_a T U_a \, dA(a)$ and for $\phi \in L^\infty(D)$, $\widehat{\phi}(z) = \int_{\mathbb{D}} \phi(\phi_a(z))\, dA(a)$. The following holds.

Corollary 3.7. Let $\phi, \psi \in L^\infty(\mathbb{D})$. If $T_{\overline{\phi}} C_a = C_a T_{\overline{\psi}}$ for all $a \in \mathbb{D}$ then $\overline{\psi} - \widehat{\overline{\phi}} \in (L^2_a(\mathbb{D}))^\perp$. Further if T_ϕ is cyclic then T_ψ is cyclic. That is, if v is a cyclic vector for T_ϕ, then $C_a^* v$ is a cyclic vector for T_ψ for all $a \in \mathbb{D}$.

Proof. Suppose $T_{\overline{\phi}} C_a = C_a T_{\overline{\psi}}$ for all $a \in \mathbb{D}$. Then from Theorem 3.2 it follows that $\widehat{T_{\overline{\phi}}} = T_{\overline{\psi}}$. But for $f, g \in L^2_a(\mathbb{D})$ and $\phi \in L^\infty(\mathbb{D})$ we have

$$\begin{aligned}\langle \widehat{T_\phi} f, g\rangle &= \int_{\mathbb{D}} \langle U_a T_\phi U_a f, g\rangle\, dA(a) = \int_{\mathbb{D}} \langle T_\phi U_a f, U_a g\rangle\, dA(a)\\ &= \int_{\mathbb{D}} \langle P(\phi U_a f), U_a g\rangle\, dA(a) = \int_{\mathbb{D}} \langle \phi U_a f, P U_a g\rangle\, dA(a)\\ &= \int_{\mathbb{D}} \langle \phi U_a f, U_a P g\rangle\, dA(a) = \int_{\mathbb{D}} \langle \phi U_a f, U_a g\rangle\, dA(a).\end{aligned}$$

But

$$\langle \phi U_a f, U_a g\rangle = \langle U_a M_\phi U_a f, g\rangle = \langle M_{\phi\circ\phi_a} f, g\rangle.$$

Thus

$$\begin{aligned}\int_{\mathbb{D}} \Big\langle \phi U_a f, U_a g\Big\rangle dA(a) &= \int_{\mathbb{D}} \Big\langle M_{\phi\circ\phi_a} f, g\Big\rangle dA(a) = \Big\langle \Big(\int_{\mathbb{D}} (\phi\circ\phi_a)\, dA(a)\Big) f, g\Big\rangle\\ &= \langle \widehat{\phi} f, g\rangle = \langle \widehat{\phi} f, Pg\rangle = \langle P(\widehat{\phi} f), g\rangle = \langle T_{\widehat{\phi}} f, g\rangle.\end{aligned}$$

Therefore $\widehat{T_{\overline{\phi}}} = T_{\widehat{\overline{\phi}}}$. Hence $T_{\overline{\psi}} = T_{\widehat{\overline{\phi}}}$. Thus $\overline{\psi} - \widehat{\overline{\phi}} \in (L^2_a(\mathbb{D}))^\perp$.

Since $T_{\overline{\phi}} C_a = C_a T_{\overline{\psi}}$ it follows that $C_a^* T_\phi = T_\psi C_a^*$. If $p(z)$ is any (analytic) polynomial, then $C_a^* p(T_\phi) = p(T_\psi) C_a^*$. Now let v be a cyclic vector for T_ϕ, so $C_a^* p(T_\phi) v = p(T_\psi) C_a^* v$. Since C_a is one-to-one, C_a^* has dense range, thus as p varies it follows that $C_a^* v$ is cyclic for T_ψ. □

Corollary 3.8. If $\phi, \psi \in h^\infty(\mathbb{D})$, then $T_{\overline{\phi}} C_a = C_a T_{\overline{\psi}}$ for all $a \in \mathbb{D}$ if and only if $\overline{\psi} = \widehat{\overline{\phi}}$.

Proof. Notice that if $\phi \in h^\infty(\mathbb{D})$ then $\widehat{\phi} \in h^\infty(\mathbb{D})$. In fact, if $\psi \in h^\infty(\mathbb{D})$ and $\psi(z) = \sum_{n=0}^\infty a_n z^n + \sum_{n=0}^\infty b_n \overline{z}^n$ then $\widehat{\psi}(z) = a_0 - (\frac{a_1}{2}) z - (\frac{b_1}{2}) \overline{z} \in h^\infty(\mathbb{D})$. If $\phi, \psi \in h^\infty(\mathbb{D})$, then by Corollary 3.7 we obtain $T_{\overline{\phi}} C_a = C_a T_{\overline{\psi}}$ for all $a \in \mathbb{D}$ if and only if $T_{\overline{\psi}} = \widehat{T_{\overline{\phi}}}$. That is, if and only if $T_{\overline{\psi}} = T_{\widehat{\overline{\phi}}}$ where $\widehat{\overline{\phi}}(z) = \int_{\mathbb{D}} \overline{\phi}(\phi_a(z))\, dA(a)$. But from [1] it follows that $T_{\overline{\psi} - \widehat{\overline{\phi}}} = 0$ if and only if $\overline{\psi} - \widehat{\overline{\phi}} = 0$. Hence the result follows. □

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