ON ASYMPTOTIC BEHAVIOR OF THE CONSTANTS IN GENERALIZED KHINTCHINE'S INEQUALITY

V. V. MYKHAYLYUK AND V. A. BALAN

ABSTRACT. We establish an asymptotic behavior of the constants in Khintchine's inequality for independent random variables of mean zero.

1. INTRODUCTION

Let $(r_n)_{n=1}^\infty$ be the sequence of Rademacher functions

$$r_n(t) = \text{sign} \sin(2^n \pi t), \quad t \in [0,1].$$

According to Khintchine's inequality [1] for each $n \in \mathbb{N}$, $p \in [1, +\infty)$, and a sequence $(a_k)_{k=1}^\infty$ of reals $a_k \in \mathbb{R}$ we have

$$(1.1) \qquad A_p^{(0)} \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}} \le \Big\| \sum_{k=1}^n a_k r_k \Big\|_p \le B_p^{(0)} \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}},$$

where $|| \cdot ||_p$ is the norm of $L_p = L_p[0,1]$,

$$A_p^{(0)} = \begin{cases} \frac{1}{\sqrt{2}}, & 1 \le p < 2, \\ 1, & 2 \le p < +\infty, \end{cases} \qquad B_p^{(0)} = \begin{cases} 1, & 1 \le p \le 2, \\ O(\sqrt{p}), & 2 < p < +\infty. \end{cases}$$

Recall that a sequence $(f_n)_{n=1}^\infty$ of measurable functions $f_n : [0,1] \to \mathbb{R}$ is called *a sequence of independent random variables (i.r.v.)* if for each $n \in \mathbb{N}$ and any intervals $[a_1, b_1], \ldots, [a_n, b_n] \subseteq \mathbb{R}$, the following equality holds:

$$\mu \Big(\bigcap_{k=1}^n f_k^{-1}([a_k, b_k]) \Big) = \prod_{k=1}^n \mu\{f_k^{-1}([a_k, b_k])\}.$$

On the other hand, Khintchine's inequality was generalized in [2] for k-tuple products of mean zero i.r.v. In particular, for each $n \in \mathbb{N}$, each collection $(y_k)_{k=1}^n$ of i.r.v. $y_k \in L_p$, $p > 1$, such that $\int_0^1 y_k d\mu = 0$ and for each collection of reals, $(a_k)_{k=1}^n$ we have

$$(1.2) \qquad \frac{\alpha_q A_p^{(0)}}{p^*} \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}} \le \Big\| \sum_{k=1}^n a_k y_k \Big\|_p \le \beta_r B_p^{(0)} p^* \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}},$$

where $r = \max\{p, 2\}$, $q = \min\{p, 2\}$, $p^* = \max\{p, \frac{p}{p-1}\} - 1$, $a_k \in \mathbb{R}$, $1 \le k \le n$, $\alpha_q = \inf\limits_{1 \le k \le n} ||y_n||_q$ and $\beta_r = \sup\limits_{1 \le k \le n} ||y_n||_r$.

We remark that inequality (1.2) can be obtained for the case of $p = 1$ ([2]).

Moreover, inequality (1.2) can be written in the following way:

$$(1.3) \qquad \alpha_q A_p \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}} \le \Big\| \sum_{k=1}^n a_k y_k \Big\|_p \le \beta_r B_p \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}},$$

where A_p and B_p are the largest and the smallest constants respectively such that this inequality holds for each $n \in \mathbb{N}$, each collection $(a_k)_{k=1}^n$ of reals a_k and each collection $(y_k)_{k=1}^n$ of mean zero independent random variables y_k.

In particular, we have

$$B_p = \sup \left\{ \frac{\| \sum\limits_{k=1}^{n} a_k y_k \|_p}{\beta_r} : n \in \mathbb{N}, \sum_{k=1}^{n} a_k^2 = 1, y_1, \dots, y_n \in L_{p^-} \text{ i.r.v.} \right\}$$

for each $p \geq 1$.

Note that (1.2) and $B_p^{(0)} \leq \sqrt{p}$ imply $B_p \leq p\sqrt{p}$. So, the question on the order of magnitude for the function $f(p) = B_p$ naturally arises.

In [3], using standard methods and some combinatorial arguments we proved that $B_{2p} \leq p$ for all $p \in \mathbb{N}$.

So, the question of a similar estimation for B_p for each $p \geq 1$ naturally arises as well.

In this paper we clarify arguments of [3], and with the help of a generalization of the Riesz-Thorin theorem we show that $B_p \leq 2\sqrt{e(p+2)}$ for all $p \geq 1$.

2. The upper estimate of B_{2p} for $p \in \mathbb{N}$

In this section we develop a method from [3] and obtain an upper estimate for B_{2p} with $p \in \mathbb{N}$.

We recall that $(a_1 + \dots + a_n)^m = \sum\limits_{k_1 + \dots + k_n = m} \gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n}$ for each reals $a_1, \dots, a_n \in \mathbb{R}$ and $m \in \mathbb{N}$, where $\gamma(k_1, \dots, k_n) = \frac{m!}{k_1! \dots k_n!}$.

We need the following combinatorial statements.

Proposition 2.1. *Let $k_1, \dots, k_n$, $m_1, \dots, m_n$ be natural positive reals such that $k_1 + \dots + k_n = m_1 + \dots + m_n = 2p$, $k_1, \dots, k_n, m_1, \dots, m_n \neq 1$ and $|k_i - m_i| \leq 1$ for $1 \leq i \leq n$. Then $\frac{\gamma(k_1, \dots, k_n)}{\gamma(m_1, \dots, m_n)} \leq \left(\frac{2e}{3}\right)^p$.*

Proof. Since $k_1 + \dots + k_n = m_1 + \dots + m_n$, the sets $\{i : k_i = m_i + 1\}$ and $\{i : k_i = m_i - 1\}$ have the same quantity of elements which we denote by s. Let $m_1 = k_1 - 1, m_2 = k_2 + 1, \dots, m_{2s-1} = k_{2s-1} - 1, m_{2s} = k_{2s} + 1$. Then

$$A = \frac{\gamma(k_1, \dots, k_n)}{\gamma(m_1, \dots, m_n)} = \frac{k_2 + 1}{k_1} \cdot \frac{k_4 + 1}{k_3} \cdots \frac{k_{2s} + 1}{k_{2s-1}}.$$

The expression A attains it's most value when $k_1, k_3, \dots, k_{2s-1}$ attain their least values. Since $m_{2i-1} \neq 1$ and $k_{2i-1} = m_{2i-1} + 1 \neq 1$, $m_{2i-1} \geq 2$ and $k_{2i-1} \geq 3$ for $1 \leq i \leq s$. Then using the Cauchy inequality we obtain

$$A \leq \frac{(k_2 + 1) \dots (k_{2s} + 1)}{3^s} \leq \frac{\left(\frac{k_2 + \dots + k_{2s} + s}{s}\right)^s}{3^s} \leq \left(\frac{k_1 + \dots + k_{2s} - 2s}{3s}\right)^s \leq \left(\frac{2p - 2s}{3s}\right)^s.$$

We consider the function $f(x) = \left(\frac{p}{x} - 1\right)^x$ where $x \in (0, \frac{p}{2}]$. Since $f(\frac{p}{2}) = \lim\limits_{x \to 0} f(x) = 1$, by Rolle's theorem, the function f reaches it's maximum at some point $x_0 \in (0, \frac{p}{2})$, where $f'(x_0) = 0$, i.e. $\ln f(x_0) = \frac{px_0}{p - x_0}$. Hence,

$$f(x_0) = e^{\frac{px_0}{p - x_0}} \leq e^p.$$

Therefore, taking into account that $s \leq p$ and $f(x) \leq 1$ for each $x \in [\frac{p}{2}, p]$ we obtain

$$A \leq \left(\frac{2}{3}\right)^s \cdot \left(\frac{p}{s} - 1\right)^s \leq \left(\frac{2}{3}\right)^s f(x_0) \leq \left(\frac{2e}{3}\right)^p.$$

$\square$

Proposition 2.2. *Let $p \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$. Then $\sum\limits_{\substack{k_1 + \dots + k_n = 2p \\ k_1, \dots, k_n \neq 1}} \gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n} \leq \left(2ep \cdot \sum\limits_{k=1}^{n} a_k^2\right)^p$.*

Proof. Without loss of generality we can assume that $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$. Let $A = \{(k_1, k_2, \dots, k_n) : k_i \neq 1, \ k_1 + k_2 + \dots + k_n = 2p\}$ and $B = \{(m_1, m_2, \dots, m_n) : m_i \text{ is even}, \ m_1 + m_2 + \dots + m_n = 2p\}$. Now we construct a mapping $\varphi : A \to B$. Fix

$(k_1, \ldots, k_n) \in A$. Denote by I the set of all numbers $1 \le i \le n$ such that k_i is even, and by J the set of all numbers $1 \le i \le n$ such that k_i is odd. Since $k_1 + \cdots + k_n = 2p$, the set J has even quantity of elements. Let $J = \{j_1, \ldots, j_{2s}\}$ where $j_1 \le j_2 \le \cdots \le j_{2s}$. Then we set

$$m_i = \begin{cases} k_i, & i \in I, \\ k_i + 1, & i = j_{2l-1}, \quad 1 \le l \le s, \\ k_i - 1, & i = j_{2l}, \quad 1 \le l \le s, \end{cases}$$

for each $1 \le i \le n$ and $(m_1, \ldots, m_n) = \varphi(k_1, \ldots, k_n)$.

Since the reals a_i decrease and the indices j_l increase,

$$(2.1) \qquad a_1^{k_1} \ldots a_n^{k_n} \le a_1^{m_1} \ldots a_n^{m_n}.$$

Now since $|k_i - m_i| \le 1$ for $1 \le i \le n$, by Proposition 2.1 we have $\gamma(k_1, \ldots, k_n) \le \left(\frac{2e}{3}\right)^p \gamma(m_1, \ldots, m_n)$.

Thus, if $(m_1, \ldots, m_n) = \varphi(k_1, \ldots, k_n)$ then

$$(2.2) \qquad \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k_n} \le \left(\frac{2e}{3}\right)^p \gamma(m_1, \ldots, m_n) a_1^{m_1} \ldots a_n^{m_n}.$$

For each $(m_1, \ldots, m_n) \in B$ we put $C = C_{m_1,\ldots,m_n} = \{(k_1, \ldots, k_n) : \varphi(k_1, \ldots, k_n) = (m_1, \ldots, m_n)\}$ and $N = N_{m_1,\ldots,m_n} = \{i \le n : m_i \ne 0\}$.

Note that for each $(k_1, \ldots, k_n) \in C$ we have $k_i = m_i$ for $i \notin N$ and $k_i \in \{m_i - 1, m_i, m_i + 1\}$ for $i \in N$. Furthermore, $m_i \ge 2$ for $i \in N$. So $|N| \le p$. Thus, by the multiplication principle of ([4]) we have $|C| \le 3^p$.

Taking into account (2.2) we obtain

$$\sum_{(k_1,\ldots,k_n)\in A} \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k_n} \le \left(\frac{2e}{3}\right)^p \sum_{(k_1,\ldots,k_n)\in A} \gamma(\varphi(k_1, \ldots, k_n)) a_1^{m_1} \ldots a_n^{m_n}$$

$$\le \left(\frac{2e}{3}\right)^p \sum_{(m_1,\ldots,m_n)\in B} |C_{m_1,\ldots,m_n}| \gamma(m_1, \ldots, m_n) a_1^{m_1} \ldots a_n^{m_n}$$

$$\le (2e)^p \sum_{l_1+\cdots+l_n=p} \gamma(2l_1, \ldots, 2l_n) a_1^{2l_1} \ldots a_n^{2l_n}.$$

Now since

$$\gamma(2l_1, \ldots, 2l_s) = \frac{2p!}{(2l_1)! \ldots (2l_s)!}$$

$$= \frac{(p+1) \ldots 2p}{(l_1 + 1) \ldots 2l_1 \cdot (2l_s + 1) \ldots 2l_s} \cdot \gamma(l_1, \ldots, l_s)$$

$$\le \frac{(2p)^p}{2^p} \gamma(l_1, \ldots, l_s) = p^p \gamma(l_1, \ldots, l_s)$$

for each collection of reals $l_1, \ldots, l_s \ge 1, l_1 + \cdots + l_s = p$, one has that

$$\sum_{(k_1,\ldots,k_n)\in A} \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k_n}$$

$$\le (2ep)^p \sum_{l_1+\cdots+l_n=p} \gamma(l_1, \ldots, l_n) a_1^{2l_1} \ldots a_n^{2l_n} = \left(2ep \cdot \sum_{k=1}^{n} a_k^2\right)^p.$$

$$\square$$

The following theorem is the main result of this section.

Theorem 2.3. $B_{2p} \le \sqrt{2ep}$ *for each* $p \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$ and a collection $(y_k)_{k=1}^n$ of i.r.v. $y_k \in L_p$ of mean zero. Note that since the random variables $y_1, \ldots, y_n$ are independent,

$$\int_0^1 y_1^{k_1} \ldots y_n^{k_n} d\mu = \int_0^1 y_1^{k_1} d\mu \cdots \int_0^1 y_n^{k_n} d\mu = 0,$$

if $k_i = 1$ for at least one of the integers. On the other hand,

$$\int_0^1 |y_1^{k_1} \ldots y_n^{k_n}| \, d\mu = \int_0^1 |y_1^{k_1}| \, d\mu \ldots \int_0^1 |y_n^{k_n}| \, d\mu$$
$$\leq ||y_1||_{k_1}^{k_1} \ldots ||y_n||_{k_n}^{k_n} \leq ||y_1||_{2p}^{k_1} \ldots ||y_n||_{2p}^{k_n} \leq \beta_{2p}^{k_1+\cdots+k_n} = \beta_{2p}^{2p}.$$

Observe that it is sufficient to consider the case of $a_1, \ldots, a_n \geq 0$.

Now we show that

$$\Big\| \sum_{k=1}^n a_k y_k \Big\|_{2p} \leq \beta_{2p} \sqrt{2ep} \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}}.$$

Using Proposition 2.2, we obtain

$$\int_0^1 \Big(\sum_{k=1}^n a_k y_k \Big)^{2p} d\mu = \sum_{k_1+\cdots+k_n=2p} \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k_n} \cdot \int_0^1 y_1^{k_1} \ldots y_n^{k_n} d\mu$$

$$= \sum_{\substack{k_1+\cdots+k_n=2p \\ k_i \neq 1}} \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k_n} \cdot \int_0^1 y_1^{k_1} \ldots y_n^{k_n} d\mu$$

$$\leq \sum_{\substack{k_1+\cdots+k_n=2p \\ k_i \neq 1}} \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k_n} \cdot \int_0^1 |y_1^{k_1} \ldots y_n^{k_n}| d\mu$$

$$\leq \beta_{2p}^{2p} \sum_{\substack{k_1+\cdots+k_n=2p \\ k_1, \ldots, k_n \neq 1}} \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k_n} \leq \beta_{2p}^{2p} \cdot \Big(2ep \cdot \sum_{k=1}^n a_k^2 \Big)^p.$$

Then

$$\Big\| \sum_{k=1}^n a_k y_k \Big\|_{2p} \leq \beta_{2p} \sqrt{2ep} \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}},$$

and thus,

$$B_{2p} \leq \sqrt{2ep}.$$

$\square$

3. A GENERALIZATION OF THE RIESZ-THORIN THEOREM

In this section we prove a version of the Riesz-Thorin interpolation theorem [5] which we will use to obtain an upper estimate of B_p for each $p \geq 1$. The proof of this version is similar to the proof of the classic theorem (see [5]).

We need Hadamard's theorem about three lines [5].

Theorem 3.1. (Hadamard). *Let* $\Pi = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$, $f : \Pi \to \mathbb{C}$ *be a bounded and analytic in* Π *function. Then for each* $\theta \in [0, 1]$ *we have*

$$M_\theta \leq M_0^{1-\theta} \cdot M_1^\theta,$$

where $M_\theta = \sup\{|f(\theta + iy)| : y \in \mathbb{R}\}$.

Now assume $n \in \mathbb{N}$, reals satisfy $1 \leq p_0^{(1)} \leq p_1^{(1)} < +\infty$, ..., $1 \leq p_0^{(n)} \leq p_1^{(n)} < +\infty$, $1 \leq q_0 \leq q_1 < +\infty$, $L_{p_j^{(i)}}, 1 \leq i \leq n, j \in \{0, 1\}$ are the spaces of $p_j^{(i)}$-integrable functions $z : [0, 1] \to \mathbb{K}$, where $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, $X_1 \subseteq L_{p_1^{(1)}}$, ..., $X_n \subseteq L_{p_1^{(n)}}$ are linear subspaces which contain all finite valued simple functions on $[0, 1]$, $Y \subseteq L_{q_1}$ is a linear subspace, $T_1 : X_1 \to Y$, ..., $T_n : X_n \to Y$ are operators, $X = \bigoplus_{i=1}^n X_i$ and $T : X \to Y$, $T(x_1 + \cdots + x_n) = T_1(x_1) + \cdots + T_n(x_n)$, where $x_1 \in X_1$, ..., $x_n \in X_n$.

For each $\alpha \in (0, 1)$ and $1 \leq k \leq n$ we denote by $p_\alpha^{(k)}$ the real $p_\alpha^{(k)} \in [p_0^{(k)}, p_1^{(k)}]$ such that $\frac{1}{p_\alpha^{(k)}} = \frac{1-\alpha}{p_0^{(k)}} + \frac{\alpha}{p_1^{(k)}}$ and by q_α the real $q_\alpha \in [q_0, q_1]$ such that $\frac{1}{q_\alpha} = \frac{1-\alpha}{q_0} + \frac{\alpha}{q_1}$.

Let $\tau : \mathbb{R}^n \to \mathbb{R}$ be any norm on $\mathbb{R}^n$. For each $\alpha \in [0,1]$ by τ_α we denote the norm on X defined by

$$\tau_\alpha(x) = \tau_\alpha(x_1 + \cdots + x_n) = \tau(\|x_1\|_{p_\alpha^{(1)}}, \ldots, \|x_n\|_{p_\alpha^{(n)}}).$$

Then we set $S_\alpha = T : (X, \tau_\alpha) \to (Y, \|\cdot\|_{q_\alpha})$.

Theorem 3.2. *Let $\mathbb{K} = \mathbb{C}$. Then $\|S_\alpha\| \leq \|S_0\|^{1-\alpha} \cdot \|S_1\|^\alpha$ for each $\alpha \in [0,1]$.*

Proof. Fix $\alpha \in [0,1]$. Denote $q = q_\alpha$, $p_1 = p_\alpha^{(1)}$, $\ldots$, $p_n = p_\alpha^{(n)}$ and $K = \|S_0\|^{1-\alpha} \cdot \|S_1\|^\alpha$. It is enough to prove that

(3.1)
$$\|S_\alpha x\|_q \leq K \cdot \tau_\alpha(x)$$

for each $x \in X$.

Since all norms on $\mathbb{R}^n$ are equivalent, the τ-convergence is equivalent to the coordinate-wise convergence in $\mathbb{R}^n$. Moreover, the set of all simple functions is dense in every space L_p. So it is sufficient to prove (3.1) for any simple functions $x_1, \ldots, x_n$.

Let $x_k = \sum_{i=1}^{m_k} a_i^{(k)} \chi_{A_{i,k}}$, $x = x_1 + \cdots + x_n$ and $\tau_\alpha(x) = 1$. Then (3.1) takes the following form:

(3.2)
$$\|S_\alpha x\|_q \leq K.$$

Hahn-Banach's theorem implies that (3.2) is equivalent to the inequality $|f(S_\alpha x)| \leq K$ for each $f \in (L_q)^*$ with $\|f\| = 1$. Thus, it is sufficient to check that for each simple function $v = \sum_{i=1}^{l} b_i \chi_{B_1}$ with $\|v\|_{q'} = 1$ where $\frac{1}{q} + \frac{1}{q'} = 1$ the following inequality holds

(3.3)
$$\left| \int_0^1 v(t) S_\alpha x(t) d\mu \right| \leq K.$$

We consider the functions $\varphi_1, \ldots, \varphi_n, \psi : \mathbb{C} \to \mathbb{C}$, $\varphi_k(z) = \frac{1-z}{p_0^{(k)}} + \frac{z}{p_1^{(k)}}$, $\psi(z) = \frac{1-z}{q_0'} + \frac{z}{q_1'}$, where $\frac{1}{q_0} + \frac{1}{q_0'} = \frac{1}{q_1} + \frac{1}{q_1'} = 1$. For each $z \in \mathbb{C}$ we set

$$x_{k,z} = \sum_{i=1}^{m_k} |a_i^{(k)}|^{p_k \cdot \varphi_k(z)} \cdot \text{sign } a_i^{(k)} \cdot (\|x_k\|_{p_k})^{1-p_k \varphi_k(z)} \cdot \chi_{A_{i,k}}$$

for $1 \leq k \leq n$ if $x_k \neq 0$ and $x_{k,z} = 0$ if $x_k = 0$,

$$x_z = (x_{1,z}, \ldots, x_{n,z})$$

and

$$v_z = \sum_{i=1}^{l} |b_i|^{q'\psi(z)} \text{sign } b_i \chi_{B_i},$$

where $\text{sign } z = \begin{cases} 0, & z = 0, \\ \frac{z}{|z|}, & z \neq 0 \end{cases}$ for any $z \in \mathbb{C}$.

Observe that $x_\alpha = x$, $v_\alpha = v$,

$$p_k \cdot \varphi_k(iy) = p_k \left(\frac{1-iy}{p_0^{(k)}} + \frac{iy}{p_1^{(k)}} \right) = \frac{p_k}{p_0^{(k)}} + iy \left(\frac{p_k}{p_1^{(k)}} - \frac{p_k}{p_0^{(k)}} \right).$$

Analogously

$$p_k \cdot \varphi_k(1+iy) = \frac{p_k}{p_1^{(k)}} + iy \left(\frac{p_k}{p_1^{(k)}} - \frac{p_k}{p_0^{(k)}} \right),$$

$$q' \cdot \psi(iy) = \frac{q'}{q_0'} + iy \left(\frac{q'}{q_1'} - \frac{q'}{q_0'} \right), \quad q' \cdot \psi(1+iy) = \frac{q'}{q_1'} + iy \left(\frac{q'}{q_1'} - \frac{q'}{q_0'} \right).$$

For each k, $1 \le k \le n$, we have

$$\|x_{k,iy}\|_{p_0^{(k)}} = \left(\int_0^1 \left| \sum_{j=1}^{m_k} |a_j^{(k)}|^{p_k \varphi_k(iy)} \cdot \operatorname{sign} a_j^{(k)} \cdot (\|x_k\|_{p_k})^{1-p_k\varphi_k(iy)} \cdot \chi_{A_{jk}} \right|^{p_0^{(k)}} d\mu \right)^{\frac{1}{p_0^{(k)}}}$$

$$= \|x_k\|_{p_k}^{1-\frac{p_k}{p_0^{(k)}}} \left(\sum_{j=1}^{m_k} \int_{A_{jk}} \left| |a_j^{(k)}|^{p_k\varphi_k(iy)} \right|^{p_0^{(k)}} d\mu \right)^{\frac{1}{p_0^{(k)}}}$$

$$= \|x_k\|_{p_k}^{1-\frac{p_k}{p_0^{(k)}}} \left(\sum_{j=1}^{m_k} \int_{A_{jk}} \left(|a_j^{(k)}|^{\frac{p_k}{p_0^{(k)}}} \right)^{p_0^{(k)}} d\mu \right)^{\frac{1}{p_0^{(k)}}}$$

$$= \|x_k\|_{p_k}^{1-\frac{p_k}{p_0^{(k)}}} \cdot \|x_k\|_{p_k}^{\frac{p_k}{p_0^{(k)}}} = \|x_k\|_{p_k}.$$

Analogously $\|x_{k,1+iy}\|_{p_1^{(k)}} = \|x_k\|_{p_k}$, $\|v_{iy}\|_{q_0'} = \|v_{1+iy}\|_{q_1'} = \|v\|_{q'} = 1$. Then $(\|x_{1,iy}\|_{p_0^{(1)}}, \ldots, \|x_{n,iy}\|_{p_0^{(n)}}) = (\|x_1\|_{p_1}, \ldots, \|x_n\|_{p_n})$, $(\|x_{1,1+iy}\|_{p_1^{(1)}}, \ldots, \|x_{n,1+iy}\|_{p_1^{(n)}}) = (\|x_1\|_{p_1}, \ldots, \|x_n\|_{p_n})$ and

$$(3.4) \qquad\qquad \tau_0(x_{iy}) = 1, \quad \tau_1(x_{1+iy}) = 1.$$

We define a function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \int_0^1 v_z T x_z \, d\mu = \int_0^1 \left(\sum_{i=1}^l |b_i|^{q'\psi(z)} \operatorname{sign} b_i \chi_{B_i} \right) \cdot T(x_{1,z} + \cdots + x_{n,z}) \, d\mu$$

$$= \sum_{i=1}^l \sum_{k=1}^n \sum_{j=1}^{m_k} (\|x_k\|_{p_k})^{1-p_k\varphi_k(z)} |b_i|^{q'\psi(z)} |a_j^{(k)}|^{p_k\varphi_k(z)} \operatorname{sign} b_i \operatorname{sign} a_j^{(k)} \int_{B_i} T \chi_{A_{j,k}} \, d\mu.$$

Since all the functions $a^z, a > 0$, are analytic and bounded in Π, the function f is the same in Π, as a linear combination of such exponential functions.

Now according to Holder's inequality and (3.4), we obtain

$$|f(iy)| = \left| \int_0^1 v_{iy} T x_{iy} \, d\mu \right| \le \|v_{iy}\|_{q_0'} \cdot \|T x_{iy}\|_{q_0} = \|v_{iy}\|_{q_0'} \cdot \|S_0 x_{iy}\|_{q_0}$$

$$\le \|v_{iy}\|_{q_0'} \cdot \|S_0\| \cdot \tau_0(x_{iy}) = \|S_0\|.$$

Analogously $|f(1+iy)| \le \|S_1\|$. Now using Theorem 3.1 and the fact that $x_\alpha = x$ and $v_\alpha = v$, we obtain

$$\left| \int_0^1 v_\alpha S_\alpha x \, d\mu \right| = \left| \int_0^1 v_\alpha T x_\alpha \, d\mu \right| = |f(\alpha)| \le \sup_{y \in \mathbb{R}} |f(\alpha + iy)|$$

$$\le \left(\sup_{y \in \mathbb{R}} |f(iy)| \right)^{1-\alpha} \cdot \left(\sup_{y \in \mathbb{R}} |f(1+iy)| \right)^\alpha \le \|S_0\|^{1-\alpha} \cdot \|S_1\|^\alpha = K.$$

$\square$

The following result gives an estimate for the norm of operator S_α in the case when $\mathbb{K} = \mathbb{R}$.

Corollary 3.3. *Let* $\mathbb{K} = \mathbb{R}$. *Then* $\|S_\alpha\| \le 2\|S_0\|^{1-\alpha} \cdot \|S_1\|^\alpha$ *for each* $\alpha \in [0,1]$.

Proof. Let for each $\alpha \in [0,1]$ $S_\alpha^{\mathbb{C}}$ be the extension of the operator S_α to the space $L_{p_\alpha}^{\mathbb{C}}$ of functions $z : [0,1] \to \mathbb{C}$.

Note that $\|S_\alpha^{\mathbb{C}}\| \ge \|S_\alpha\|$ and $\|S_\alpha^{\mathbb{C}}\| \le \sup_{\|z\| \le 1} (\|S_\alpha x\| + \|S_\alpha y\|) \le 2\|S_\alpha\|$. By Theorem 2.2 we have

$$\|S_\alpha\| \le \|S_\alpha^{\mathbb{C}}\| \le \|S_0^{\mathbb{C}}\|^{1-\alpha} \cdot \|S_1^{\mathbb{C}}\|^\alpha \le 2 \cdot \|S_0\|^{1-\alpha} \cdot \|S_1\|^\alpha.$$

$\square$

4. The main result

The following theorem gives an upper estimate for B_p.

Theorem 4.1. *Let $2 \leq p_0 < p_1$, $q \geq 2$, $\beta \in [0,1]$ and $\frac{1}{q} = \frac{1-\beta}{p_0} + \frac{\beta}{p_1}$. Then $B_q \leq 2B_{p_0}^{1-\beta} \cdot B_{p_1}^{\beta}$.*

Proof. Fix $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$ such that $\sum_{k=1}^{n} a_k^2 = 1$ and i.r.v. $x_1, \ldots, x_n \in L_q$. It is sufficient to prove that

$$\frac{\|\sum_{k=1}^{n} a_k x_k\|_q}{\sup_{1 \leq k \leq n} \|x_k\|_q} \leq 2B_{p_0}^{1-\beta} \cdot B_{p_1}^{\beta}.$$

At first we consider the case $x_1, \ldots, x_n \in L_{p_1}$.

For each $1 \leq k \leq n$ we set $\psi_k : L_{p_1}[0,1] \to L_{p_1}([0,1]^n)$, $\psi_k(x)(t_1, \ldots, t_n) = x(t_k)$. Observe that all the mappings ψ_k are isometric embeddings.

Since the spaces $L_{p_1}[0,1]$ and $L_{p_1}([0,1]^n)$ are isometrically isomorphic [6], there exists a linear isometry $\psi : L_{p_1}([0,1]^n) \to L_{p_1}[0,1]$. Now let $\varphi_k = \psi \circ \psi_k$ for each $1 \leq k \leq n$.

Note that for any $z_1, \ldots, z_n \in L_{p_1}$ the functions $\psi_1(z_1), \ldots, \psi_n(z_n)$ are i.r.v. Moreover, for each i.r.v. $z_1, \ldots, z_n \in L_{p_1}$ and reals $b_1, \ldots, b_n \in \mathbb{R}$ we have

(4.1)
$$\|\sum_{k=1}^{n} b_k z_k\|_s = \|\sum_{k=1}^{n} b_k \varphi_k(z_k)\|_s = \|\sum_{k=1}^{n} b_k \psi_k(z_k)\|_s$$

for any $s \in [p_0, p_1]$.

Let $q_0 = p_0^{(1)} = p_0^{(2)} = \cdots = p_0^{(n)} = p_0$, $q_1 = p_1^{(1)} = p_1^{(2)} = \cdots = p_1^{(n)} = p_1$, $X_1 = X_2 = \cdots = X_n = L_{p_1}$, $Y = L_{p_1}$, $T_1 = a_1 \varphi_1$, $\ldots$, $T_n = a_n \varphi_n$, $X = \overset{n}{\underset{k=1}{\oplus}} X_k$ and operator $T : X \to Y$ acts by the following rule $T(z_1 + \cdots + z_n) = T_1(z_1) + \cdots + T_n(z_n) = \sum_{k=1}^{n} a_k \varphi_k(z_k)$, where $z_1 \in X_1, \ldots, z_n \in X_n$. Furthermore, by τ we denote the maximum-norm on $\mathbb{R}^n$. Let for each $\alpha \in [0,1]$ and $1 \leq k \leq n$ reals $p_\alpha^{(k)}$ and q_α, norm τ_α and operator S_α are defined analogously as in Section 3.

Taking into account that $\psi_1(z_1), \ldots, \psi_n(z_n)$ are i.r.v. and (4.1) we have

$$\|S_0\| = \sup_{z \neq 0} \frac{\|S_0(z)\|_{p_0}}{\tau_0(z)} = \sup_{z \neq 0} \frac{\|\sum_{k=1}^{n} a_k \varphi_k(z_k)\|_{p_0}}{\sup_{1 \leq k \leq n} \|z_k\|_{p_0}} = \sup_{z \neq 0} \frac{\|\sum_{k=1}^{n} a_k \psi_k(z_k)\|_{p_0}}{\sup_{1 \leq k \leq n} \|\psi_k(z_k)\|_{p_0}} \leq B_{p_0},$$

where $z = z_1 + \cdots + z_n$. Similarly $\|S_1\| \leq B_{p_1}$.

Now by Corollary 3.3, equality (4.1) and equality $q = q_\beta$ we have

$$\frac{\|\sum_{k=1}^{n} a_k x_k\|_q}{\sup_{1 \leq k \leq n} \|x_k\|_q} \overset{(4.1)}{=} \frac{\|\sum_{k=1}^{n} a_k \varphi_k(x_k)\|_q}{\sup_{1 \leq k \leq n} \|x_k\|_q} = \frac{\|S_\beta(x_1 + \cdots + x_n)\|_q}{\tau_\beta(x_1 + \cdots + x_n)}$$

$$\leq \|S_\beta\| \leq 2\|S_0\| \cdot \|S_1\| \leq 2B_{p_0}^{1-\beta} \cdot B_{p_1}^{\beta}.$$

Now we prove the general case $x_1, \ldots, x_n \in L_p$. We consider a sequence $\left((x_k^{(i)})_{k=1}^{n}\right)_{i=1}^{\infty}$ of collections $x_1^{(i)}, \ldots, x_n^{(i)}$ of i.r.v. $x_k^{(i)} \in L_\infty$ such that $\|x_k^{(i)} - x_k\|_q \leq \frac{1}{i}$ for each $i \in \mathbb{N}$ and $k = 1, \ldots, n$. Accordingly to the proved above we have

$$\frac{\|\sum_{k=1}^{n} a_k x_k^{(i)}\|_q}{\sup_{1 \leq k \leq n} \|x_k^{(i)}\|_q} \leq 2B_{p_0}^{1-\beta} \cdot B_{p_1}^{\beta}.$$

It remains to pass i to infinity. $\square$

The following result gives an upper and a lower estimates for B_p if $p \geq 1$.

Theorem 4.2. $\sqrt{\frac{[\frac{p}{2}]}{2}} \leq B_p \leq 2\sqrt{e(p+2)}$ *for each $p \geq 1$.*

Proof. We remark that $B_p \geq B_p^{(0)}$ for each $p \geq 1$. We will show that $B_p^{(0)} \geq \sqrt{\frac{[\frac{p}{2}]}{2}}$.

Fix $p, n \in \mathbb{N}$, $n \geq 2p$ and reals $a_1 = \cdots = a_n = 1$. Observe that

$$\|a_1 r_1 + \cdots + a_n r_n\|_{2p}^{2p} = \|r_1 + \cdots + r_n\|_{2p}^{2p} = \sum_{k_1 + \cdots + k_n = p} \gamma(2k_1, \ldots, 2k_n)$$

$$\geq C_n^p \gamma(2, \ldots, 2) = \frac{n!}{p!(n-p)!} \cdot \frac{(2p)!}{(2!)^p}$$

$$= \frac{n(n-1)\ldots(n-p+1)(p+1)\ldots 2p}{2^p} \geq \left(\frac{np}{2}\right)^p,$$

that is,

$$\|r_1 + \cdots + r_n\|_{2p} \geq \sqrt{\frac{np}{2}} = \left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}} \cdot \sqrt{\frac{p}{2}}.$$

Then for every $p \geq 1$ we have

$$\|r_1 + \cdots + r_n\|_{2p} \geq \|r_1 + \cdots + r_n\|_{2[\frac{p}{2}]} \geq \left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}} \cdot \sqrt{\frac{[\frac{p}{2}]}{2}}$$

and thus, $B_p^{(0)} \geq \sqrt{\frac{[\frac{p}{2}]}{2}}$.

Now we will show that $B_p \leq 2\sqrt{e(p+2)}$. Let $p \geq 2$. Then there exists a number $q \in \mathbb{N}$ such that $2q \leq p \leq 2(q+1)$. Now choose $\alpha \in [0,1]$ such that $\frac{1}{p} = \frac{1-\alpha}{2q} + \frac{\alpha}{2q+2}$. Using theorems 2.3 and 4.1 we obtain

$$B_p \leq 2B_{2q}^{1-\alpha} \cdot B_{2q+2}^{\alpha} \leq 2\left(\sqrt{2eq}\right)^{1-\alpha} \cdot \left(\sqrt{2e(q+1)}\right)^{\alpha}$$

$$\leq 2\sqrt{e(2q+2)} \leq 2\sqrt{e(p+2)}.$$

Now let $1 \leq p < 2$. Then

$$B_p \leq B_2 \leq \sqrt{2e} < 2\sqrt{e(p+2)}.$$

$\square$

The following question naturally arises in a connection with Theorem 4.2.

Question 4.3. *Does there exist $\lim_{p \to \infty} \frac{B_p}{\sqrt{p}}$?*

References

1. J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces* I, Springer-Verlag, Berlin—Heidelberg—New York, 1977.
2. I. K. Matsak, A. N. Plichko, *The Khinchin inequality for k-multiple products of independent random variables*, Mat. Zametki **44** (1988), no. 3, 378–384. (Russian); English transl. Math. Notes **44** (1988), no. 3, 690–694.
3. V. V. Mykhaylyuk, V. A. Kholomenyuk, *Adjusting of the estimate in Khintchine inequality for independent random variables*, Scientific Bulletin of Chernivtsy University, Series Mathematics, Issue 336–337, Chernivtsy, Ruta, 2007, pp. 133–136. (Ukrainian)
4. M. L. Krasnov, A. I. Kiselyov, G. I. Makarenko, *All Higher Mathematics,* Vol. 5, *Probability, Mathematical Statistics, Game Theory*, LKI Publishing Group, Moscow, 2007. (Russian)
5. V. M. Kadets, *A Course in Functional Analysis*, V. N. Karazin Kharkiv National University, Kharkiv, 2006. (Russian)
6. H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, Berlin—Heidelberg—New York, 1974.

DEPARTMENT OF MATHEMATICAL ANALYSIS, CHERNIVTSY NATIONAL UNIVERSITY
E-mail address: `math.analysis.chnu@gmail.com`

DEPARTMENT OF MATHEMATICAL ANALYSIS, CHERNIVTSY NATIONAL UNIVERSITY
E-mail address: `trufaldinka@gmail.com`