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A NEW METRIC IN THE STUDY OF SHIFT INVARIANT SUBSPACES OF $L^2(\mathbb{R}^n)$

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Abstract. A new metric on the set of all shift invariant subspaces of $L^2(\mathbb{R}^n)$ is defined and the properties are studied. The limit of a sequence of principal shift invariant subspaces under this metric is principal shift invariant is proved. Also, the uniform convergence of a sequence of local trace functions is characterized in terms of convergence under this new metric.

1. Introduction

Shift invariant subspaces (SIS) are closed subspaces of $L^2(\mathbb{R}^n)$ that are invariant under all integer translations (also called shifts). They play an important role in various areas of mathematical analysis and its applications, especially in the areas of wavelets, splines, Gabor systems and approximation theory. The general structure of such spaces was revealed in the work of Carl de Boor, DeVore, and Ron [7] and also in the work of Bownik [1]. Using the fiberization techniques based on range functions, a characterization of finite shift invariant spaces was provided in [7] and later it was extended to an arbitrary shift invariant subspace by Bownik in [1].

Motivated by Bownik's association of range functions with shift invariant subspaces [1], we introduce a natural metric, namely *shift metric* on the set of all closed shift invariant subspaces of $L^2(\mathbb{R}^n)$ and use it to learn more about shift invariant subspaces like completeness of SIS (Theorem 3.4).

Local trace function [LTF], introduced by Dutkay [8] and spectral function, introduced by Rzeszotnik [2] are recent tools for the analysis of SIS. We study the relationship among shift metric, LTF and spectral function in Theorems 3.6, 3.7 and 3.9. Consequently, the uniform convergence of a sequence of local trace functions has been characterized in terms of shift metric. With these tools in hand, we have been able to answer an important question "Is the limit of a sequence of principal shift invariant (PSI) spaces, a PSI?" (Theorem 3.11, Theorem 3.13, Theorem 3.15).

The final section provides a non trivial example of a sequences of SIS converging under shift metric.

2. Preliminaries

Some of the important known concepts and results used in the subsequent sections are given here for easy reference ([1], [2], [7], [8]).

A closed subspace $V \subseteq L^2(\mathbb{R}^n)$ is called a *shift invariant subspace* (SIS) if for every $f \in V$, we also have $T_k f \in V$, where the *shift operator* T_k on $L^2(\mathbb{R}^n)$ is given by $T_k f(x) = f(x-k)$ for $x \in \mathbb{R}^n, k \in \mathbb{Z}^n$. Every SIS V has a countable subset $\mathcal{A}$ of generators in the sense that $V = \overline{\operatorname{span}}\{T_k f : f \in \mathcal{A}, k \in \mathbb{Z}^n\}$. The latter set is also denoted by $S(\mathcal{A})$. SIS with a single generator ψ is called a *principal shift invariant*

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(PSI) space and is denoted by $< \psi >$. Every SIS has an orthogonal decomposition in terms of countable number of PSI spaces where the generator of each PSI space is *quasi orthogonal*.

Range function $J : \mathbb{T}^n \longrightarrow C(l^2(\mathbb{Z}^n))$, the set of all closed subspaces of $l^2(\mathbb{Z}^n)$, is an important tool in the characterization of SIS. Using the isometric isomorphism $\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$ given by $\tau f(\xi) = (\hat{f}(\xi+k))_{k\in\mathbb{Z}^n}$, with $f \in L^2(\mathbb{R}^n),\ \xi \in \mathbb{T}^n$, the following characterization of SIS is proved in [1].

Proposition 2.1. *V is a SIS $\Longleftrightarrow$ there exists some measurable range function J such that $V = \{f \in L^2(\mathbb{R}^n) : \tau f(x) \in J(x)$ a.e $x \in \mathbb{T}^n\}$. Identifying the range functions that are equal almost everywhere, the correspondence between shift invariant subspaces and measurable range functions is one-one and onto.*

Since then, various tools were being developed to extract information about the structure of SIS. The dimension function $dim_V(\xi) := \dim(J(\xi)), \xi \in \mathbb{T}^n$ measures the "size" of V by counting the 'fibers' $J(\xi)$ of V whereas the *spectral function*, whose definition is given below, measures the "localized size "of the SIS.

Definition 2.2. *Suppose $V \subset L^2(\mathbb{R}^n)$ is shift invariant with range function J with projection $P_{J(\xi)},\ \xi \in \mathbb{T}^n$. The spectral function of V is the measurable mapping $\sigma_V : \mathbb{R}^n \longrightarrow [0,1]$ given by $\sigma_V(\xi+k) = \|P_{J(\xi)}e_k\|^2, \quad \xi \in \mathbb{T}^n,\ k \in \mathbb{Z}^n$ where $(e_k)_k$ is the standard orthonormal basis of $l^2(\mathbb{Z}^n)$.*

For computing the spectral function of a PSI space $V = < \phi >$, the formula

$$\sigma_V(\xi) = \begin{cases} |\hat{\phi}(\xi)|^2(\sum_k |\hat{\phi}(\xi+k)|^2)^{-1}, & \xi \in \operatorname{supp}(\hat{\phi}) \\ 0, & \text{otherwise} \end{cases}$$

can be used. It is additive on countable orthogonal sums.

The *Local Trace Function* (LTF) $\tau_{V,T}$ associated with a SIS V and a positive operator T on $l^2(\mathbb{Z}^n)$ is the map $\tau_{V,T}(\xi) = \mathrm{Trace}(TP_{J(\xi)})$ from $\mathbb{T}^n$ to $[0,\infty]$ where $P_{J(\xi)}$ is the projection onto the fiber space $J(\xi)$ of V. In the special case when $T = P_f$ where $P_f(v) = \langle v, f\rangle f$, it is called *restricted local trace function* and is denoted by $\tau_{V,f}$. Besides being more general than both dimension function and spectral function, the local trace function completely determines the SIS: two SIS are equal if and only if their local trace functions are equal.

3. The Shift Metric

$SI(\mathbb{R}^n)$ is the collection of all shift invariant subspaces of $L^2(\mathbb{R}^n)$ and $PSI(\mathbb{R}^n)$, the collection of all principal shift invariant spaces of $L^2(\mathbb{R}^n)$.

In the following proposition, we introduce *shift metric*, a new tool for the analysis of shift invariant subspaces.

Lemma 3.1. *Let V and W be two shift invariant subspaces of $L^2(\mathbb{R}^n)$ and let J_V and J_W be the measurable range functions associated with these subspaces (identified as equal if $J_V(\xi) = J_W(\xi)$ for a.e $\xi \in \mathbb{T}^n$). For $\xi \in \mathbb{T}^n$, let $P_{J_V(\xi)},\ P_{J_W(\xi)}$ be the orthogonal projections onto $J_V(\xi)$ and $J_W(\xi)$ respectively. Define θ as*

$$\theta(V,W) = inf\{\alpha > 0 : m(\{\xi \in \mathbb{T}^n : \|P_{J_V(\xi)} - P_{J_W(\xi)}\| > \alpha\}) = 0\},$$

where $\|.\|$ denotes the operator norm and m the Lebesgue measure. Then θ is a metric on $SI(\mathbb{R}^n)$. θ is called the shift metric.

Proof. That $\theta(V,W) \geq 0$ follows from definition.

For $V, W \in SI(\mathbb{R}^n)$, if $\theta(V,W) = 0$, one can find a sequence (α_n) of positive numbers converging to 0 and a set E of zero measure such that $\|P_{J_V(\xi)} - P_{J_W(\xi)}\| \leq \alpha_n\ \forall\ n \in \mathbb{N}$ and for $\xi \notin E$. It follows that $\|P_{J_V(\xi)} - P_{J_W(\xi)}\| = 0$ for a.e $\xi \in \mathbb{T}^n$ and hence $V = W$.

On the other hand, $V = W$ implies $J_V(\xi) = J_W(\xi)$ a.e $\xi \in \mathbb{T}^n$ which in turn implies that $\|P_{J_V(\xi)} - P_{J_W(\xi)}\| > 0$ only on a set of measure 0. Hence $\theta(V, W) = 0$.

For $U, V, W \in SI(\mathbb{R}^n)$ and $\epsilon > 0$, one can get $M_1, M_2 > 0$ such that

$$M_1 < \theta(V,U) + \frac{\epsilon}{2}, \quad M_2 < \theta(U,W) + \frac{\epsilon}{2},$$
$$m(\{\xi \in \mathbb{T}^n : \|P_{J_V(\xi)} - P_{J_U(\xi)}\| > M_1\}) = 0,$$
$$m(\{\xi \in \mathbb{T}^n : \|P_{J_U(\xi)} - P_{J_W(\xi)}\| > M_2\}) = 0.$$

Applying triangle inequality for norm gives $\|P_{J_V(\xi)} - P_{J_W(\xi)}\| \leq M_1 + M_2$ for a.e $\xi \in \mathbb{T}^n$. It follows that $\theta(V, W) \leq \theta(V, U) + \theta(U, W)$

$\theta(V, U) = \theta(U, V)$ follows from the property of the norm. □

Remark 3.2. *Let θ denote the shift metric on $SI(\mathbb{R}^n)$. For $V, W \in SI(\mathbb{R}^n)$, we have $\theta(V, W) \leq \epsilon \iff \|P_{J_V(\xi)} - P_{J_W(\xi)}\| \leq \epsilon$ a.e $\xi \in \mathbb{T}^n$.*

The following proposition is used in the proof of next theorem.

Proposition 3.3. *Let $(J_n)_n$ be a sequence of measurable range functions and $(P_n)_n$ be the corresponding sequence of orthogonal projections onto J_n's. Suppose $(P_n(\xi))$ converge to the orthogonal projection $P(\xi)$ under the operator norm for every $\xi \in \mathbb{T}^n$. If $J(\xi)$ is the range of $P(\xi)$, then J is a measurable range function.*

Proof. Let $a \in l^2(\mathbb{Z}^n)$ be arbitrary. Setting $F_n(\xi) = P_n(\xi)a$ and $F(\xi) = P(\xi)a$, we have $\|F_n(\xi) - F(\xi)\| \leq \|P_n(\xi) - P(\xi)\|\|a\|$. It now follows that $F(\xi) = \lim F_n(\xi)$. Thus F is the limit of a sequence (F_n) of vector measurable functions and hence vector measurable. That is J is measurable. □

Theorem 3.4. *$SI(\mathbb{R}^n)$, the collection of all shift invariant subspaces of $L^2(\mathbb{R}^n)$, is complete under shift metric.*

Proof. Suppose (V_n) is Cauchy in $SI(\mathbb{R}^n)$. Then $(P_{J_{V_n}(\xi)})$ is Cauchy in the Banach space $BL(l^2(\mathbb{Z}^n))$ and hence converges to an orthogonal projection $P(\xi)$ for a.e $\xi \in \mathbb{T}^n$. If $J(\xi)$ is the closed subspace of $l^2(\mathbb{Z}^n)$ associated with the orthogonal projection $P(\xi)$, then

$$V := \{f \in L^2(\mathbb{R}^n) | \ \tau f(\xi) \in J(\xi) \text{ a.e } \xi \in \mathbb{T}^n\}$$

is a SIS. From uniqueness of range functions, we have $J_V(\xi) = J(\xi)$ for a.e $\xi \in \mathbb{T}^n$ and hence $P_{J_V}(\xi) = P(\xi)$ for a.e $\xi \in \mathbb{T}^n$. Consequently, (V_n) converges to V under the shift metric. □

Proposition 3.5. *$SI(\mathbb{R}^n)$ is not compact under shift metric topology.*

Proof. It is enough to show that $SI(\mathbb{R}^n)$ is not totally bounded under shift metric. First choose a countable basis $\{\phi_1, \phi_2, \phi_3, \ldots, \ldots, \ldots\}$ for $L^2(\mathbb{R}^n)$ which exists as $L^2(\mathbb{R}^n)$ is a separable Hilbert space. Set $V_m = \mathcal{S}(\mathcal{A}_m)$ where $\mathcal{A}_m = \{\phi_1, \phi_2, \ldots, \phi_m\}$. Then $V_m \subset V_{m+1} \ \forall \ m$ and hence $\|P_{J_{V_m}(\xi)} - P_{J_{V_{m+1}}(\xi)}\| = 1, \ \forall \ \xi \in \mathbb{T}^n$ (Theorem 4.30 (c) [11]). That is $\theta(V_m, V_{m+1}) = 1 \ \forall \ m$. Hence for $\epsilon = \frac{1}{2}$, no finite collection of $\epsilon-$ balls can contain all V_m 's. □

Now we discuss the behavior of spectral function and local trace functions with respect to limits under the hypothesis of convergence of the spaces in the shift metric.

Theorem 3.6. *Let (V_n) be a sequence of shift invariant subspaces converging to a shift invariant subspace V under the shift metric. For any $f \in l^2(\mathbb{Z}^n)$, the restricted local trace function $\tau_{V_n,f}$ converges uniformly to $\tau_{V,f}$ in $\mathbb{T}^n$, except possibly on a set of measure 0.*

Proof. Since $\tau_{V,f}(\xi) = \langle f, P_{J_V(\xi)} f\rangle \quad \text{a.e.} \quad \xi \in \mathbb{T}^n$,

$$\begin{aligned} |\tau_{V_n,f}(\xi) - \tau_{V,f}(\xi)| &\leq \|f\|^2(\|P_{J_{V_n}(\xi)} - P_{J_V(\xi)}\|) \ \text{ a.e } \ \xi \in \mathbb{T}^n \\ &\leq \|f\|^2\theta(V_n, V) \ \text{ a.e } \ \xi \in \mathbb{T}^n. \end{aligned}$$

The uniform convergence of local trace functions follows from this. □

Theorem 3.7. *Let (V_n) be a sequence of shift invariant subspaces converging to a shift invariant subspace V under shift metric. Then the corresponding sequence of spectral functions of V_n converge uniformly to the spectral function of V on $\mathbb{R}^n$ a.e.*

Proof. Let σ_V denote the spectral function of V. Then $\tau_{V,e_m}(\xi) = \sigma_V(\xi+m)$ for $\xi \in \mathbb{T}^n$. One get the required conclusion using previous result. □

The converse of the above result is not true as illustrated in the example below.

Example 3.8. Let $\phi, \psi \in L^2(\mathbb{R})$ be given by $\hat{\phi} = \frac{1}{\sqrt{2}}(\mathbf{1}_{(0,1)} + \mathbf{1}_{(1,2)})$ and $\hat{\psi} = \frac{1}{\sqrt{2}}(\mathbf{1}_{(0,1)} - \mathbf{1}_{(1,2)})$. Then ϕ and ψ are quasi orthogonal generator of $V = \mathcal{S}(\phi)$ and $W = \mathcal{S}(\psi)$ respectively. Both V and W have the same spectral function, namely $\sigma_V = \sigma_W = \frac{1}{2}\mathbf{1}_{(0,2)}$ but $V \perp W$. Hence, if we consider the sequence $(V, W, V, W, \dots, \dots, \dots)$ then it can not converge under shift metric but the corresponding sequence of spectral functions converge uniformly.

However, if we are assured the uniform convergence of $\tau_{V_n,f}$ to $\tau_{V,f}$ on the unit circle of both $\mathbb{R}^n$ and $l^2(\mathbb{Z}^n)$, we have the converse statement.

Theorem 3.9. *Let $V, V_m \in SI(\mathbb{R}^n)$ $\forall$ $m \in \mathbb{N}$. Assume that the restricted local trace function $\tau_{V_m,f}$ converges uniformly to $\tau_{V,f}$ on $\mathbb{T}^n$, except possibly on a set of measure 0 and for all $f \in l^2(\mathbb{Z}^n)$ with $\|f\| = 1$. Then (V_m) converges to V under shift metric.*

Proof. We have, $\mid \tau_{V_n,f}(\xi) - \tau_{V,f}(\xi) \mid = \mid \langle f, (P_{J_{V_n}(\xi)} - P_{J_V(\xi)})f\rangle \mid$ a.e. $\xi \in \mathbb{T}^n$. Hence

$$\begin{aligned} \sup_{\|f\|=1} \mid \tau_{V_n,f}(\xi) - \tau_{V,f}(\xi) \mid &= \sup_{\|f\|=1} \mid \langle f, (P_{J_{V_n}(\xi)} - P_{J_V(\xi)})f\rangle \mid \\ &= \|P_{J_{V_n}(\xi)} - P_{J_V(\xi)}\|. \end{aligned}$$

The result now follows from the assumption of the theorem. □

Let us say that a sequence $(\tau_{V_n,f})_n$ of local trace functions is *uniformly Cauchy* on the unit circle of $\mathbb{R}^n$ and $l^2(\mathbb{Z}^n)$, if for each $\epsilon > 0$, $\exists$ $k \in \mathbb{N}$ such that $|\tau_{V_n,f}(\xi) - \tau_{V_m,f}(\xi)| < \epsilon$ $\forall$ $\xi \in \mathbb{T}^n$ and $\forall$ $f \in l^2(\mathbb{Z}^n)$ with $\|f\| = 1$ whenever $n, m \geq k$. We have the following corollary.

Corollary 3.10. *Suppose that (V_n) is a sequence of shift invariant subspaces for which the sequence $(\tau_{V_n,f})_n$ of local trace functions is uniformly Cauchy on the unit circle of both $\mathbb{R}^n$ and $l^2(\mathbb{Z}^n)$. Then the limit is a local trace function.*

PSI spaces, being the building blocks of all SIS, it is natural to give a special consideration in our study to this collection. Below, we shall prove that $PSI(\mathbb{R}^n)$ is a closed subspace of $SI(\mathbb{R}^n)$ under shift metric.

Theorem 3.11. *$PSI(\mathbb{R}^n)$ is complete under the shift metric θ.*

Proof. Suppose that (V_n) is a Cauchy sequence of elements of $PSI(\mathbb{R}^n)$. By Theorem (3.4), sequence (V_n) converges to some $V \in SI(\mathbb{R}^n)$. We need only to show that V has a single generator.

For $0 < \epsilon < 1$, choose $p \in \mathbb{N}$ such that $\theta(V_n, V) < \epsilon$, $\forall$ $n \geq p$. This implies that $\|P_{J_{V_n}}(\xi) - P_{J_V}(\xi)\| < \epsilon$ for a.e ξ whenever $n \geq p$. Hence $\dim J_V(\xi) = \dim J_{V_n}(\xi) = 1$ for a.e. ξ (Theorem 4.35(a) [11]). This proves that V can be generated by a single function and hence $V \in PSI(\mathbb{R}^n)$. □

Theorem 3.12. *The space of all SIS with a fixed dimension function is complete under shift metric.*

Proof. The proof is similar to the proof of Theorem 3.11. □

A proof of different flavor and yielding information about the generator of the limit space is given below.

Theorem 3.13. *Suppose that (ϕ_n) is a Cauchy sequence in $L^2(\mathbb{R}^n)$ and that each ϕ_n is a quasi orthogonal generator of $V_n \in PSI(\mathbb{R}^n)$. If (V_n) is Cauchy under shift metric, then the limit is also a principal shift invariant space and is generated by the limit of (ϕ_n).*

Proof. Let (ϕ_n) converge to ϕ in $L^2(\mathbb{R}^n)$. Then we have a subsequence (n_k) on natural numbers such that $(\tau\phi_{n_k})$ converges point wise to $\tau\phi$ in $L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$. Consequently, $\|\tau\phi(\xi)\| = lim_{k\to\infty}\|\tau\phi_{n_k}(\xi)\| = \mathbf{1}_E(\xi)$ for some $E \subseteq \mathbb{T}^n$. Therefore ϕ is a quasi orthogonal generator for the principal shift invariant space $V := \mathcal{S}(\phi)$.

As $SI(\mathbb{R}^n)$ is complete under θ, there exists a $W \in SI(\mathbb{R}^n)$ to which V_n converges. Hence, $\tau_{V_n,f} \to \tau_{W,f}$ uniformly for a.e ξ and for each $f \in l^2(\mathbb{Z}^n)$ as $n \longrightarrow \infty$. Now,

$$\begin{aligned} |\ \tau_{V_{n_k},f}(\xi) - \tau_{V,f}(\xi)\ | &= |\ |\langle f,\ \tau\phi_{n_k}(\xi)\rangle|^2\ -\ |\langle f,\ \tau\phi(\xi)\rangle|^2\ | \\ &\le 2\|f\|^2\|\tau\phi_{n_k}(\xi) - \tau\phi(\xi)\|_{l^2} \quad \forall\ f \in l^2(\mathbb{Z}^n). \end{aligned}$$

From this we get that $\tau_{V_{n_k},f}(\xi) \longrightarrow \tau_{V,f}(\xi)$ as $k \to \infty$ for a.e ξ. This implies $\tau_{W,f} = \tau_{V,f}\ \forall\ f \in l^2(\mathbb{Z}^n)$, thereby proving the theorem. □

Corollary 3.14. *Let $\phi_n \in L^2(\mathbb{R}^n)$ be a quasi orthogonal generator of $V_n \in PSI(\mathbb{R}^n)$, $n \in \mathbb{N}$ and let $(\hat{\phi}_n)$ converge pointwise. Further suppose that*

(1) *support of all $\hat{\phi}_n$'s are contained in a compact set E and*
(2) *sequence (V_n) is Cauchy under shift metric θ.*

Then sequence (V_n) converges to a PSI space generated by limit(ϕ_n).

Proof. Suppose $(\hat{\phi}_n)$ converge pointwise to the function ψ. The characterization of quasi orthogonal generators gives $|\hat{\phi}_n(\xi)| \le 1$ for a.e $\xi \in \mathbb{R}^n$. Also using condition 1, for all n, we have

$$|\hat{\phi}_n|(\xi) \le g(\xi) := \begin{cases} 1, & \xi\ \in\ E \\ 0, & \text{otherwise} \end{cases}.$$

As $g \in L^2(\mathbb{R}^n)$, $\psi \in L^2(\mathbb{R}^n)$ and $(\hat{\phi}_n)$ converge to ψ in $L^2(\mathbb{R}^n)$. The result now follows from previous theorem. □

Theorem 3.15. *Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz class of rapidly decreasing functions on $\mathbb{R}^n$. Suppose (ϕ_n) converges to ϕ in $\mathcal{S}(\mathbb{R}^n)$ where ϕ_n is a quasi orthogonal generator of the PSI space V_n. Then V_n converges to the PSI space $V := \mathcal{S}(\phi)$ under shift metric.*

Proof. Let $\psi_n = \phi_n - \phi$. Its Fourier transform $\hat{\psi}_n \in \mathcal{S}(\mathbb{R}^n)$. Let $C_n = \sup_{\xi\in\mathbb{R}^n}\{(1+|\xi|)|\hat{\psi}_n(\xi)|\}$.Then $\|\tau\phi_n(\xi) - \tau\phi(\xi)\| = \sum_k |\hat{\phi}_n(\xi) - \hat{\phi}(\xi)|^2 \le MC_n^2$ where $M = \sum_k \frac{1}{(1+|k|)^2}$. From the convergence of ϕ_n to ϕ, we conclude the uniform convergence of $\tau\phi_n$ to $\tau\phi$ in $\mathbb{T}^n$. This, in turn, implies convergence of V_n to V under shift metric. □

4. Example

Example 4.1. Here we give an example of a sequence of SIS converging under shift metric. Let $a, a_n, b, b_n > 0$, be such that $a^2 + b^2 = 1,\ a_n^2 + b_n^2 = 1$. Define ϕ_0 and ϕ_n by

$$\hat{\phi}_0(\xi) := \begin{cases} a, & \xi \in (0,1) \\ b, & \xi \in (1,2) \\ 0, & \text{otherwise} \end{cases},$$

$$\hat{\phi}_n(\xi) := \begin{cases} a_n, & \xi \in (0,1) \\ b_n, & \xi \in (1,2) \\ 0, & \text{otherwise} \end{cases}.$$

Further assume that $a_n \longrightarrow a$ so that $b_n \longrightarrow b$. We claim that $V_n := \mathcal{S}(\phi_n)$ converges to $V := \mathcal{S}(\phi_0)$ under shift metric.

A simple calculation shows that for $\xi \in (0,1)$

$$J_V(\xi) = \text{span}\{ae_0 + be_1\} \quad \text{and} \quad J_{V_n}(\xi) = \text{span}\{a_n e_0 + b_n e_1\} \quad \forall\, n.$$

Hence, for any $f \in l^2(\mathbb{Z}^n)$, there exist scalars $k_{f,0}$ and $k_{f,n}$ such that

$$\begin{aligned} P_{J_{V_n}(\xi)} f - P_{J_V(\xi)} f &= k_{f,n}(a_n e_0 + b_n e_1) - k_{f,0}(ae_0 + be_1) \\ &= (k_{f,n}a_n - k_{f,0}a)e_0 + (k_{f,n}b_n - k_{f,0}b)e_1. \end{aligned} \tag{1}$$

An evaluation using spectral function formula gives

$$k_{e_0,0} = a, \quad k_{e_1,0} = b, \quad k_{e_0,n} = a_n, \quad k_{e_1,n} = b_n \quad \text{and} \quad k_{e_p,n} = 0 \quad \text{for} \quad p \neq 0, 1. \tag{2}$$

Also for any $f \in l^2(\mathbb{Z})$ with $\|f\|_2 = 1$

$$\|P_{J_{V_n}(\xi)} f - P_{J_V(\xi)} f\| \le \|(P_{J_{V_n}(\xi)} e_0 - P_{J_V(\xi)} e_0)\| + \|(P_{J_{V_n}(\xi)} e_1 - P_{J_V(\xi)} e_1)\|.$$

Evaluating the RHS using (1) and (2), we can conclude the convergence of (V_n) to V under shift metric.

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