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# ON COMMUTING SYMMETRIC OPERATORS

S. M. ZAGORODNYUK

*Dedicated to the memory of A. G. Kostyuchenko*

Abstract. In this paper we present some conditions for a pair of commuting symmetric operators with a joint invariant dense domain in a Hilbert space, to have a commuting self-adjoint extension in the same space. The remarkable Godič-Lucenko theorem allows to get a convenient description of all such extensions.

## 1. Introduction

The extension problems for commuting operators have been studied for a long time and have numerous applications (e.g. [1, Chapter VIII, Section 2], [2]–[8], etc.). We shall give some conditions for a pair of commuting symmetric operators with a joint invariant dense domain in a Hilbert space, to have a commuting self-adjoint extension in the same space. A convenient description of all such extensions can be obtained by the remarkable Godič-Lucenko theorem [9] (also see [10] for another proof and a generalization). Our conditions form an analog of Theorem 2.6 in [1, Chapter VIII] obtained in 1945 by Livshic, and independently later by Eskin. Notice that Theorem 2.6 regarded operators acting in the tensor products of Hilbert spaces and having a fixed structure. Our proof below is analogous to that of Theorem 2.6 except its part related to the Godič-Lucenko Theorem. Also, our result generalizes the result of Ismagilov in [3, Theorem 1]. A close result to Theorem 2.1 below, was given by Slinker in [4, Theorem 3.1].

**Notations.** We denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. Everywhere in this paper, all Hilbert spaces are assumed to be separable. By $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ we denote the scalar product and the norm in a Hilbert space $H$, respectively. The indices may be omitted in obvious cases. For a set $M$ in $H$, by $\overline{M}$ we mean the closure of $M$ in the norm $\|\cdot\|_H$. The identity operator in $H$ is denoted by $E_H$. For an arbitrary linear operator $A$ in $H$, the operators $A^*$,$\overline{A}$,$A^{-1}$ mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator $A$.

## 2. An extension of commuting symmetric operators in the same space

**Theorem 2.1.** *Let* $\mathbf{A}$ *be a symmetric operator and* $\mathbf{B}$ *be an essentially self-adjoint operator with a common domain* $\mathcal{D} = D(\mathbf{A}) = D(\mathbf{B})$ *in a Hilbert space* $\mathbf{H}$*,* $\overline{\mathcal{D}} = \mathbf{H}$*, and*

$$\mathbf{A}\mathcal{D} \subseteq \mathcal{D}, \quad \mathbf{B}\mathcal{D} \subseteq \mathcal{D};$$

$$\mathbf{AB} = \mathbf{BA}.$$

*Suppose also that for some* $z_0 \in \mathbb{C}\backslash\mathbb{R}$*, the operator* $\mathbf{B}$ *restricted to the domain* $(\mathbf{A} - z_0 E_{\mathbf{H}})\mathcal{D}$ *is essentially self-adjoint in a Hilbert space* $(\overline{\mathbf{A}} - z_0 E_{\mathbf{H}})D(\overline{\mathbf{A}})$.

*If there exists a conjugation* $\mathbf{J}$ *in* $\mathbf{H}$ *such that* $\mathbf{J}\mathcal{D} \subseteq \mathcal{D}$*, and*

$$\mathbf{AJ} = \mathbf{JA}, \quad \mathbf{BJ} = \mathbf{JB},$$

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*then there exists a self-adjoint operator* $\widetilde{\mathbf{A}} \supseteq \mathbf{A}$, *which commutes with* $\overline{\mathbf{B}}$.

*Proof.* Consider the Cayley transformations of the operators $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$

$$V_{\mathbf{A}} := (\overline{\mathbf{A}} - \overline{z_0}E_{\mathbf{H}})(\overline{\mathbf{A}} - z_0E_{\mathbf{H}})^{-1} = E + (z_0 - \overline{z_0})(\overline{\mathbf{A}} - z_0E_{\mathbf{H}})^{-1},$$
$$U_{\mathbf{B}} := (\overline{\mathbf{B}} + iE_{\mathbf{H}})(\overline{\mathbf{B}} - iE_{\mathbf{H}})^{-1} = E + 2i(\overline{\mathbf{B}} - iE_{\mathbf{H}})^{-1}.$$

Set

$$\mathbf{H}_1 = (\overline{\mathbf{A}} - z_0E_{\mathbf{H}})D(\overline{\mathbf{A}}), \quad \mathbf{H}_2 = \mathbf{H} \ominus \mathbf{H}_1, \quad \mathbf{H}_3 = (\overline{\mathbf{A}} - \overline{z_0}E_{\mathbf{H}})D(\overline{\mathbf{A}}), \quad \mathbf{H}_4 = \mathbf{H} \ominus \mathbf{H}_3.$$

Observe that $V_{\mathbf{A}}$ maps isometrically $\mathbf{H}_1$ onto $\mathbf{H}_3$, and $U_{\mathbf{B}}$ is a unitary operator in $\mathbf{H}$. For arbitrary $h \in (\mathbf{B} - iE_{\mathbf{H}})(\mathbf{A} - z_0E_{\mathbf{H}})\mathcal{D}$, $h = (\mathbf{B} - iE_{\mathbf{H}})(\mathbf{A} - z_0E_{\mathbf{H}})f$, $f \in \mathcal{D}$, we may write

$$U_{\mathbf{B}}V_{\mathbf{A}}h = U_{\mathbf{B}}(\overline{\mathbf{A}} - \overline{z_0}E_{\mathbf{H}})(\mathbf{B} - iE_{\mathbf{H}})f = (\mathbf{B} + iE_{\mathbf{H}})(\mathbf{A} - \overline{z_0}E_{\mathbf{H}})f;$$
$$V_{\mathbf{A}}U_{\mathbf{B}}h = V_{\mathbf{A}}(\overline{\mathbf{B}} + iE_{\mathbf{H}})(\mathbf{A} - z_0E_{\mathbf{H}})f = (\overline{\mathbf{A}} - \overline{z_0}E_{\mathbf{H}})(\overline{\mathbf{B}} + iE_{\mathbf{H}})f.$$

Therefore

$$U_{\mathbf{B}}V_{\mathbf{A}}h = V_{\mathbf{A}}U_{\mathbf{B}}h, \quad h \in (\mathbf{B} - iE_{\mathbf{H}})(\mathbf{A} - z_0E_{\mathbf{H}})\mathcal{D}. \tag{1}$$

Since $\mathbf{B}$ restricted to the domain $(\mathbf{A} - z_0E_{\mathbf{H}})\mathcal{D}$ is essentially self-adjoint, the sets $(\mathbf{B} \pm iE_{\mathbf{H}})(\mathbf{A} - z_0E_{\mathbf{H}})\mathcal{D}$ are dense in $\mathbf{H}_1$. By continuity, from (1) we derive

$$U_{\mathbf{B}}V_{\mathbf{A}}h = V_{\mathbf{A}}U_{\mathbf{B}}h, \quad h \in \mathbf{H}_1. \tag{2}$$

Observe that

$$U_{\mathbf{B}}(\mathbf{B} - iE_{\mathbf{H}})(\mathbf{A} - z_0E_{\mathbf{H}})f = (\mathbf{B} + iE_{\mathbf{H}})(\mathbf{A} - z_0E_{\mathbf{H}})f, \quad f \in \mathcal{D}.$$

Thus, $U_{\mathbf{B}}$ maps a dense set in $\mathbf{H}_1$ on a dense set in $\mathbf{H}_1$. Therefore $U_{\mathbf{B}}$ is a unitary operator in $\mathbf{H}_1$. Then $U_{\mathbf{B}}$ is a unitary operator in $\mathbf{H}_2$, as well. By (2) we may write

$$U_{\mathbf{B}}\mathbf{H}_3 = U_{\mathbf{B}}V_{\mathbf{A}}\mathbf{H}_1 = V_{\mathbf{A}}U_{\mathbf{B}}\mathbf{H}_1 = V_{\mathbf{A}}\mathbf{H}_1 = \mathbf{H}_3.$$

Therefore $U_{\mathbf{B}}$ is a unitary operator in subspaces $\mathbf{H}_3$ and $\mathbf{H}_4$, as well.

Let us construct an isometric operator $U_{2,4}$ which maps $\mathbf{H}_2$ onto $\mathbf{H}_4$, and commutes with $U_{\mathbf{B}}$

$$U_{\mathbf{B}}U_{2,4}x = U_{2,4}U_{\mathbf{B}}x, \quad x \in \mathbf{H}_2. \tag{3}$$

The operator $U_{\mathbf{B}}$ restricted to $\mathbf{H}_j$ we denote by $U_{\mathbf{B},j}$, $1 \le j \le 4$. Using the definition of the closure we get

$$\mathbf{J}\overline{\mathbf{A}}h = \overline{\mathbf{A}}\mathbf{J}h, \quad h \in D(\overline{\mathbf{A}}),$$
$$\mathbf{J}\overline{\mathbf{B}}h = \overline{\mathbf{B}}\mathbf{J}h, \quad h \in D(\overline{\mathbf{B}}).$$

Observe that

$$\mathbf{J}U_{\mathbf{B}}h = U_{\mathbf{B}}^{-1}\mathbf{J}h, \quad h \in D(U_{\mathbf{B}}).$$

To check this relation we may write for an arbitrary $x \in \mathcal{D}$ and $g := (\mathbf{B} - iE_{\mathbf{H}})x$

$$(\mathbf{B} - iE_{\mathbf{H}})\mathbf{J}x = \mathbf{J}(\mathbf{B} + iE_{\mathbf{H}})x = \mathbf{J}U_{\mathbf{B}}g.$$

Multiply the latter equality by $U_{\mathbf{B}}$

$$(\mathbf{B} + iE_{\mathbf{H}})\mathbf{J}x = U_{\mathbf{B}}\mathbf{J}U_{\mathbf{B}}g,$$

and we get

$$\mathbf{J}g = U_{\mathbf{B}}\mathbf{J}U_{\mathbf{B}}g, \quad g \in (\mathbf{B} - iE_{\mathbf{H}})\mathcal{D}.$$

Moreover, we have

$$\overline{\mathbf{A}}^*\mathbf{J}x = \mathbf{J}\overline{\mathbf{A}}^*x, \quad x \in D(\overline{\mathbf{A}}^*). \tag{4}$$

Indeed, for arbitrary $f_A \in D(\overline{\mathbf{A}})$ and $g_{A^*} \in D(\overline{\mathbf{A}}^*)$ we may write

$$\overline{(\overline{\mathbf{A}}f_A, \mathbf{J}g_{A^*})} = (\mathbf{J}\overline{\mathbf{A}}f_A, g_{A^*}) = (\overline{\mathbf{A}}\mathbf{J}f_A, g_{A^*}) = (\mathbf{J}f_A, \overline{\mathbf{A}}^*g_{A^*}) = \overline{(f_A, \mathbf{J}\overline{\mathbf{A}}^*g_{A^*})},$$

and (4) follows.
Choose an arbitrary $x \in \mathbf{H}_2$. We have $\overline{\mathbf{A}}^* x = \overline{z_0} x$, and therefore $\overline{\mathbf{A}}^* \mathbf{J} x = \mathbf{J} \overline{\mathbf{A}}^* x = z_0 x$. Thus, we have $\mathbf{J}\mathbf{H}_2 \subseteq \mathbf{H}_4$. In a similar manner we get $\mathbf{J}\mathbf{H}_4 \subseteq \mathbf{H}_2$, and therefore

$$\mathbf{J}\mathbf{H}_2 = \mathbf{H}_4, \quad \mathbf{J}\mathbf{H}_4 = \mathbf{H}_2. \tag{5}$$

By the Godič-Lucenko Theorem we have a representation: $U_{\mathbf{B},2} = KL$, where $K$ and $L$ are some conjugations in $\mathbf{H}_2$. We set

$$U_{2,4} := \mathbf{J}K. \tag{6}$$

From (5) it follows that $U_{2,4}$ maps isometrically $\mathbf{H}_2$ onto $\mathbf{H}_4$. Notice that $U_{2,4}^{-1} = K\mathbf{J}$. Then

$$U_{2,4} U_{\mathbf{B},2} U_{2,4}^{-1} x = \mathbf{J}KKLK\mathbf{J}x = \mathbf{J}LK\mathbf{J}x = \mathbf{J}U_{\mathbf{B},2}^{-1}\mathbf{J}x = \mathbf{J}U_{\mathbf{B}}^{-1}\mathbf{J}x = U_{\mathbf{B},4}x, \quad x \in \mathbf{H}_4.$$

Therefore relation (3) is true. Set $\widetilde{U} = V_{\mathbf{A}} \oplus U_{2,4}$. It is straightforward to check that $\widetilde{U}$ commutes with $U_{\mathbf{B}}$. Moreover, it is easy to check that all unitary operators $U \supseteq V_{\mathbf{A}}$ which commute with $U_{\mathbf{B}}$ have the following form:

$$U = V_{\mathbf{A}} \oplus W_{2,4}, \tag{7}$$

where $W_{2,4}$ is an isometric operator which maps $\mathbf{H}_2$ onto $\mathbf{H}_4$, and commutes with $U_{\mathbf{B}}$. Finally, we set $\widetilde{\mathbf{A}}$ to be the inverse Cayley transformation of $\widetilde{U}$. □

*Remark* 2.1. Notice that by (7),(6) we easily obtain that the following relation:

$$W_{2,4} = U_{2,4} U_2,$$

where $U_2$ is an arbitrary unitary operator in $\mathbf{H}_2$ commuting with $U_{\mathbf{B}}$, provides all admissible operators for (7). Operators $U_2$ may be described in terms of the decomposable operators with respect to the spectral measure of the operator $U_{\mathbf{B}}$ restricted to $\mathbf{H}_2$.

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School of Mathematics and Mechanics, Karazin Kharkiv National University, 4 Svobody sq., Kharkiv, 61077, Ukraine
*E-mail address*: Sergey.M.Zagorodnyuk@univer.kharkov.ua

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