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ON $*$-REPRESENTATIONS OF A CLASS OF ALGEBRAS WITH POLYNOMIAL GROWTH RELATED TO COXETER GRAPHS

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Abstract. For a Hilbert space $\mathcal{H}$, we study configurations of its subspaces related to Coxeter graphs $\mathbb{G}_{s_1,s_2}$, $s_1, s_2 \in \{4, 5\}$, which are arbitrary trees such that one edge has type s_1, another one has type s_2 and the rest are of type 3. We prove that such irreducible configurations exist only in a finite dimensional $\mathcal{H}$, where the dimension of $\mathcal{H}$ does not exceed the number of vertices of the graph by more than twice. We give a description of all irreducible nonequivalent configurations; they are indexed with a continuous parameter. As an example, we study irreducible configurations related to a graph that consists of three vertices and two edges of type s_1 and s_2.

0. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_i \subset \mathcal{H}$, $i = 0, \dots, n-1$, be a set of its subspaces. Many publications (see [6, 2, 1, 4, 12, 3, 5] and others) are dedicated to an investigation of systems of subspaces,

$$S = (\mathcal{H}; \mathcal{H}_0, \dots, \mathcal{H}_{n-1}).$$

For any system of subspaces S we can introduce a set of orthogonal projections $\{P_i\}$, $i = 0, \dots, n-1$, where P_i denotes an orthogonal projection on the Hilbert subspace H_i of the Hilbert space H.

A system of subspaces $S = (\mathcal{H}; \mathcal{H}_0, \dots, \mathcal{H}_{n-1})$ is called a *simple* system if all subspaces are different and for any pair of subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$, $i \neq j$, following relations hold:

$$P_iP_jP_i = \tau_{ij}P_i \quad \text{and} \quad P_jP_iP_j = \tau_{ji}P_j, \tag{1}$$

where

$$0 < \tau_{ij} = \tau_{ji} = \cos^2\theta_{ij} < 1 \quad (0 < \theta_{ij} < \frac{\pi}{2}),$$

or the subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$ are orthogonal, i.e.,

$$P_iP_j = P_jP_i = 0 \quad (\theta_{ij} = \frac{\pi}{2}). \tag{2}$$

For more details about simple systems of subspaces see the review [11].

A more general class of systems of subspaces is a class of systems such that for any pair of subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$, $i < j$, one of following relations hold:

$$\prod_{k=0}^{m_{ij}-1} \left(P_iP_jP_i - \tau_{ij}^k P_i\right) = 0 \quad \text{and} \quad \prod_{k=0}^{m_{ij}-1} \left(P_jP_iP_j - \tau_{ij}^k P_j\right) = 0, \tag{3}$$

or

$$\prod_{k=0}^{m_{ij}-1} \left(P_iP_jP_i - \tau_{ij}^k P_i\right) P_j = 0 \quad \text{and} \quad \prod_{k=0}^{m_{ij}-1} \left(P_jP_iP_j - \tau_{ij}^k P_j\right) P_i = 0, \tag{4}$$

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where

$$m_{ij} \in \mathbb{N}, \quad 0 < \tau_{ij}^k = \cos^2 \theta_{ij}^k \leqslant 1 \quad (0 \leqslant \theta_{ij}^k < \frac{\pi}{2}), \quad 0 \leqslant k \leqslant m_{ij} - 1,$$

or the subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$ are orthogonal, i.e.,

$$P_i P_j = P_j P_i = 0 \quad (\theta_{ij} = \frac{\pi}{2}). \tag{5}$$

Suppose that $\tau_{ij}^{k_1} = \tau_{ij}^{k_2}$, $k_1 \neq k_2$. Let us show that relations (3) imply the following relations:

$$\prod_{\substack{k=0\\k\neq k_2}}^{m_{ij}-1} \left(P_i P_j P_i - \tau_{ij}^k P_i\right) = 0 \quad \text{and} \quad \prod_{\substack{k=0\\k\neq k_2}}^{m_{ij}-1} \left(P_j P_i P_j - \tau_{ij}^k P_j\right) = 0, \tag{6}$$

and relations (4) imply the relations

$$\prod_{\substack{k=0\\k\neq k_2}}^{m_{ij}-1} \left(P_i P_j P_i - \tau_{ij}^k P_i\right) P_j = 0 \quad \text{and} \quad \prod_{\substack{k=0\\k\neq k_2}}^{m_{ij}-1} \left(P_j P_i P_j - \tau_{ij}^k P_j\right) P_i = 0. \tag{7}$$

Indeed, if the first equality of relations (6) does not hold, then there exists a vector $y \in \mathcal{H}$ such that

$$z = \prod_{\substack{k=0\\k\neq k_2}}^{m_{ij}-1} \left(P_i P_j P_i - \tau_{ij}^k P_i\right) y \neq 0,$$

then

$$\begin{aligned}
\langle z, z \rangle &= \langle \prod_{\substack{k=0\\k\neq k_2}}^{m_{ij}-1} \left(P_i P_j P_i - \tau_{ij}^k P_i\right) y, \prod_{\substack{k=0\\k\neq k_2}}^{m_{ij}-1} \left(P_i P_j P_i - \tau_{ij}^k P_i\right) y \rangle \\
&= \langle \prod_{k=0}^{m_{ij}-1} \left(P_i P_j P_i - \tau_{ij}^k P_i\right) y, \prod_{\substack{k=0\\k\neq k_2\\k\neq k_1}}^{m_{ij}-1} \left(P_i P_j P_i - \tau_{ij}^k P_i\right) y \rangle = 0,
\end{aligned}$$

which contradicts to $z \neq 0$. The rest of equalities of relations (6) and (7) can be proved in the same way.

Taking into account the above, we will assume that

$$1 \geqslant \tau_{ij}^0 > \tau_{ij}^1 > \cdots > \tau_{ij}^{m_{ij}-1} > 0.$$

Moreover, in the case of $m_{ij} = 1$ and $\tau_{ij}^0 = 1$, relations (3) imply $\mathcal{H}_i = \mathcal{H}_j$. This case will be excluded from our consideration. So it will be supposed that $0 < \tau_{ij}^0 < 1$ considering relations (3) in the case where $m_{ij} = 1$.

For further considerations it will be suitable to determine m_{ji} and τ_{ji}^k in the case where $i < j$ and $0 \leqslant k \leqslant m_{ij} - 1$ by equalities $m_{ji} = m_{ij}$ and $\tau_{ji}^k = \tau_{ij}^k$.

It is convenient to represent considered systems with finite non-oriented Coxeter graphs $\mathbb{G} = (V, R)$ without multiple edges and loops (here $V = \{0, \dots, n-1\}$ is the set of vertices of the graph and $R = \{\gamma_{ij} = \gamma_{ji}\}$ is the set of edges of the graph which are split into the types $R = \sqcup R_s$, $s \in \mathbb{N}$, $s \geqslant 3$) and a mapping f which maps edges of the graph into polynomials $f : R \to \mathbb{R}[x]$. More precisely, each vertex i of the graph corresponds to a subspace $\mathcal{H}_i$ and vertices i and j are connected with an edge γ_{ij} having type $s = 2m_{ij} + 1$ or $s = 2(m_{ij} + 1)$ if and only if condition (3) or (4), correspondingly, holds for the subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$,

$$f : \gamma_{ij} \mapsto f_{ij}(x) = \prod_{k=0}^{m_{ij}-1} (x - \tau_{ij}^k)$$

in the first case and

$$f : \gamma_{ij} \mapsto f_{ij}(x) = x \prod_{k=0}^{m_{ij}-1} (x - \tau_{ij}^k)$$

in the second one. If the subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$ are orthogonal, the vertices are not connected. It can be supposed that such "missed edges" have type 2 and the related mapping f maps these "edges" into the polynomials $f_{ij}(x) = x$.

Note that relations (3) can be rewritten as

$$f_{ij}(P_i P_j) P_i = 0, \quad f_{ij}(P_j P_i) P_j = 0,$$

and relations (4), (5) can be rewritten in the following form:

$$f_{ij}(P_i P_j) = 0, \quad f_{ij}(P_j P_i) = 0.$$

Moreover, after opening the parentheses, the left-hand side of each equation in relations (3) and (4) will be a linear combination of products of the projections P_i and P_j, and type s will be equal to the length of the longest product in the linear combination.

Evidently, the considered systems of subspaces are $*$-representations, in Hilbert spaces, of $*$-algebras

$$\begin{aligned} TL_{\mathbb{G},f,\perp} = \mathbb{C}\langle\, p_0, \dots, p_{n-1} \mid p_i^2 = p_i^* = p_i,\ i \in V;& \\ f_{ij}(p_i p_j) p_i^{\sigma_{ij}} = 0,\ f_{ij}(p_j p_i) p_j^{\sigma_{ij}} = 0,\ i \neq j \in V \,\rangle&, \end{aligned}$$

here $\sigma_{ij} = 1$ if the type of the edge is an odd number and $\sigma_{ij} = 0$ otherwise. The equality $\mathcal{H}_i = \operatorname{Im} \pi(p_i)$, $i \in V$, gives a correspondence between the class of systems of subspaces, $S = (\mathcal{H}; \mathcal{H}_0, \dots, \mathcal{H}_{n-1})$, and $*$-representations π of the $*$-algebras $TL_{\mathbb{G},f,\perp}$ in the Hilbert space $\mathcal{H}$.

In papers [9, 10, 8], the algebras defined above have been denoted by $TL_{\mathbb{G},g,\perp}$, where g is also a mapping from the set of edges into the set of polynomials defined in such a way that $f_{ij}(x) = x^{m_{ij} - \sigma_{ij} + 1} - g_{ij}(x)$.

It was shown in [9] that the algebra $TL_{\mathbb{G},f,\perp}$ is finite dimensional if and only if the graph $\mathbb{G}$ is a tree and the number of edges that have the type grater than 3 is less or equal to one; the algebra is infinite dimensional and has polynomial growth if and only if the graph $\mathbb{G}$ has one cycle and all its edges have type 3, or the graph $\mathbb{G}$ is a tree and the type of any edge is less than 6 and only two of the edges have the type greater than 3. $*$-Representations of finite dimensional algebras have been studied in paper [10]. In paper [8] there have been studied $*$-representations of algebras that have polynomial growth and two edges of the related Coxeter graph have type 4. In the present paper, we consider algebras that have polynomial growth and at least one edge of the related Coxeter graph has type 5.

By $\mathbb{G}_{s_1,s_2}$, $s_1, s_2 \in \{4, 5\}$, we denote a Coxeter graph such that it is a tree and its edges have type 3 except for two edges the types of which are s_1 and s_2.

In the first section we will show that any irreducible $*$-representation of the algebra $TL_{\mathbb{G}_{s_1,s_2},f,\perp}$ is finite dimensional, moreover, a strict estimation of the dimension will be obtained (Theorem 8).

In the second section we will prove three simple propositions which describe pairs of orthogonal projections on a finite dimensional Hilbert space connected with edges of type 3, 4 or 5.

In the third section we will provide a procedure which allows to construct irreducible $*$-representations π_ν on Hilbert spaces $\mathcal{H}_\nu$, where ν is a parameter with the values chosen in such a way that some sesquilinear form is nonnegative definite (Lemma 14). For different values of ν, the related $*$-representations are unitarily nonequivalent (Proposition 15).

In the forth section it will be shown that any irreducible proper $*$-representation is unitarily equivalent to π_ν for some ν.

In the fifth section we will consider $*$-algebras related to Coxeter graphs with three vertices and two edges where the first edge has type 5 and the second one has type 4 or 5.

1. Irreducible $*$-representations are finite dimensional

A path of length m in a Coxeter graph $\mathbb{G}$,

$$l = l(i_0) = (i_0, i_1, \dots, i_m), \quad \gamma_{i_{k-1}, i_k} \in R,$$

will be called a *path without repetitions* if $i_k \neq i_j$ for $k, j = 0, \dots, m$, $k \neq j$. The path $l = (i_0)$ is considered as a path of length 0 without repetitions, and it is convenient to consider the path $l = ()$ as an "empty" one. For a path $l = (i_0, i_1, \dots, i_m)$, define $l^* = (i_m, i_{m-1}, \dots, i_0)$. A *union of paths* $l_1 = (i_0, \dots, i_{k-1}, i_k)$ and $l_2 = (i_k, i_{k+1} \dots, i_t)$ is defined to be the path $l_1 \cup l_2 = (i_0, \dots, i_{k-1}, i_k, i_{k+1}, \dots, i_t)$. To any path $l = (i_0, i_1, \dots, i_m)$, we make correspond the product $\Pi_l = p_{i_0} \dots p_{i_m}$ in the algebra, to the "empty" path, we set $\Pi_l = e$.

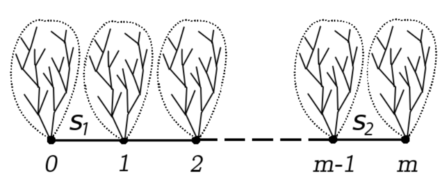
To be specific let us enumerate vertices of the Coxeter graph $\mathbb{G}_{s_1,s_2}$ such that edge $\gamma_{0,1}$ has type s_1, $\gamma_{m-1,m}$ has type s_2 and the vertices 1 and $m-1$ are connected by the path

$$\hat{l} = (1, 2, \dots, m-1).$$

All vertices of the graph can be naturally splitted into three parts

$$V = V_0 \cup V_{in} \cup V_m,$$

where any two vertices of each part are connected with a path which consists of type 3 edges only.



Denote by $\mathcal{N}$ the set of all paths l such that Π_l is a normal word and denote by $\mathcal{N}_i$ the set of all paths $l \in \mathcal{N}$ which end at vertex i. For normal words, Groebner bases, the composition lemma, we refer to e.g. [13]. For the algebra $TL_{\mathbb{G},f,\perp}$, normal words are precisely the words that do not contain, as subwords, the leading words of the defining relations of the algebra $TL_{\mathbb{G},f,\perp}$, see [9]. That is, a normal word should not contain, as subwords, the following words:

$p_i^2, i \in V;$

$p_i p_j, \;\; p_j p_i,$ if $\gamma_{ij} \notin R;$

$(p_i p_j)^k p_i^\sigma, \quad (p_j p_i)^k p_j^\sigma, \quad$ if $\quad \gamma_{ij} \in R_s, \quad s = 2k + \sigma \geqslant 3, \quad \sigma \in \{0, 1\}.$

Let $\pi : TL_{\mathbb{G}_{s_1,s_2},f,\perp} \to \mathcal{B}(\mathcal{H})$ be a $*$-representation of the algebra $TL_{\mathbb{G}_{s_1,s_2},f,\perp}$. Denote $\mathcal{H}_i = \operatorname{Im} P_i$, $i \in V$.

Proposition 1. *Let $0 \neq x_i \in \mathcal{H}_i$. Then the closure of the linear span of the vectors $\{\pi(\Pi_l)x_i\}_{l\in\mathcal{N}_i}$ is invariant with respect to π.*

Proof. Indeed, either the product $p_j\Pi_l$ is equal to 0 or $p_j\Pi_l = \Pi_{l'}$, where the path l' ends at the vertex i. In the second case, either $l' \in N_i$ or $\Pi_{l'} = \sum_{l''} \lambda_{l''}\Pi_{l''}$, where $l'' \in N_i$. Actually, it follows from the relations in the algebra that if a normal word ends with p_i then the product of the word by p_j on the left is either equal to 0 or it is equal to some linear combination of words which end with p_i. □

Denote

$$\begin{aligned}
&d = \begin{cases} p_1, & m = 2,\\ \frac{p_{m-1}p_{m-2}}{\sqrt{\tau_{m-1,m-2}}} \cdot \frac{p_{m-2}p_{m-3}}{\sqrt{\tau_{m-2,m-3}}} \cdot \ldots \cdot \frac{p_2p_1}{\sqrt{\tau_{2,1}}}, & m > 2,\end{cases}\\
&D = \pi(d),\\
&b_1 = p_1p_0p_1, \quad B_1 = \pi(b_1)|_{\mathcal{H}_1},\\
&b_2 = d^*p_md, \quad B_2 = \pi(b_2)|_{\mathcal{H}_1}.
\end{aligned}$$

It is evident that $d^*d = p_1$ and $dd^* = p_{m-1}$.

Proposition 2. *The following identities hold:*

$$f_{0,1}(B_1) = 0, \quad f_{m-1,m}(B_2) = 0.$$

Proof. Indeed,

$$0 = f_{0,1}(P_1P_0)P_1 = f_{0,1}(P_1P_0P_1)P_1,$$

so $f_{0,1}(B_1) = 0$. Further,

$$\begin{aligned}
0 &= f_{m-1,m}(P_{m-1}P_m)P_{m-1}\\
&= D^*f_{m-1,m}(P_{m-1}P_m)P_{m-1}D = f_{m-1,m}(D^*P_mD)P_1,
\end{aligned}$$

which means $f_{m-1,m}(B_2) = 0$. □

Consider a $*$-algebra

$$\mathcal{A} = \mathbb{C}\langle\, c_1, c_2 \,|\, c_i^* = c_i,\ f_{0,1}(c_1) = 0,\ f_{m-1,m}(c_2) = 0\,\rangle.$$

For any $*$-representation $\pi : TL_{\mathbb{G}_{s_1,s_2},f,\perp} \to \mathcal{B}(\mathcal{H})$, we can construct $*$-representation of the $*$-algebra $\mathcal{A}$ by the formulas

$$\hat{\pi} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_1), \quad c_1 \mapsto B_1, \quad c_2 \mapsto B_2.$$

Proposition 3. *Irreducible $*$-representations of the $*$-algebra $\mathcal{A}$ can be one- or two-dimensional only.*

Proof. If $\deg f_{0,1} = 2$, $\deg f_{m-1,m} = 2$ and each polynomial has distinct roots, then the algebra $\mathcal{A}$ is isomorphic to the algebra generated by two orthogonal projections,

$$\mathbb{C}\langle\, q_1, q_2 \,|\, q_i^* = q_i = q_i^2\,\rangle.$$

It is known that this algebra has one- and two-dimensional irreducible $*$-representations only (see, for example, [7]). □

Lemma 4. *If a $*$-representation π is irreducible then the $*$-representation $\hat{\pi}$ is irreducible too.*

Proof. Let $\hat{\pi}$ be reducible. Then $\mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12}$, where $\mathcal{H}_{11}$ and $\mathcal{H}_{12}$ are nontrivial invariant subspaces of $\mathcal{H}_1$ with respect to the $*$-representation $\hat{\pi}$. Consider

$$0 \neq x \in \mathcal{H}_{11}, \quad 0 \neq y \in \mathcal{H}_{12}.$$

By Proposition 1, the closure of the linear span of the set of vectors,

$$\{\pi(\Pi_l)x\}_{l\in\mathcal{N}_1},$$

is invariant in respect to π, then it is equal to $\mathcal{H}$. On the other hand,

$$\langle \pi(\Pi_l)x, y \rangle = \langle P_1\pi(\Pi_l)P_1x, y \rangle = \sum_{\tilde{l}:\tilde{l}\in\mathcal{N}_1,\tilde{l}^*\in\mathcal{N}_1} \lambda_{\tilde{l}}\langle \pi(\Pi_{\tilde{l}})x, y \rangle = 0.$$

Indeed, if $\tilde{l} \in \mathcal{N}_1, \tilde{l}^* \in \mathcal{N}_1$ then either $\pi(\Pi_{\tilde{l}}) = P_1$ or $\pi(\Pi_{\tilde{l}})$ is equal, up to a scalar, to

$$(P_1P_0P_1)^{\sigma_1}(D^*P_mDP_1P_0P_1)^r(D^*P_mD)^{\sigma_2},$$

where $\sigma_1, \sigma_2 \in \{0,1\}$, $r \in \mathbb{N}\cup\{0\}$, and $\sigma_1+\sigma_2+r > 0$. Then $\pi(\Pi_{\tilde{l}})x \in \mathcal{H}_{11}$. So it has been shown that $\langle z, y \rangle = 0$ for any $z \in \mathcal{H}$. If $y \in \mathcal{H}_{12} \subset \mathcal{H}_1 \subset \mathcal{H}$, then $\langle y, y \rangle = 0$ and this contradicts to $y \neq 0$. □

Corollary 5. *If the $*$-representation π is irreducible and $P_0D^*P_mDP_0 \neq 0$, then there exists $0 \neq x \in \mathcal{H}_0$ and $\xi > 0$ such that $P_0D^*P_mDP_0x = \xi x$.*

Proof. By the previous proposition, $*$-representation $\hat{\pi}$ is irreducible because the $*$-representation π is irreducible. Then

$$\dim \mathcal{H}_1 \leqslant 2, \quad \dim \overline{\operatorname{Im} P_0D^*P_mDP_0} \leqslant 2.$$

Because $P_0D^*P_mDP_0 \neq 0$, we have that the self-adjoint finite dimensional non negative operator $P_0D^*P_mDP_0$ has positive eigenvalue ξ and the related eigenvector x. □

Let us introduce a map

$$\psi_i : V \to \mathcal{N}_i$$

that maps every vertex into the unique path without repetitions from this vertex into vertex i. It is evident that $\hat{l} = \psi_0(m)$. Let introduce two paths with repetitions $l_0 = (0,1,0)$ and $l_m = (m, m-1, m)$. Consider the sets of paths,

$$\begin{aligned} \mathcal{S} &= \{\psi_0(i) \,|\, i \in V\}, \\ \mathcal{L}_0 &= \{\psi_0(i) \cup l_0 \,|\, i \in V_0\}, \\ \mathcal{L}_{in} &= \{\psi_m(i) \cup \hat{l} \,|\, i \in V_{in}\}, \\ \mathcal{L}_m &= \{\psi_m(i) \cup l_m \cup \hat{l} \,|\, i \in V_m\}. \end{aligned}$$

It is evident that any $l \in \mathcal{N}_0$ can be represented in one of the two following forms:

$$\begin{aligned} l &= l' \cup \underbrace{(\hat{l}^* \cup \hat{l}) \cup \cdots \cup (\hat{l}^* \cup \hat{l})}_{k \text{ times}}, \\ l &= l' \cup \underbrace{(\hat{l}^* \cup \hat{l}) \cup \cdots \cup (\hat{l}^* \cup \hat{l})}_{k \text{ times}} \cup l_0, \end{aligned}$$

where $k \in \mathbb{N}\cup\{0\}$ and $l' \in \mathcal{P} = \mathcal{S}\cup\mathcal{L}_0\cup\mathcal{L}_{in}\cup\mathcal{L}_m$.

Proposition 6. *Let π be an irreducible $*$-representation, $P_0P_1 \neq 0$, and $P_{m-1}P_m \neq 0$. Then $P_0D^*P_mDP_0 \neq 0$.*

Proof. Since $P_0P_1 \neq 0$, there exists x_0 such that $P_0x_0 = x_0$, $P_1P_0x_0 \neq 0$. Suppose that $P_0D^*P_mDP_0 = 0$. Then $\pi(\Pi_{\hat{l}^*\cup\hat{l}}) = 0$, so $\pi(\Pi_{\hat{l}}) = 0$. This means that $\pi(\Pi_l) = 0$ for any path $l \in \mathcal{N}_0$ such that Π_l contains $\Pi_{\hat{l}}$ as a subword.

Consider a linear span $\mathcal{H}'$ of a finite set of vectors $\{\pi(\Pi_l)x_0\}_{l\in\mathcal{N}_0'}$, where $\mathcal{N}_0' \subset \mathcal{N}_0$ is a set of paths l which do not contain $\hat{l}$ as a subword. It is evident that $\mathcal{H}'$ is invariant with respect to the representation π. So it is equal to Hilbert space $\mathcal{H}$.

Let us show that for any path $l \in \mathcal{N}_0'$ the identity $P_mP_{m-1}\pi(\Pi_l)x_0 = 0$ holds. Indeed, $P_{m-1}\pi(\Pi_l) \neq 0$ if and only if the initial vertex of the path l is $m-1$ or it is connected with the vertex $m-1$ by an edge. So,

$$l = \psi_0(j), \quad l = \psi_0(j) \cup l_0,$$

where either $j = m-1$ or j is connected with the vertex $m-1$ by an edge of type 3. If $j = m-1$ or $j = m-2$, then $P_m P_{m-1}\pi(\Pi_l) = \pi(\Pi_{\hat{l}}) = 0$ and $P_m P_{m-1}\pi(\Pi_l) = \pi(\Pi_{\hat{l}})P_0 P_1 P_0 = 0$. If j is a vertex, other than $m-2$, connected with the vertex $m-1$ by an edge of type 3, then $P_m P_{m-1}\pi(\Pi_l) = \tau_{j,m-1}\pi(\Pi_{\hat{l}}) = 0$ or $P_m P_{m-1}\pi(\Pi_l) = \tau_{j,m-1}\pi(\Pi_{\hat{l}})P_0 P_1 P_0 = 0$. Thus

$$P_m P_{m-1}\mathcal{H} = \{0\},$$

which contradicts to $P_m P_{m-1} \neq 0$. □

Denote

$$\hat{\mathcal{P}} = \begin{cases} \mathcal{S} \cup \mathcal{L}_{in}, & s_1 = 4, \quad s_2 = 4; \\ \mathcal{S} \cup \mathcal{L}_0 \cup \mathcal{L}_{in}, & s_1 = 5, \quad s_2 = 4; \\ \mathcal{S} \cup \mathcal{L}_{in} \cup \mathcal{L}_m, & s_1 = 4, \quad s_2 = 5; \\ \mathcal{S} \cup \mathcal{L}_0 \cup \mathcal{L}_{in} \cup \mathcal{L}_m, & s_1 = 5, \quad s_2 = 5. \end{cases}$$

Proposition 7. *Let $P_0 D^* P_m D P_0 \neq 0$ then the linear span of the set of vectors, $\{\pi(\Pi_l)x\}_{l\in\hat{\mathcal{P}}}$, is invariant with respect to the representation π.*

Proof. Let $\mathcal{P} = \mathcal{S} \cup \mathcal{L}_0 \cup \mathcal{L}_{in} \cup \mathcal{L}_m$ and $\mathcal{H}'$ be the linear span of the set of vectors $\{\pi(\Pi_l)x\}_{l\in\mathcal{P}}$, then

$$\mathcal{H}' = \sum_{i\in V} \mathcal{H}'_i,$$

where $\mathcal{H}'_i$ is the linear span of the pair of vectors $\{\pi(\Pi_l)x\}$, where

$$\begin{aligned} &l \in \{\psi_0(i), \psi_0(i) \cup l_0\}, \quad i \in V_0; \\ &l \in \{\psi_0(i), \psi_m(i) \cup \hat{l}\}, \quad i \in V_{in}; \\ &l \in \{\psi_0(i), \psi_m(i) \cup l_m \cup \hat{l}\}, \quad i \in V_m. \end{aligned}$$

For any path $l \in \mathcal{N}_0$ there exists a path $l' \in \mathcal{P}$ and numbers $k \in \mathbb{N}\cup\{0\}$, $\sigma \in \{0,1\}$ such that

$$\Pi_l = \Pi_{l'}\Pi^k_{\hat{l}^*\cup\hat{l}}\Pi^\sigma_{l_0}.$$

Let us show that $\mathcal{H}'_0$ is invariant with respect to the set of operators $\pi(\Pi^k_{\hat{l}^*\cup\hat{l}}\Pi^\sigma_{l_0})$.

First of all, the vector $P_1\pi(\Pi_{l_0})x$ belongs to $\mathcal{H}'_1$, indeed,

$$\begin{aligned} P_1 P_0 P_1 P_0 x &= \xi^{-1} P_1 P_0 P_1 P_0 P_1 D^* P_m D P_0 x \\ &= \xi^{-1} g_{01}(P_1 P_0) P_1 D^* P_m D P_0 x \\ &= \xi^{-1}(\lambda_1 P_1 P_0 D^* P_m D P_0 x + \lambda_2 D^* P_m D P_0 x) \\ &= \lambda_1 P_1 P_0 x + \xi^{-1}\lambda_2 D^* P_m D P_0 x \\ &= \lambda_1 \pi(\Pi_{\psi_0(1)})x + \lambda'_2 \pi(\Pi_{\psi_m(1)\cup\hat{l}})x. \end{aligned}$$

It follows that $\pi(\Pi_{\hat{l}^*\cup\hat{l}})^k(P_0P_1P_0)^\sigma x$ belongs to $\mathcal{H}_0'$,

$$\begin{aligned}
\pi(\Pi_{\hat{l}^*\cup\hat{l}})x &= \xi\tau_{1,2}\dots\tau_{m-2,m-1}x,\\
\pi(\Pi_{\hat{l}^*\cup\hat{l}})P_0P_1P_0x &= \lambda P_0D^*P_mDP_1P_0P_1P_0x\\
&= \lambda_1''P_0D^*P_mDP_1P_0x + \lambda_2''P_0D^*P_mDD^*P_mDP_0x\\
&= \lambda_1'''x + \lambda_2''P_0D^*P_{m-1}P_mP_{m-1}P_mP_{m-1}DP_0x\\
&= \lambda_1'''x + \lambda_2'''P_0D^*P_mDP_0x + \lambda_3P_0D^*P_{m-1}DP_0x\\
&= \lambda_1'''x + \lambda_2'''\xi x + \lambda_3P_0P_1P_0x\\
&= \lambda_1''''\pi(\Pi_{\psi_0(0)})x + \lambda_3\pi(\Pi_{\psi_0(0)\cup l_0})x.
\end{aligned}$$

Now let us show that $\pi(\Pi_{l'})x \in \mathcal{H}'$ for any path $l' \in \mathcal{P}$. The vector $\pi(\Pi_{\hat{l}}\Pi_{l_0})x$ belongs to $\mathcal{H}_m'$, indeed,

$$\begin{aligned}
\pi(\Pi_{\hat{l}})P_0P_1P_0x &= \pi(\Pi_{\psi_1(m)})P_1P_0P_1P_0x\\
&= \lambda_1\pi(\Pi_{\psi_1(m)}\Pi_{\psi_0(1)})x + \lambda_2'\pi(\Pi_{\psi_1(m)}\Pi_{\psi_m(1)\cup\hat{l}})x\\
&= \lambda_1\pi(\Pi_{\hat{l}})x + \lambda_2''''\pi(\Pi_{l_m\cup\hat{l}})x.
\end{aligned}$$

Furthermore, it is evident that $\mathcal{H}_0'$ is invariant with respect to $\pi(\Pi_{l_0})$ and $\mathcal{H}_m'$ is invariant with respect to $\pi(\Pi_{l_m})$, thus, for any path $l' \in \mathcal{P}$, the vector $\pi(\Pi_{l'})x$ belongs to $\mathcal{H}_i'$ if $i \in V_0$ and $i \in V_m$.

For any vertex $i \in V_{in}$,

$$\begin{aligned}
\pi(\Pi_{\psi_0(i)})P_0P_1P_0x &= \pi(\Pi_{\psi_1(i)})P_1P_0P_1P_0x\\
&= \lambda_1\pi(\Pi_{\psi_1(i)}\Pi_{\psi_0(1)})x + \lambda_2'\pi(\Pi_{\psi_1(i)}\Pi_{\psi_m(1)\cup\hat{l}})x\\
&= \lambda_1\pi(\Pi_{\psi_0(i)})x + \lambda_2'''''\pi(\Pi_{\psi_m(i)\cup\hat{l}})x,\\
\pi(\Pi_{\psi_m(i)}\Pi_{\hat{l}})P_0P_1P_0x &= \lambda_1\pi(\Pi_{\psi_m(i)})\pi(\Pi_{\hat{l}})x + \lambda_2''''''\pi(\Pi_{\psi_m(i)})\pi(\Pi_{l_m\cup\hat{l}})x\\
&= \lambda_1'\pi(\Pi_{\psi_m(i)\cup\hat{l}})x + \lambda_5''\pi(\Pi_{\psi_0(i)})x,
\end{aligned}$$

as far as

$$\begin{aligned}
\pi(\Pi_{\psi_m(i)})\pi(\Pi_{l_m\cup\hat{l}})x &= \pi(\Pi_{\psi_{m-1}(i)})P_{m-1}P_mP_{m-1}P_mP_{m-1}\pi(\Pi_{\psi_0(m-1)})x\\
&= \lambda_4\pi(\Pi_{\psi_{m-1}(i)})P_{m-1}P_mP_{m-1}\pi(\Pi_{\psi_0(m-1)})x\\
&\quad+ \lambda_5\pi(\Pi_{\psi_{m-1}(i)})P_{m-1}\pi(\Pi_{\psi_0(m-1)})x\\
&= \lambda_4\pi(\Pi_{\psi_m(i)\cup\hat{l}})x + \lambda_5'\pi(\Pi_{\psi_0(i)})x,
\end{aligned}$$

so the vectors $\pi(\Pi_{\psi_0(i)}\Pi_{l_0})x$ and $\pi(\Pi_{\psi_m(i)\cup\hat{l}}\Pi_{l_0})x$ belong to $\mathcal{H}_i'$.

This means that $\mathcal{H}'$ is invariant with respect to the representation π.

Let us show that, in the case of $s_1 = 4$, $\dim\mathcal{H}_i' = 1$ for any vertex $i \in V_0$. Indeed,

$$\begin{aligned}
P_0P_1P_0x &= \xi^{-1}P_0P_1P_0P_1D^*P_mDP_0x\\
&= \xi^{-1}\tau_{0,1}P_0P_1D^*P_mDP_0x = \tau_{0,1}x.
\end{aligned}$$

If $s_2 = 4$, $\dim\mathcal{H}_i' = 1$ for any $i \in V_m$ as well, indeed,

$$\begin{aligned}
P_mP_{m-1}P_m\pi(\Pi_{\hat{l}})x &= P_mP_{m-1}P_mP_{m-1}\pi(\Pi_{\psi_0(m-1)})x\\
&= \tau_{m-1,m}P_mP_{m-1}\pi(\Pi_{\psi_0(m-1)})x\\
&= \tau_{m-1,m}\pi(\Pi_{\hat{l}})x.
\end{aligned}$$

Thus we have proved that the linear span of the set of vectors $\{\pi(\Pi_l)x\}_{l\in\hat{\mathcal{P}}}$ coincides with the Hilbert space $\mathcal{H}'$. $\square$

The following theorems are corollaries of the previous proposition.

Theorem 8. *For any irreducible $*$-representation*

$$\pi : TL_{\mathbb{G}_{s_1,s_2},f,\perp} \to \mathcal{B}(\mathcal{H})$$

the following inequality holds:

$$\dim \mathcal{H} \leqslant 2|V|.$$

Moreover,

$$\begin{aligned} \dim \mathcal{H} &\leqslant |V_0| + 2|V_{in}| + |V_m| && \text{if } s_1 = 4 \text{ and } s_2 = 4; \\ \dim \mathcal{H} &\leqslant |V_0| + 2|V_{in}| + 2|V_m| && \text{if } s_1 = 4 \text{ and } s_2 = 5; \\ \dim \mathcal{H} &\leqslant 2|V_0| + 2|V_{in}| + |V_m| && \text{if } s_1 = 5 \text{ and } s_2 = 4. \end{aligned}$$

Theorem 9. *For any non trivial irreducible $*$-representation*

$$\pi : TL_{\mathbb{G}_{s_1,s_2},f,\perp} \to \mathcal{B}(\mathcal{H})$$

the following inequality holds:

$$\operatorname{rank} P_i \leqslant 2.$$

Moreover, $\operatorname{rank} P_i = 1$ *in the cases of* $i \in V_0$, $s_1 = 4$, *and* $i \in V_m$, $s_2 = 4$.

Proof. For any $l \in \hat{\mathcal{P}}$ the vector $P_i\pi(\Pi_l)x$ belongs to H_i' which was defined in the proof of Proposition 7. So, $\operatorname{rank} P_i = \dim H_i'$ □

2. $*$-Representations of a pair of projections connected with an edge of type 3, 4, or 5

The results here are not new, but they will be needed when we describe irreducible proper $*$-representations of the algebras $TL_{\mathbb{G}_{5,s},f,\perp}$.

Let P_0, P_1 be nonzero projections on a finite dimensional Hilbert space $\mathcal{H}$ related with one of the next types (8), (9) or (12). As earlier $\mathcal{H}_i = \operatorname{Im} P_i$, $i = 0, 1$. We describe $\mathcal{H}_0$ and $\mathcal{H}_1$ in each case.

1. Let $f(x) = x - \tau$, where $\tau \in (0;1)$. And relations $f(P_0P_1)P_0 = 0$, $f(P_1P_0)P_1 = 0$ hold, i.e.,

$$P_0P_1P_0 = \tau P_0, \quad P_1P_0P_1 = \tau P_1. \tag{8}$$

This means that P_0, P_1 correspond to vertices joined with an edge of type 3.

Consider the operators

$$A_{i,j} = \frac{P_iP_j}{\sqrt{\tau}}\lceil_{\mathcal{H}_j} : \mathcal{H}_j \to \mathcal{H}_i, \quad i, j = 0, 1, \quad i \neq j.$$

Proposition 10. *The subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are isomorphic, and the operators $A_{0,1}, A_{1,0}$ are unitary.*

Proof. Relations (8) imply that $\dim \mathcal{H}_0 = \dim \mathcal{H}_1$. Then

$$A_{0,1}^* = \left(\frac{P_0P_1}{\sqrt{\tau}}\lceil_{\mathcal{H}_1}\right)^* = \frac{P_1P_0}{\sqrt{\tau}}\lceil_{\mathcal{H}_0} = A_{1,0}.$$

We have

$$A_{0,1}^*A_{0,1} = A_{1,0}A_{0,1} = Id_{\mathcal{H}_1}, \quad A_{0,1}A_{0,1}^* = A_{0,1}A_{1,0} = Id_{\mathcal{H}_0},$$

so,

$$A_{0,1}^* = A_{0,1}^{-1}, \quad A_{1,0}^* = A_{1,0}^{-1}.$$

□

2. Let $f(x) = (x-\tau)x$, where $\tau \in (0;1]$. And the relations $f(P_0P_1) = 0$, $f(P_1P_0) = 0$ hold, i.e.,

$$(P_0P_1P_0 - \tau P_0)P_0P_1 = 0, \quad (P_1P_0P_1 - \tau P_1)P_1P_0 = 0. \tag{9}$$

This means that P_0, P_1 correspond to the vertices joined with an edge of type 4.

Proposition 11. *The subspaces $\mathcal{H}_0, \mathcal{H}_1$ can be decomposed as*

$$\mathcal{H}_0 = \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1} \quad \textit{and} \quad \mathcal{H}_1 = \mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1} \tag{10}$$

such that

$$A_{0,1} = \frac{P_0P_1}{\sqrt{\tau}}\lceil_{\mathcal{H}_{1,0}} : \mathcal{H}_{1,0} \to \mathcal{H}_{0,0} \tag{11}$$

is correctly defined and is unitary, where

$$\mathcal{H}_{0,1} = \ker(P_1P_0) \cap \mathcal{H}_0,$$
$$\mathcal{H}_{1,1} = \ker(P_0P_1) \cap \mathcal{H}_1.$$

Proof. We define

$$\mathcal{H}_{0,0} = \ker(P_0P_1P_0 - \tau P_0) \cap \mathcal{H}_0,$$
$$\mathcal{H}_{1,0} = \ker(P_1P_0P_1 - \tau P_1) \cap \mathcal{H}_1.$$

Let us show that $\mathcal{H}_{0,0} \perp \mathcal{H}_{0,1}$. Indeed, let $x \in \mathcal{H}_{0,0}$ and $y \in \mathcal{H}_{0,1}$. Then

$$\langle x, y \rangle = \frac{1}{\tau}\langle P_0P_1P_0x, y \rangle = \frac{1}{\tau}\langle x, P_0P_1P_0y \rangle = 0.$$

In the same way we have $\mathcal{H}_{1,0} \perp \mathcal{H}_{1,1}$.

It is clear that $\mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1} \neq \{0\}$ and $\mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1} \neq \{0\}$. We prove that $\mathcal{H}_0 = \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1}$. If not, then there exists $z \in \mathcal{H}_0$, $z \neq 0$ and $z \perp \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1}$.

Then $(P_0P_1P_0 - \tau P_0)z \neq 0$. Put $z_1 = (P_0P_1P_0 - \tau P_0)z$. Then $z_1 \perp \mathcal{H}_{0,1}$, indeed, for any $x \in \mathcal{H}_{0,1}$,

$$\langle z_1, x \rangle = \langle z, (P_0P_1P_0 - \tau P_0)x \rangle = -\tau\langle z, x \rangle = 0.$$

Then $P_1P_0z_1 \neq 0$, so

$$P_1P_0z_1 = P_1P_0(P_0P_1P_0 - \tau P_0)z = (P_1P_0P_1P_0 - \tau P_1P_0)z \neq 0,$$

which is a contradiction. So, $\mathcal{H}_0 = \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1}$. Similarly $\mathcal{H}_1 = \mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1}$.

We prove that $P_0P_1(\mathcal{H}_{1,0}) \subset \mathcal{H}_{0,0}$. Indeed, for any $y \in \mathcal{H}_{1,0}$, we have

$$(P_0P_1P_0 - \tau P_0)P_0P_1y = 0.$$

So, the operator $A_{0,1}$ is correctly defined. Let us prove that it is unitary. we have,

$$A_{0,1}^* = \frac{P_1P_0}{\sqrt{\tau}}\lceil_{\mathcal{H}_{0,0}} : \mathcal{H}_{0,0} \to \mathcal{H}_{1,0} = A_{1,0},$$
$$A_{0,1}A_{0,1}^* = \frac{P_0P_1}{\sqrt{\tau}}\frac{P_1P_0}{\sqrt{\tau}}\lceil_{\mathcal{H}_{0,0}} = P_0\lceil_{\mathcal{H}_{0,0}} = Id_{\mathcal{H}_{0,0}},$$
$$A_{0,1}^*A_{0,1} = \frac{P_1P_0}{\sqrt{\tau}}\frac{P_0P_1}{\sqrt{\tau}}\lceil_{\mathcal{H}_{1,0}} = P_1\lceil_{\mathcal{H}_{1,0}} = Id_{\mathcal{H}_{1,0}}.$$

□

3. Let $f(x) = (x-\tau^0)(x-\tau^1)$, where $\tau^0, \tau^1 \in (0;1]$ and $\tau^0 \neq \tau^1$. And the relations $f(P_0P_1)P_0 = 0$, $f(P_1P_0)P_1 = 0$ hold, i.e.,

$$(P_0P_1P_0 - \tau^0P_0)(P_0P_1P_0 - \tau^1P_0) = 0, \quad (P_1P_0P_1 - \tau^0P_1)(P_1P_0P_1 - \tau^1P_1) = 0. \tag{12}$$

This means that the projections correspond to the vertices joined with an edge of type 5.

Proposition 12. *There are the decompositions*

$$\mathcal{H}_0 = \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1} \quad \text{and} \quad \mathcal{H}_1 = \mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1} \tag{13}$$

such that the operators

$$A^0_{0,1} = \frac{P_0P_1}{\sqrt{\tau^0}}\lceil_{\mathcal{H}_{1,0}} : \mathcal{H}_{1,0} \to \mathcal{H}_{0,0},$$
$$A^1_{0,1} = \frac{P_0P_1}{\sqrt{\tau^1}}\lceil_{\mathcal{H}_{1,1}} : \mathcal{H}_{1,1} \to \mathcal{H}_{0,1}$$

are correctly defined and unitary.

Proof. We define

$$\mathcal{H}_{0,i} = \ker(P_0P_1P_0 - \tau^iP_0) \cap \mathcal{H}_0,$$
$$\mathcal{H}_{1,i} = \ker(P_1P_0P_1 - \tau^iP_1) \cap \mathcal{H}_1, \quad i = 0, 1.$$

Obviously, $\mathcal{H}_{0,0} \perp \mathcal{H}_{0,1}$. Indeed, let $x \in \mathcal{H}_{0,0}$ and $y \in \mathcal{H}_{0,1}$. Then

$$\langle x, y\rangle = \frac{1}{\tau^0}\langle P_0P_1P_0x, y\rangle = \frac{1}{\tau^0}\langle x, P_0P_1P_0y\rangle = \frac{\tau^1}{\tau^0}\langle x, y\rangle,$$

so $\langle x, y\rangle = 0$. Similarly, $\mathcal{H}_{1,0} \perp \mathcal{H}_{1,1}$.

Notice that $\mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1} \neq \{0\}$. We prove $\mathcal{H}_0 = \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1}$. Otherwise there exists $z \in \mathcal{H}_0, z \neq 0$ and $z \perp \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1}$. Put $z_1 = (P_0P_1P_0 - \tau^1P_0)z$, then $z_1 \neq 0$ and $z_1 \perp \mathcal{H}_{0,1}$. Then $z_1 \perp \mathcal{H}_{0,0}$. Indeed, for any $x \in \mathcal{H}_{0,0}$ we have

$$\langle z_1, x\rangle = \langle z, (P_0P_1P_0 - \tau^1P_0)x\rangle = (\tau^0 - \tau^1)\langle z, x\rangle = 0.$$

We obtain $z_1 \perp \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1}$, which implies that $(P_0P_1P_0 - \tau^0P_0)z_1 \neq 0$ and $(P_0P_1P_0 - \tau^0P_0)(P_0P_1P_0 - \tau^1P_0)z \neq 0$, which is a contradiction. So, $\mathcal{H}_0 = \mathcal{H}_{0,0} \oplus \mathcal{H}_{0,1}$. In a similar way, $\mathcal{H}_1 = \mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1}$.

Let us prove $P_0P_1(\mathcal{H}_{1,i}) \subset \mathcal{H}_{0,i}$ where $i = 0, 1$. For $x \in \mathcal{H}_{1,i}$, we define $y = P_0P_1x$. Then

$$(P_0P_1P_0 - \tau^iP_0)y = (P_0P_1P_0 - \tau^iP_0)P_0P_1x = P_0(P_1P_0P_1 - \tau^iP_1)x = 0.$$

Similarly, $P_1P_0(\mathcal{H}_{0,i}) \subset \mathcal{H}_{1,i}$, $i = 0, 1$.

We have shown that $A^i_{0,1}, i = 0, 1$ was defined correctly. Obviously,

$$(A^i_{0,1})^* = A^i_{1,0} = \frac{P_1P_0}{\sqrt{\tau^i}}\lceil_{\mathcal{H}_{0,i}} : \mathcal{H}_{0,i} \to \mathcal{H}_{1,i}, \quad i = 0, 1.$$

For any $x \in \mathcal{H}_{0,i}$, we have

$$A^i_{0,1}(A^i_{0,1})^*x = \frac{P_0P_1}{\sqrt{\tau^i}}\frac{P_1P_0}{\sqrt{\tau^i}}x = \frac{\tau^i}{\tau^i}x = x.$$

And for any $y \in \mathcal{H}_{1,i}$, we obtain

$$(A^i_{0,1})^*A^i_{0,1}y = \frac{P_1P_0}{\sqrt{\tau^i}}\frac{P_0P_1}{\sqrt{\tau^i}}y = \frac{\tau^i}{\tau^i}y = y.$$

Which implies that $A^i_{0,1}, i = 0, 1$ are unitary. □

3. A description of proper $*$-representations of the algebras $TL_{\mathbb{G}_{5,s},f,\perp}$ where $s=4$ or $s=5$.

As earlier, $\mathbb{G}_{5,s}$ is a tree where the edge $\gamma_{0,1}$ has type 5 and $\gamma_{m-1,m}$ has type s, $s\in\{4,5\}$.

We call a $*$-representation π of the algebra $TL_{\mathbb{G}_{5,s},f,\perp}$ *proper*, if any of next relations does not hold:

$$\begin{aligned}
&P_0P_1P_0=\tau_{0,1}^0P_0, && P_1P_0P_1=\tau_{0,1}^0P_1, &&\\
&P_0P_1P_0=\tau_{0,1}^1P_0, && P_1P_0P_1=\tau_{0,1}^1P_1, &&\\
&P_mP_{m-1}P_m=\tau_{m-1,m}^0P_m, && P_{m-1}P_mP_{m-1}=\tau_{m-1,m}^0P_{m-1}, &&\\
&P_mP_{m-1}=0, && P_{m-1}P_m=0, && \text{if } s=4,\\
&P_mP_{m-1}P_m=\tau_{m-1,m}^1P_m, && P_{m-1}P_mP_{m-1}=\tau_{m-1,m}^1P_{m-1}, && \text{if } s=5.
\end{aligned}$$

If any of these relations holds then the irreducible $*$-representation is a lifting of some $*$-representation of the corresponding quotient algebra, which is finite dimensional (for representations of finite dimensional algebras $TL_{\mathbb{G},f,\perp}$, see [10]). For example, relations between p_0 and p_1 imply that, if $p_0p_1p_0=\tau_{0,1}^ip_0$ holds, then $p_1p_0p_1=\tau_{0,1}^ip_1$, $i=0,1$ holds too (and vice versa). And quotient algebra $TL_{\mathbb{G}_{5,s},f,\perp}/\langle p_0p_1p_0-\tau_{0,1}^ip_0,p_1p_0p_1-\tau_{0,1}^ip_1\rangle$ is finite dimensional.

Note that in the case $s=4$, if for an irreducible $*$-representation $P_{m-1}P_m\neq 0$ holds, then $P_mP_{m-1}P_m=\tau_{m-1,m}^0P_m$ is true (can be proved using 7).

We consider a linear space L generated by $|\hat{\mathcal{P}}|$ vectors $\hat{x}_i,\hat{y}_j$, where $i\in V$, $j\in\tilde{V}$, where $\tilde{V}$ is the following subset of the set of vertices:

$$\tilde{V}=\begin{cases} V_0\cup V_{in}, & s=4,\\ V, & s=5.\end{cases}$$

For any $\nu\in(0;1)$, we consider, on L, a sesquilinear form $B^{\nu}_{\mathbb{G}_{5,s},f}$ defined on the vectors of basis in the following way:

$$B^{\nu}_{\mathbb{G}_{5,s},f}(\hat{x}_i,\hat{x}_i)=B^{\nu}_{\mathbb{G}_{5,s},f}(\hat{y}_j,\hat{y}_j)=1,\quad i\in V,\quad j\in\tilde{V};$$

$$B^{\nu}_{\mathbb{G}_{5,s},f}(\hat{x}_i,\hat{x}_j)=B^{\nu}_{\mathbb{G}_{5,s},f}(\hat{x}_j,\hat{x}_i)=\begin{cases}\sqrt{\tau_{i,j}}, & \gamma_{i,j}\in R_3,\\ \sqrt{\tau_{0,1}^0}, & \gamma_{i,j}=\gamma_{0,1},\\ \sqrt{\nu\tau_{m-1,m}^0}, & \gamma_{i,j}=\gamma_{m-1,m};\end{cases}$$

$$B^{\nu}_{\mathbb{G}_{5,s},f}(\hat{y}_i,\hat{y}_j)=B^{\nu}_{\mathbb{G}_{5,s},f}(\hat{y}_j,\hat{y}_i)=\begin{cases}\sqrt{\tau_{i,j}}, & \gamma_{i,j}\in R_3,\\ \sqrt{\tau_{0,1}^1}, & \gamma_{i,j}=\gamma_{0,1},\\ -\sqrt{\nu\tau_{m-1,m}^1}, & \gamma_{i,j}=\gamma_{m-1,m},\quad s=5;\end{cases}$$

$$B^{\nu}_{\mathbb{G}_{5,s},f}(\hat{x}_i,\hat{y}_j)=B^{\nu}_{\mathbb{G}_{5,s},f}(\hat{y}_j,\hat{x}_i)=\begin{cases}\sqrt{(1-\nu)\tau_{m-1,m}^1}, & (i,j)=(m-1,m),\quad s=5,\\ \sqrt{(1-\nu)\tau_{m-1,m}^0}, & (i,j)=(m,m-1),\\ 0, & \text{otherwise.}\end{cases}$$

On the other pairs of basis vectors, $B^{\nu}_{\mathbb{G}_{5,s},f}$ equals to 0.

Let $\Sigma_{\mathbb{G}_{5,s},f}$ be the set of those ν for which the form $B^{\nu}_{\mathbb{G}_{5,s},f}$ is non-negative definite.

For $\nu \in \Sigma_{\mathbb{G}_{5,s,f}}$, denote by $\mathcal{H}_\nu$ the Hilbert space obtained by equipping the linear space $L/L_{0,\nu}$, where $L_{0,\nu}$ is the set of those $\hat{x} \in L$ that $B^{\nu}_{\mathbb{G}_{5,s,f}}(\hat{x},\hat{x}) = 0$, with the scalar product $\langle \hat{x} + L_{0,\nu}, \hat{y} + L_{0,\nu} \rangle = B^{\nu}_{\mathbb{G}_{5,s,f}}(\hat{x},\hat{y})$.

Denote by $x_\nu = x = \hat{x} + L_{0,\nu}$. Since, by the definition of $B^{\nu}_{\mathbb{G}_{5,s,f}}$, any $\hat{x}_i, \hat{y}_j$, $i \in V, j \in \tilde{V}$, are not in $L_{0,\nu}$, the corresponding $x_i = \hat{x}_i + L_{0,\nu}, y_j = \hat{y}_j + L_{0,\nu}$ generate the space $\mathcal{H}_\nu$. But in the case when the form is not positive definite, the set x_i, y_j, where $i \in V, j \in \tilde{V}$, is not the set of linearly independent vectors.

For an arbitrary vertex $i \in V$ define an operator $P_{i,\nu} = P_i$ to be the orthogonal projection onto the linear span of the pair of vectors x_i, y_i, if $i \in \tilde{V}$, and for an arbitrary vertex $i \in V\backslash\tilde{V}$ the operator $P_{i,\nu} = P_i$ is defined to be an orthogonal projection onto the linear span of vector x_i.

Proposition 13. *For any $x \in \mathcal{H}_\nu$, we have the formula*

$$P_i x = \begin{cases} \langle x, x_i \rangle x_i + \langle x, y_i \rangle y_i, & i \in \tilde{V}, \\ \langle x, x_i \rangle x_i, & i \in V\backslash\tilde{V}. \end{cases}$$

Proof. It sufficient to notice that $\langle P_i x, x_i \rangle = \langle x, x_i \rangle$ for any $i \in V$, and $\langle P_i x, y_i \rangle = \langle x, y_i \rangle$ for any $i \in \tilde{V}$. □

We denote $\mathcal{H}_i = \operatorname{Im} P_i$, and fix the basis $\{x_i, y_i\}$, if $i \in \tilde{V}$, and $\{x_i\}$, if $i \in V\backslash\tilde{V}$. The operator $X_{i,j} : \mathcal{H}_j \to \mathcal{H}_i$, $i \neq j$ is defined to be the restriction of $P_i P_j$ to $\mathcal{H}_j$. By simple calculations, we have that, in the fixed basis,

(1) $X_{i,j} = 0$, if the vertices i and j are not connected with an edge;
(2) $X_{i,j} = \left(\sqrt{\tau_{i,j}}\right)$, if $s = 4$ and $i, j \in V_m$;
(3) $X_{i,j} = \begin{pmatrix} \sqrt{\tau_{i,j}} & 0 \\ 0 & \sqrt{\tau_{i,j}} \end{pmatrix}$, $i, j \in \tilde{V}, \gamma_{ij} \in R_3$;
(4) $X_{0,1} = \begin{pmatrix} \sqrt{\tau^0_{0,1}} & 0 \\ 0 & \sqrt{\tau^1_{0,1}} \end{pmatrix}$;
(5) $X_{m-1,m} = \begin{pmatrix} \sqrt{\nu \tau^0_{m-1,m}} \\ \sqrt{(1-\nu)\tau^0_{m-1,m}} \end{pmatrix} = \sqrt{\tau^0_{m-1,m}} \begin{pmatrix} \sqrt{\nu} \\ \sqrt{1-\nu} \end{pmatrix}$, if $s = 4$;
(6) $X_{m-1,m} = \begin{pmatrix} \sqrt{\nu \tau^0_{m-1,m}} & \sqrt{(1-\nu)\tau^1_{m-1,m}} \\ \sqrt{(1-\nu)\tau^0_{m-1,m}} & -\sqrt{\nu \tau^1_{m-1,m}} \end{pmatrix}$, if $s = 5$.

Notice that, if we denote

$$\tilde{D} = \begin{pmatrix} \tau^0_{m-1,m} & 0 \\ 0 & \tau^1_{m-1,m} \end{pmatrix}, \quad U = \begin{pmatrix} \sqrt{\nu} & \sqrt{1-\nu} \\ \sqrt{1-\nu} & -\sqrt{\nu} \end{pmatrix},$$

then $X_{m-1,m} = U\sqrt{\tilde{D}}$, for the case of $s = 5$. It is clear that U is a unitary, self-adjoint matrix and $X^*_{i,j} = X_{j,i}$.

Lemma 14. *For each $\nu \in \Sigma_{\mathbb{G}_{5,s,f}}$, the mapping*

$$\pi_\nu : TL_{\mathbb{G}_{5,s,f,\perp}} \to \mathcal{B}(\mathcal{H}_\nu) : p_i \mapsto P_i$$

is an irreducible proper $*$-representation.

Proof. Let us show that π_ν is a $*$-representation.

It is clear that $P_i^2 x = P_i x$, since $\langle x_i, y_i \rangle = 0$.

Any relation of the form $f(P_i P_j) = 0$, $f(0) = 0$, or $f(P_j P_i) P_j = 0$ is sufficient to be verified on the vectors of $\mathcal{H}_j$, since on the vectors of $\mathcal{H}_j^\perp$ they are clearly satisfied. Let us fix some vector $x \in \mathcal{H}_j$. We denote by $\alpha = \langle x, x_j \rangle$, $j \in V$, and $\beta = \langle x, y_j \rangle$, $j \in \tilde{V}$, the coordinates of x in the basis of $\mathcal{H}_j$.

If the vertices i and j are not connected with an edge, then

$$P_iP_jx = X_{i,j}\alpha = 0, \quad s = 4, \quad j \in V_m;$$
$$P_iP_jx = X_{i,j}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0, \quad j \in \tilde{V}.$$

Let now the vertices i and j be connected with an edge of type 3, then the following relations hold:

$$(P_jP_iP_j - \tau_{ij}P_j)x = (X_{j,i}X_{i,j} - \tau_{ij})\alpha = 0, \quad s = 4, \quad j \in V_m;$$
$$(P_jP_iP_j - \tau_{ij}P_j)x = (X_{j,i}X_{i,j} - \tau_{ij}I_j)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0, \quad j \in \tilde{V}.$$

Let $i = 1$, $j = 0$, then, for the projections P_0, P_1, the following is true:

$$\begin{aligned}(P_0P_1P_0 - &\tau^0_{0,1}P_0)(P_0P_1P_0 - \tau^1_{0,1}P_0)x\\ &= (X_{0,1}X_{1,0} - \tau^0_{0,1}I_0)(X_{0,1}X_{1,0} - \tau^1_{0,1}I_0)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ &= \begin{pmatrix}0 & 0\\ 0 & \tau^1_{0,1} - \tau^0_{0,1}\end{pmatrix}\begin{pmatrix}\tau^0_{0,1} - \tau^1_{0,1} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0.\end{aligned}$$

Notice, that

$$P_0P_1P_0 - \tau^0_{0,1}P_0 \neq 0, \quad P_0P_1P_0 - \tau^1_{0,1}P_0 \neq 0.$$

The case $i = 0$, $j = 1$ is similar.

Let $s = 4$ and $i = m-1$, $j = m$. Then

$$\begin{aligned}(P_mP_{m-1}P_m - \tau^0_{m-1,m}P_m)x &= (X_{m,m-1}X_{m-1,m} - \tau^0_{m-1,m})\alpha\\ &= (\tau^0_{m-1,m}\begin{pmatrix}\sqrt{\nu} & \sqrt{1-\nu}\end{pmatrix}\begin{pmatrix}\sqrt{\nu}\\ \sqrt{1-\nu}\end{pmatrix} - \tau^0_{m-1,m})\alpha = 0.\end{aligned}$$

Which implies that

$$P_mP_{m-1}P_mP_{m-1} - \tau^0_{m-1,m}P_mP_{m-1} = 0,$$
$$P_{m-1}P_mP_{m-1}P_m - \tau^0_{m-1,m}P_{m-1}P_m = 0.$$

But

$$\begin{aligned}(P_{m-1}P_mP_{m-1} - \tau^0_{m-1,m}P_{m-1})x_{m-1} &= (X_{m-1,m}X_{m,m-1} - \tau^0_{m-1,m}I_{m-1})\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \tau^0_{m-1,m}\left(\begin{pmatrix}\sqrt{\nu}\\ \sqrt{1-\nu}\end{pmatrix}\begin{pmatrix}\sqrt{\nu} & \sqrt{1-\nu}\end{pmatrix} - I_{m-1}\right)\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \tau^0_{m-1,m}\begin{pmatrix}\nu - 1\\ \sqrt{\nu(1-\nu)}\end{pmatrix} \neq 0.\end{aligned}$$

Let $s = 5$, $i = m$, $j = m-1$, then

$$\begin{aligned}(P_{m-1}&P_mP_{m-1} - \tau^0_{m-1,m}P_{m-1})(P_{m-1}P_mP_{m-1} - \tau^1_{m-1,m}P_{m-1})x\\ &= (X_{m-1,m}X_{m,m-1} - \tau^0_{m-1,m}I_{m-1})(X_{m-1,m}X_{m,m-1} - \tau^1_{m-1,m}I_{m-1})\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ &= (U\tilde{D}U - \tau^0_{m-1,m}I_{m-1})(U\tilde{D}U - \tau^1_{m-1,m}I_{m-1})\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ &= U(\tilde{D} - \tau^0_{m-1,m}I_{m-1})(\tilde{D} - \tau^1_{m-1,m}I_{m-1})U\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0.\end{aligned}$$

Notice, that

$$P_{m-1}P_mP_{m-1} - \tau^0_{m-1,m}P_{m-1} \neq 0, \quad P_{m-1}P_mP_{m-1} - \tau^1_{m-1,m}P_{m-1} \neq 0.$$

In the case of $s = 5$, $i = m-1$, $j = m$, we have

$$\begin{aligned}(P_mP_{m-1}P_m &- \tau^0_{m-1,m}P_m)(P_mP_{m-1}P_m - \tau^1_{m-1,m}P_m)x \\ &= (X_{m,m-1}X_{m-1,m} - \tau^0_{m-1,m}I_m)(X_{m,m-1}X_{m-1,m} - \tau^1_{m-1,m}I_m)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\ &= (\tilde{D} - \tau^0_{m-1,m}I_m)(\tilde{D} - \tau^1_{m-1,m}I_m)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0\end{aligned}$$

and

$$P_mP_{m-1}P_m - \tau^0_{m-1,m}P_m \neq 0, \quad P_mP_{m-1}P_m - \tau^1_{m-1,m}P_m \neq 0.$$

So, we have shown that π_ν is a proper $*$-representation.

Let us prove that π_ν is irreducible. Assume that an operator $C \in B(\mathcal{H})$ commutes with all $P_i, i \in V$. We are going to show that C is a multiple of the identity. Since $CP_i = P_iC$, we have $C(\mathcal{H}_i) \subset \mathcal{H}_i, i \in V$. So that

$$P_0P_1P_0Cx_0 = CP_0P_1P_0x_0 = \sqrt{\tau^0_{0,1}}CP_0x_1 = \tau^0_{0,1}Cx_0,$$

$$P_0P_1P_0Cy_0 = CP_0P_1P_0y_0 = \sqrt{\tau^1_{0,1}}CP_0y_1 = \tau^1_{0,1}Cy_0$$

and $Cx_0 = \lambda_0x_0$ and $Cy_0 = \lambda_1y_0$ for some $\lambda_0, \lambda_1 \in \mathbb{C}$.

From

$$\sqrt{\tau^0_{0,1}}Cx_1 = CP_1x_0 = P_1Cx_0 = \lambda_0P_1x_0 = \lambda_0\sqrt{\tau^0_{0,1}}x_1,$$

$$\sqrt{\tau^1_{0,1}}Cy_1 = CP_1y_0 = P_1Cy_0 = \lambda_1P_1y_0 = \lambda_1\sqrt{\tau^1_{0,1}}y_1,$$

it follows that $Cx_1 = \lambda_0x_1$ and $Cy_1 = \lambda_1y_1$.

Let the vertices i and j be joined with an edge of type 3 and we have $Cx_i = \lambda_0x_i$ and $Cy_i = \lambda_1y_i$. Then $P_jP_ix_i = \sqrt{\tau_{i,j}}x_j$, $P_jP_iy_i = \sqrt{\tau_{i,j}}y_j$, and

$$\sqrt{\tau_{i,j}}Cx_j = CP_jx_i = P_jCx_i = \lambda_0P_jx_i = \lambda_0\sqrt{\tau_{i,j}}x_j, \tag{14}$$

$$\sqrt{\tau_{i,j}}Cy_j = CP_jy_i = P_jCy_i = \lambda_1P_jy_i = \lambda_1\sqrt{\tau_{i,j}}y_j, \tag{15}$$

which implies that $Cx_j = \lambda_0x_j$ and $Cy_j = \lambda_1y_j$.

In the coordinates of the subspace $\mathcal{H}_{m-1}$,

$$P_{m-1}P_mP_{m-1}x_{m-1} = X_{m-1,m}X_{m,m-1}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix},$$

it can be shown by a simple calculation that $\alpha \neq 0$ and $\beta \neq 0$ in the both cases $s = 4$ and $s = 5$. And in the coordinates of the subspace $\mathcal{H}_{m-1}$, we have

$$\begin{pmatrix}\lambda_0\alpha\\ \lambda_1\beta\end{pmatrix} = CP_{m-1}P_mP_{m-1}x_{m-1} = \lambda_0P_{m-1}P_mP_{m-1}x_{m-1} = \begin{pmatrix}\lambda_0\alpha\\ \lambda_0\beta\end{pmatrix},$$

which implies that $\lambda_0 = \lambda_1$.

We have shown that $Cx_{m-1} = \lambda_0x_{m-1}$ and $Cy_{m-1} = \lambda_0y_{m-1}$. Let us prove that $Cx_m = \lambda_0x_m$ and $Cy_m = \lambda_0y_m$. If $s = 4$, then in the coordinates of the subspace $\mathcal{H}_{m-1}$ we put

$$\tilde{x} = \begin{pmatrix}\sqrt{\nu}\\ \sqrt{1-\nu}\end{pmatrix} \in \mathcal{H}_{m-1},$$

then

$$P_m\tilde{x} = P_mP_{m-1}\tilde{x} = X_{m,m-1}\begin{pmatrix}\sqrt{\nu}\\ \sqrt{1-\nu}\end{pmatrix} = \sqrt{\tau^0_{m-1,m}}x_m.$$

In the case where $s = 5$, we consider the vectors $\tilde{x}_{m-1} = Ux_{m-1}$, $\tilde{y}_{m-1} = Uy_{m-1}$, then

$$P_m\tilde{x}_{m-1} = P_mP_{m-1}\tilde{x}_{m-1} = X_{m,m-1}U\begin{pmatrix}1\\0\end{pmatrix} = \sqrt{\tilde{D}}\begin{pmatrix}1\\0\end{pmatrix} = \sqrt{\tau^0_{m-1,m}}x_m,$$
$$P_m\tilde{y}_{m-1} = P_mP_{m-1}\tilde{y}_{m-1} = X_{m,m-1}U\begin{pmatrix}0\\1\end{pmatrix} = \sqrt{\tilde{D}}\begin{pmatrix}0\\1\end{pmatrix} = \sqrt{\tau^1_{m-1,m}}y_m.$$

As earlier in (14) and (15) we obtain that $Cx_m = \lambda_0x_m$ and $Cy_m = \lambda_0y_m$. So, we have shown that $C = \lambda_0I$, which means that π_ν is irreducible. □

Proposition 15. *∗-Representations π_{ν_1} and π_{ν_2} are unitary equivalent if and only if $\nu_1 = \nu_2$.*

Proof. Let π_{ν_1} be unitary equivalent to π_{ν_2}, i.e., there exists a unitary operator $\tilde{U} : \mathcal{H}_{\nu_1} \to \mathcal{H}_{\nu_2}$, such that $\tilde{U}\pi_{\nu_1}(a) = \pi_{\nu_2}(a)\tilde{U}$ for any $a \in TL_{\mathbb{G}_{5,s},f,\perp}$. Since $\tilde{U}\pi_{\nu_1}(p_{m-1}) = \pi_{\nu_2}(p_{m-1})\tilde{U}$, the restriction of operator $\tilde{U}$ onto $\mathcal{H}_{m-1,\nu_1}$ is correctly defined,

$$\tilde{U}_{m-1} : \mathcal{H}_{m-1,\nu_1} \to \mathcal{H}_{m-1,\nu_2}.$$

The operator $P_{m-1}P_{m-2}\dots P_1P_0P_1\dots P_{m-2}P_{m-1}$ in the coordinates of $\mathcal{H}_{m-1}$ has a diagonal form,

$$X_{m-1,m-2}\dots X_{1,0}X_{1,0}\dots X_{m-2,m-1} = \prod_{k=1}^{m-2}\tau_{k,k+1}\begin{pmatrix}\tau^0_{0,1} & 0\\ 0 & \tau^1_{0,1}\end{pmatrix}.$$

It is easy to show that in the bases of $\mathcal{H}_{m-1,\nu_1}$, $\mathcal{H}_{m-1,\nu_2}$ the unitary operator $\tilde{U}_{m-1}$ is of the form

$$\begin{pmatrix}e^{i\varphi_1} & 0\\ 0 & e^{i\varphi_2}\end{pmatrix}, \quad \varphi_1, \varphi_2 \in [0, 2\pi).$$

On the other hand, $P_{m-1,\nu_2}P_{m,\nu_2}P_{m-1,\nu_2} = \tilde{U}_{m-1}P_{m-1,\nu_1}P_{m,\nu_1}P_{m-1,\nu_1}\tilde{U}^*_{m-1}$. In a coordinate representation there are 2×2-matrices on the left- the right-hand sides, by a direct computation of the value in the first row and the first column, in the case of $s = 4$, we obtain $\nu_1\tau^0_{m-1,m} = \nu_2\tau^0_{m-1,m}$, and in the case of $s = 5$ we have the equality

$$\nu_1\tau^0_{m-1,m} + (1-\nu_1)\tau^1_{m-1,m} = \nu_2\tau^0_{m-1,m} + (1-\nu_2)\tau^1_{m-1,m},$$

i.e.,

$$\nu_1(\tau^0_{m-1,m} - \tau^1_{m-1,m}) = \nu_2(\tau^0_{m-1,m} - \tau^1_{m-1,m}).$$

Since $\tau^0_{m-1,m} \neq 0$ and $\tau^0_{m-1,m} \neq \tau^1_{m-1,m}$, in the two cases we have $\nu_1 = \nu_2$. □

4. A description of all irreducible proper ∗-representations of the algebra $TL_{\mathbb{G}_{5,s},f,\perp}$

Let π be an irreducible proper ∗-representation of the algebra $TL_{\mathbb{G}_{5,s},f,\perp}$. We will show that there exists a number $\nu \in \Sigma_{\mathbb{G}_{5,s},f}$ such that the ∗-representation π is unitarily equivalent to the ∗-representation π_ν.

Proposition 16. *Let π be an irreducible proper ∗-representation of the algebra $TL_{\mathbb{G}_{5,s},f,\perp}$ on a Hilbert space $\mathcal{H}$. Then there exist a number $\nu \in (0,1)$ and vectors $u_i, v_j \in \mathcal{H}$, $i \in V$, $j \in \tilde{V}$, such that P_i is an orthogonal projection on a subspace of $\mathcal{H}$ generated by a pair of vectors u_i and v_i if $i \in \tilde{V}$ or generated by the vector u_i if $i \in V \setminus \tilde{V}$. Moreover, the Hilbert space $\mathcal{H}$ is a linear span of the set of vectors $\{u_i, v_j\}_{i\in V, j\in\tilde{V}}$ and the Gram matrix of this set of vectors is equal to the matrix of sesquilinear form $B^\nu_{\mathbb{G}_{5,s},f}$.*

Proof. If the vertices i and j are connected with an edge of type 3 then, by Proposition 10, the operator $A_{i,j} = \frac{P_iP_j}{\sqrt{\tau_{i,j}}} \restriction_{\mathcal{H}_j} : \mathcal{H}_j \to \mathcal{H}_i$ is unitary. Consider the case of $i, j \in \tilde{V}$. Suppose $u_j, v_j \in \mathcal{H}_j$, $u_j \perp v_j$, and $\|u_j\| = \|v_j\| = 1$. Define $u_i = A_{i,j}u_j$ and $v_i = A_{i,j}v_j$. Then $u_i \perp v_i$, $\|u_i\| = \|v_i\| = 1$ and the following identities hold:

$$\langle u_i, u_j\rangle = \langle \frac{P_iP_j}{\sqrt{\tau_{i,j}}}u_j, u_j\rangle = \sqrt{\tau_{i,j}}\langle u_j, u_j\rangle = \sqrt{\tau_{i,j}},$$
$$\langle v_i, v_j\rangle = \langle \frac{P_iP_j}{\sqrt{\tau_{i,j}}}v_j, v_j\rangle = \sqrt{\tau_{i,j}}\langle v_j, v_j\rangle = \sqrt{\tau_{i,j}},$$
$$\langle u_i, v_j\rangle = \langle \frac{P_iP_j}{\sqrt{\tau_{i,j}}}u_j, v_j\rangle = \sqrt{\tau_{i,j}}\langle u_j, v_j\rangle = 0,$$
$$\langle v_i, u_j\rangle = \langle \frac{P_iP_j}{\sqrt{\tau_{i,j}}}v_j, u_j\rangle = \sqrt{\tau_{i,j}}\langle v_j, u_j\rangle = 0.$$

The same reasoning applied to the case of $i, j \in V\setminus\tilde{V}$ allows us to define $u_i = A_{i,j}u_j \in \mathcal{H}_i$, $\|u_i\| = 1$, if we have already defined $u_j \in \mathcal{H}_j$, $\|u_j\| = 1$.

So, to construct a set of vectors u_i, $i \in V$ and v_j, $j \in \tilde{V}$, it is enough to find vectors u_i, $i \in \{0, 1, m\}$ and v_j, $j \in \{0, 1, m\} \cap \tilde{V}$.

By Theorem 9 for an irreducible $*$-representation, $\operatorname{rank} P_i = 1$ for $i \in V \setminus \tilde{V}$, and $\operatorname{rank} P_i \leqslant 2$ otherwise. If $\operatorname{rank} P_0 = 1$, then $P_0P_1P_0 = \lambda P_0$ for some $\lambda \in \mathbb{C}$, on the other hand $(P_0P_1P_0 - \tau^0_{0,1}P_0)(P_0P_1P_0 - \tau^1_{0,1}P_0) = 0$, and so either $\lambda = \tau^0_{0,1}$ or $\lambda = \tau^1_{0,1}$. This means that if $\operatorname{rank} P_0 = 1$ then the representation cannot be proper. Thus $\operatorname{rank} P_0 = 2$ and so $\operatorname{rank} P_i = 2$, $i \in \tilde{V}$, as far as all projections P_i have the same rank for $i \in \tilde{V}$.

By Proposition 12, the projection P_1 has two eigenvectors $u_1 \in \mathcal{H}_{1,0}$ and $v_1 \in \mathcal{H}_{1,1}$ such that $\|u_1\| = \|v_1\| = 1$ and $u_1 \perp v_1$. Let $u_0 = A^0_{0,1}u_1$ and $v_0 = A^1_{0,1}v_1$. Then $u_0 \in \mathcal{H}_{0,0}$, $v_0 \in \mathcal{H}_{0,1}$ and $u_0 \perp v_0$, $\|u_0\| = \|v_0\| = 1$. It is evident that the following identities hold:

$$\langle u_0, u_1\rangle = \left\langle \frac{P_0P_1}{\sqrt{\tau^0_{0,1}}}u_1, u_1 \right\rangle = \sqrt{\tau^0_{0,1}}\langle u_1, u_1\rangle = \sqrt{\tau^0_{0,1}},$$
$$\langle v_0, v_1\rangle = \left\langle \frac{P_0P_1}{\sqrt{\tau^1_{0,1}}}v_1, v_1 \right\rangle = \sqrt{\tau^1_{0,1}}\langle v_1, v_1\rangle = \sqrt{\tau^1_{0,1}},$$
$$\langle u_0, v_1\rangle = \left\langle \frac{P_0P_1}{\sqrt{\tau^0_{0,1}}}u_1, v_1 \right\rangle = \sqrt{\tau^0_{0,1}}\langle u_1, v_1\rangle = 0,$$
$$\langle u_1, v_0\rangle = \left\langle u_1, \frac{P_0P_1}{\sqrt{\tau^1_{0,1}}}v_1 \right\rangle = \sqrt{\tau^1_{0,1}}\langle u_1, v_1\rangle = 0.$$

a) Let us now consider the case of $s = 5$. By Proposition 12 there exists a pair of vectors $\tilde{u}_{m-1} \in \mathcal{H}_{m-1,0}$ and $\tilde{v}_{m-1} \in \mathcal{H}_{m-1,1}$ such that $\|\tilde{u}_{m-1}\| = \|\tilde{v}_{m-1}\| = 1$ and $\tilde{u}_{m-1} \perp \tilde{v}_{m-1}$. Then, for some numbers $\nu \in [0, 1]$, $\varphi, \psi, \theta \in [0, 2\pi]$, the following identities hold:

$$\tilde{u}_{m-1} = e^{-i\varphi}\sqrt{\nu}u_{m-1} + e^{i\psi}\sqrt{1-\nu}v_{m-1},$$
$$\tilde{v}_{m-1} = (e^{-i\psi}\sqrt{1-\nu}u_{m-1} - e^{i\varphi}\sqrt{\nu}v_{m-1})e^{i\theta}.$$

Define u_m and v_m by the formulas

$$u_m = \frac{P_mP_{m-1}}{\sqrt{\tau^0_{m-1,m}}}\tilde{u}_{m-1}, \quad v_m = \frac{P_mP_{m-1}}{\sqrt{\tau^1_{m-1,m}}}\tilde{v}_{m-1},$$

then, evidently, $\|u_m\| = \|v_m\| = 1$ and $u_m \perp v_m$.

If $\nu = 1$, then the set of vectors $\{u_i\}$, $i \in V$, is invariant with respect to the $*$-representation. If $\nu = 0$, then the set of vectors $\{u_i\} \cup \{v_j\}$, $i \in V \setminus V_m$, $j \in V_m$, is invariant with respect to the $*$-representation. So, in both cases, the linear span of the set does not coincide with $\mathcal{H}$ thus the $*$-representation π cannot be irreducible and we obtained a contradiction. So $\nu \neq 0$ and $\nu \neq 1$.

Evidently, $\tilde{u}_{m-1}$ and $\tilde{v}_{m-1}$ can be chosen in such a way that

$$\tilde{u}_{m-1} = \sqrt{\nu}u_{m-1} + e^{i(\varphi+\psi)}\sqrt{1-\nu}v_{m-1},$$
$$\tilde{v}_{m-1} = \sqrt{1-\nu}u_{m-1} - e^{i(\varphi+\psi)}\sqrt{\nu}v_{m-1}.$$

Moreover, replacing v_1 with $e^{i(\varphi+\psi)}v_1$ we will get that following identities hold:

$$\tilde{u}_{m-1} = \sqrt{\nu}u_{m-1} + \sqrt{1-\nu}v_{m-1},$$
$$\tilde{v}_{m-1} = \sqrt{1-\nu}u_{m-1} - \sqrt{\nu}v_{m-1}.$$

Then

$$\langle u_{m-1}, u_m\rangle = \langle u_{m-1}, \frac{P_{m-1}P_mP_{m-1}}{\sqrt{\tau^0_{m-1,m}}}\tilde{u}_{m-1}\rangle = \sqrt{\tau^0_{m-1,m}\nu},$$
$$\langle u_{m-1}, v_m\rangle = \langle u_{m-1}, \frac{P_{m-1}P_mP_{m-1}}{\sqrt{\tau^1_{m-1,m}}}\tilde{v}_{m-1}\rangle = \sqrt{\tau^1_{m-1,m}(1-\nu)},$$
$$\langle v_{m-1}, u_m\rangle = \langle v_{m-1}, \frac{P_{m-1}P_mP_{m-1}}{\sqrt{\tau^0_{m-1,m}}}\tilde{u}_{m-1}\rangle = \sqrt{\tau^0_{m-1,m}(1-\nu)},$$
$$\langle v_{m-1}, v_m\rangle = \langle v_{m-1}, \frac{P_{m-1}P_mP_{m-1}}{\sqrt{\tau^1_{m-1,m}}}\tilde{v}_{m-1}\rangle = -\sqrt{\tau^1_{m-1,m}\nu}.$$

So we have found a set of vectors $\{u_i, v_i\}_{i\in V}$ such that their Gram matrix is equal to the matrix of the sesquilinear form $B^{\nu}_{\mathbb{G}_{5,5,f}}$ and the image of the projection P_i is a linear span of the pair of vectors $\{u_i, v_i\}$. Since the linear span of the set is invariant with respect to the $*$-representation, it coincides with $\mathcal{H}$.

b) The case of $s = 4$ is almost the same. In this case, $\operatorname{rank} P_m = 1$ and, by Proposition 11, there exist vectors $\tilde{u}_{m-1} \in \mathcal{H}_{m-1,0}$ and $\tilde{v}_{m-1} \in \mathcal{H}_{m-1,1}$ such that $\|\tilde{u}_{m-1}\| = \|\tilde{v}_{m-1}\| = 1$ and $\tilde{u}_{m-1} \perp \tilde{v}_{m-1}$. Then for some numbers $\nu \in [0,1]$, $\varphi, \psi \in [0, 2\pi]$, the following identity holds:

$$\tilde{u}_{m-1} = e^{-i\varphi}\sqrt{\nu}u_{m-1} + e^{i\psi}\sqrt{1-\nu}v_{m-1}.$$

Define u_m by the formula

$$u_m = \frac{P_mP_{m-1}}{\sqrt{\tau^0_{m-1,m}}}\tilde{u}_{m-1},$$

then, evidently, $\|u_m\| = 1$.

In the same way as in the case of $s = 5$, we can show that, for an irreducible $*$-representation, $\nu \in (0,1)$ and the vectors $\tilde{u}_{m-1}$ and v_1 can be selected in such a way that

$$\tilde{u}_{m-1} = \sqrt{\nu}u_{m-1} + \sqrt{1-\nu}v_{m-1}.$$

Then $u_m \in \mathcal{H}_{m,0}$ and

$$\langle u_{m-1}, u_m\rangle = \langle u_{m-1}, \frac{P_{m-1}P_mP_{m-1}}{\sqrt{\tau^0_{m-1,m}}}\tilde{u}_{m-1}\rangle = \sqrt{\tau^0_{m-1,m}\nu},$$

$$\langle v_{m-1}, u_m\rangle = \langle v_{m-1}, \frac{P_{m-1}P_mP_{m-1}}{\sqrt{\tau^0_{m-1,m}}}\tilde{u}_{m-1}\rangle = \sqrt{\tau^0_{m-1,m}(1-\nu)}.$$

So, we have found a set of vectors $\{u_i, v_j\}_{i\in V, j\in\tilde{V}}$ such that their Gram matrix is equal to the matrix of the sesquilinear form $B^{\nu}_{\mathbb{G}_{5,4},f}$ and the image of the projection P_i is a linear span of the pair of vectors $\{u_i, v_i\}$ for $i \in \tilde{V}$ or the single vector u_i for $i \in V_m$. Because the linear span of the set is invariant with respect to the $*$-representation, it coincides with $\mathcal{H}$. □

Theorem 17. *For any proper irreducible $*$-representation π of the algebra $TL_{\mathbb{G}_{5,s},f,\perp}$ there exists $\nu \in \Sigma_{\mathbb{G}_{5,s},f}$ such that π is unitarily equivalent to π_ν.*

Proof. For any proper irreducible $*$-representation π, according to the previous proposition there exists a number $\nu \in (0,1)$ and a set of vectors $\{u_i, v_j\}_{i\in V, j\in\tilde{V}}$ such that their Gram matrix equals to the matrix of the sesquilinear form $B^{\nu}_{\mathbb{G}_{5,s},f}$. So, the sesquilinear form $B^{\nu}_{\mathbb{G}_{5,s},f}$ is non-negative definite, i.e., $\nu \in \Sigma_{\mathbb{G}_{5,s},f}$. Let us show that π is unitarily equivalent to π_ν.

We define an operator $C : \mathcal{H} \to \mathcal{H}_\nu$ by $Cu_i = x_i$, $Cv_j = y_j$, $i \in V$, $j \in \tilde{V}$. It is clear, that C is a unitary operator.

Then, for any u_k, v_k and $i \in \tilde{V}$, the next relations hold:

$$\begin{aligned}
CP_iu_k &= \langle u_k, u_i\rangle Cu_i + \langle u_k, v_i\rangle Cv_i = \langle x_k, x_i\rangle x_i + \langle x_k, y_i\rangle y_i,\\
CP_iv_k &= \langle v_k, u_i\rangle Cu_i + \langle v_k, v_i\rangle Cv_i = \langle y_k, x_i\rangle x_i + \langle y_k, y_i\rangle y_i,\\
P_{i,\nu}Cu_k &= P_{i,\nu}x_k = \langle x_k, x_i\rangle x_i + \langle x_k, y_i\rangle y_i,\\
P_{i,\nu}Cv_k &= P_{i,\nu}y_k = \langle y_k, x_i\rangle x_i + \langle y_k, y_i\rangle y_i.
\end{aligned}$$

And for any u_k, v_k and $i \in V \setminus \tilde{V}$, we have

$$\begin{aligned}
CP_iu_k &= \langle u_k, u_i\rangle Cu_i = \langle x_k, x_i\rangle x_i,\\
CP_iv_k &= \langle v_k, u_i\rangle Cu_i = \langle y_k, x_i\rangle x_i,\\
P_{i,\nu}Cu_k &= P_{i,\nu}x_k = \langle x_k, x_i\rangle x_i,\\
P_{i,\nu}Cv_k &= P_{i,\nu}y_k = \langle y_k, x_i\rangle x_i.
\end{aligned}$$

So, we have shown that $CP_i = P_{i,\nu}C$ for any $i \in V$, which implies that the $*$-representations π and π_ν are unitarily equivalent. □

5. Examples

As examples we consider the graphs $\hat{\mathbb{G}}_{5,4}$ and $\hat{\mathbb{G}}_{5,5}$ such that the sets of their vertices consist of precisely three elements $\{0,1,2\}$. For these algebras we will describe the sets $\Sigma_{\hat{\mathbb{G}}_{5,4},f}$ and $\Sigma_{\hat{\mathbb{G}}_{5,5},f}$.

For $\hat{\mathbb{G}}_{5,4}$, the matrix of the related sesquilinear form in the basis $\{x_0, y_0, x_1, y_1, x_2\}$ is

$$\begin{pmatrix}
1 & 0 & \sqrt{\tau^0_{01}} & 0 & 0\\
0 & 1 & 0 & \sqrt{\tau^1_{01}} & 0\\
\sqrt{\tau^0_{01}} & 0 & 1 & 0 & \sqrt{\nu\tau^0_{12}}\\
0 & \sqrt{\tau^1_{01}} & 0 & 1 & \sqrt{(1-\nu)\tau^0_{12}}\\
0 & 0 & \sqrt{\nu\tau^0_{12}} & \sqrt{(1-\nu)\tau^0_{12}} & 1
\end{pmatrix}$$

and, for $\hat{\mathbb{G}}_{5,5}$, the matrix of the related sesquilinear form in the basis $\{x_0, y_0, x_1, y_1, x_2, y_2\}$ is

$$\begin{pmatrix} 1 & 0 & \sqrt{\tau_{01}^0} & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{\tau_{01}^1} & 0 & 0 \\ \sqrt{\tau_{01}^0} & 0 & 1 & 0 & \sqrt{\nu\tau_{12}^0} & \sqrt{(1-\nu)\tau_{12}^1} \\ 0 & \sqrt{\tau_{01}^1} & 0 & 1 & \sqrt{(1-\nu)\tau_{12}^0} & -\sqrt{\nu\tau_{12}^1} \\ 0 & 0 & \sqrt{\nu\tau_{12}^0} & \sqrt{(1-\nu)\tau_{12}^0} & 1 & 0 \\ 0 & 0 & \sqrt{(1-\nu)\tau_{12}^1} & -\sqrt{\nu\tau_{12}^1} & 0 & 1 \end{pmatrix}.$$

To find when these matrices are nonnegative definite let us calculate the principal diagonal minors of these matrices. Note that the principal diagonal minors of the first matrix are principal diagonal minors of the second one as well. Let us denote the minors by M_i, $i = 1, \ldots, 5$ ($i = 1, \ldots, 6$ for the second matrix).

To calculate the determinants, we use the fact that if to some row we add a linear combination of others rows then the determinant of the matrix does not change. Such a transformation of the matrix will be called an allowed transformation.

First of all, consider the case of $\tau_{01}^0 = 1$. Then, evidently, $M_1 = M_2 = 1$, $M_3 = M_4 = 0$. Furthermore, by using allowed transformations, we will get the identity

$$M_5 = \det \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{\tau_{01}^1} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\nu\tau_{12}^0} \\ 0 & 0 & 0 & 1-\tau_{01}^1 & \sqrt{(1-\nu)\tau_{12}^0} \\ 0 & 0 & \sqrt{\nu\tau_{12}^0} & \sqrt{(1-\nu)\tau_{12}^0} & 1 \end{pmatrix}.$$

So $M_5 = -\nu(1-\tau_{01}^1)\tau_{12}^0 < 0$, because $1 = \tau_{01}^0 > \tau_{01}^1$, $\tau_{12}^0 > 0$, $\nu \in (0,1)$. This means that in the case of $\tau_{01}^0 = 1$ the matrices of these forms cannot be nonnegative definite, so, there do not exist proper representations.

If $\tau_{01}^0 < 1$ then, evidently, $M_1 = M_2 = 1$, $M_3 = 1 - \tau_{01}^1 > 0$, $M_4 = 1 - \tau_{01}^1 > 0$, and using the allowed transformations we can get the identities

$$M_5 = \det \begin{pmatrix} 1 & 0 & \sqrt{\tau_{01}^0} & 0 & 0 \\ 0 & 1 & 0 & \sqrt{\tau_{01}^1} & 0 \\ 0 & 0 & 1-\tau_{01}^0 & 0 & \sqrt{\nu\tau_{12}^0} \\ 0 & 0 & 0 & 1-\tau_{01}^1 & \sqrt{(1-\nu)\tau_{12}^0} \\ 0 & 0 & 0 & 0 & 1 - \frac{\nu\tau_{12}^0}{1-\tau_{01}^0} - \frac{(1-\nu)\tau_{12}^0}{1-\tau_{01}^1} \end{pmatrix}$$

and

$$M_6 = \det \begin{pmatrix} 1 & 0 & \sqrt{\tau_{01}^0} & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{\tau_{01}^1} & 0 & 0 \\ 0 & 0 & 1-\tau_{01}^0 & 0 & \sqrt{\nu\tau_{12}^0} & \sqrt{(1-\nu)\tau_{12}^1} \\ 0 & 0 & 0 & 1-\tau_{01}^1 & \sqrt{(1-\nu)\tau_{12}^0} & -\sqrt{\nu\tau_{12}^1} \\ 0 & 0 & 0 & 0 & X_0 & Y \\ 0 & 0 & 0 & 0 & Y & X_1 \end{pmatrix},$$

correspondingly, where

$$X_i = 1 - \frac{\nu\tau_{12}^i}{1-\tau_{01}^i} - \frac{(1-\nu)\tau_{12}^i}{1-\tau_{01}^{1-i}}, \quad i = 0, 1,$$

$$Y = \frac{\sqrt{\nu(1-\nu)\tau_{12}^0\tau_{12}^1}}{1-\tau_{01}^1} - \frac{\sqrt{\nu(1-\nu)\tau_{12}^0\tau_{12}^1}}{1-\tau_{01}^0}.$$

Thus the first form is nonnegative definite if and only if

$$X_0 = 1 - \frac{\nu\tau_{12}^0}{1-\tau_{01}^0} - \frac{(1-\nu)\tau_{12}^0}{1-\tau_{01}^1} \geqslant 0,$$

i.e.,

$$\frac{1-\tau_{01}^1-\tau_{12}^0}{1-\tau_{01}^1} - \nu\tau_{12}^0\frac{\tau_{01}^0-\tau_{01}^1}{(1-\tau_{01}^0)(1-\tau_{01}^1)} \geqslant 0,$$

which is equivalent to the inequality

$$\nu \leqslant \frac{(1-\tau_{01}^1-\tau_{12}^0)(1-\tau_{01}^0)}{\tau_{12}^0(\tau_{01}^0-\tau_{01}^1)} = \nu_0 = \nu_0(f).$$

Moreover, it is positive definite if and only if $\nu < \nu_0$.

For the second form in the case of $M_5 = 0$, we will get the identity

$$\begin{aligned} M_6 &= -Y^2(1-\tau_{01}^0)(1-\tau_{01}^1) \\ &= -\nu(1-\nu)\tau_{12}^0\tau_{12}^1(1-\tau_{01}^0)(1-\tau_{01}^1)\left(\frac{1}{1-\tau_{01}^1} - \frac{1}{1-\tau_{01}^0}\right)^2. \end{aligned}$$

So $M_6 < 0$ and the condition $\nu < \nu_0$ is required for the second form to be nonnegative definite. In the case of $\nu < \nu_0$, the form is nonnegative definite if and only if

$$X_0X_1 - Y^2 \geqslant 0. \tag{16}$$

It is evident that

$$\begin{aligned} X_i &= \tau_{12}^i\left(\frac{1}{\tau_{12}^i} - \frac{1}{1-\tau_{01}^{1-i}}\right) - \tau_{12}^i\nu\left(\frac{1}{1-\tau_{01}^i} - \frac{1}{1-\tau_{01}^{1-i}}\right), \quad i = 0,1, \\ Y^2 &= \nu(1-\nu)\tau_{12}^0\tau_{12}^1\left(\frac{1}{1-\tau_{01}^1} - \frac{1}{1-\tau_{01}^0}\right)^2. \end{aligned}$$

Let us introduce

$$\begin{aligned} X_i' &= \frac{1}{\tau_{12}^i} - \frac{1}{1-\tau_{01}^{1-i}} = \frac{1-\tau_{01}^{1-i}-\tau_{12}^i}{\tau_{12}^i(1-\tau_{01}^{1-i})}, \quad i = 0,1, \\ Z &= \frac{1}{1-\tau_{01}^0} - \frac{1}{1-\tau_{01}^1}, \end{aligned}$$

then inequality (16) is equivalent to the following inequalities:

$$(X_0' - \nu Z)(X_1' + \nu Z) - \nu(1-\nu)Z^2 \geqslant 0,$$

$$X_0'X_1' - \nu Z(X_1' - X_0' + Z) \geqslant 0.$$

Because

$$\begin{aligned} Z(X_1' - X_0' + Z) &= \left(\frac{1}{1-\tau_{01}^0} - \frac{1}{1-\tau_{01}^1}\right)\left(\frac{1}{\tau_{12}^1} - \frac{1}{\tau_{12}^0}\right) \\ &= \frac{(\tau_{01}^0-\tau_{01}^1)(\tau_{12}^0-\tau_{12}^1)}{\tau_{01}^0\tau_{01}^1(1-\tau_{01}^0)(1-\tau_{01}^1)} > 0, \end{aligned}$$

inequality (16) is equivalent to

$$\nu \leqslant \frac{X_0'X_1'}{Z(X_1'-X_0'+Z)},$$

or

$$\nu \leqslant \frac{(1-\tau_{01}^1-\tau_{12}^0)(1-\tau_{01}^0-\tau_{12}^1)}{(\tau_{01}^0-\tau_{01}^1)(\tau_{12}^0-\tau_{12}^1)} = \nu_1 = \nu_1(f).$$

So we have shown that $\Sigma_{\hat{\mathbb{G}}_{5,4},f} = (0, \nu_0(f)] \cap (0,1)$ and $\Sigma_{\hat{\mathbb{G}}_{5,5},f} = (0, \nu_0(f)) \cap (0, \nu_1(f)] \cap (0,1)$.

Note that $\Sigma_{\hat{\mathbb{G}}_{5,4},f}$ is an empty set if and only if $\nu_0(f) \leqslant 0$ and $\Sigma_{\hat{\mathbb{G}}_{5,5},f}$ is empty if and only if $\nu_0(f) \leqslant 0$ or $\nu_1(f) \leqslant 0$. Thus we have proved the following proposition.

Theorem 18. 1. *The algebra $TL_{\hat{\mathbb{G}}_{5,4},f,\perp}$ has no proper irreducible ∗-representations in the case where*

$$\tau_{01}^0 = 1 \quad \text{or} \quad \tau_{01}^1 + \tau_{12}^0 \geqslant 1.$$

Otherwise, all proper irreducible unitarily nonequivalent ∗-representations are π_ν, where

$$\nu \in (0,1) \cap (0, \nu_0(f)].$$

2. *The algebra $TL_{\hat{\mathbb{G}}_{5,5},f,\perp}$ has no proper irreducible ∗-representations in the case where*

$$\tau_{01}^1 + \tau_{12}^0 \geqslant 1 \quad \text{or} \quad \tau_{01}^0 + \tau_{12}^1 \geqslant 1.$$

Otherwise, all proper irreducible unitarily nonequivalent ∗-representations are π_ν, where

$$\nu \in (0,1) \cap (0, \nu_0(f)) \cap (0, \nu_1(f)].$$

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