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# ON ONE CLASS OF NONSELFADJOINT OPERATORS WITH A DISCRETE SPECTRUM

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Abstract. In this work completely continious nondissipative operators with two-dimensional imaginary parts, acting in separable Hilbert space are studied. The criteria of completeness and unconditional basis property of root vectors of such operators are obtained. The results are formulated in terms of characteristic matrix-valued functions of nonselfadjoint operators and proved using analysis of functional models in de Branges spaces.

**1.** The bounded nonselfadjoint operators $K$ with finite-dimensional imaginary parts $\operatorname{Im} K := \frac{1}{2i}(K - K^*)$ and discrete spectrum are found in different fields of analysis. For example, such operators appear while studying nonselfadjoint boundary problems for canonic systems of differential equations, in theory of control of systems with distributed parameters, while building unconditional bases of Hilbert spaces from the values of entire vector-functions [1–2]. The operators $K$ present exceptional interest when $\operatorname{Im} K$ has as positive as negative eigenvalues (nondissipative operators). Let us remark that the main difficulties while researching such operators appear even in case of $\operatorname{rank} \operatorname{Im} K = 2$.

In this paper nondissipative, nonselfadjoint operators $K$ with two-dimensional imaginary parts and with discrete spectrum are studied. The main results are formulated in terms of characteristic matrix-valued functions of the considered operators.

Let the nondissipative completely continious operator $K$ act in separable Hilbert space $\mathfrak{H}$, moreover $\operatorname{rank} \operatorname{Im} K = 2$. Then there exist linearly independent vectors $e_1, e_2 \in \mathfrak{H}$ such that

$$(i)^{-1}(K - K^*)h = i(h, e_1)e_2 - i(h, e_2)e_1, \quad h \in \mathfrak{H}. \tag{1}$$

Second-order meromorphic matrix-valued function $w(z)$, which is determined by equalities

$$\begin{gathered} w(z) = E + z\Delta(z)j, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \Delta_{\alpha\beta}(z) = \left((I - zK)^{-1} e_\beta, e_\alpha\right), \quad 1 \le \alpha, \quad \beta \le 2 \end{gathered} \tag{2}$$

is called characteristic for the operator $K$ [3]. Matrix-valued function $w(z)$ with elements $w_{kj}(z), 1 \le k, j \le 2$ is called perfect de Branges matrix [4], if the function

$$\Phi(z) := (w_{11}(z)i + w_{12}(z))/(w_{21}(z)i + w_{22}(z))$$

is holomorphic on a real line and

$$\lim_{y \to +\infty} y^{-1}\Phi(iy) = 0.$$

In the sequel the set of completely continious nondissipative operators $K$ in Hilbert space $\mathfrak{H}$ is denoted by $\mathcal{K}$. These operators satisfy the folowing conditions:

1) $\operatorname{rank} \operatorname{Im} K = 2$;

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2) the characteristic matrix-valued function $w$ of the operator $K$ is a perfect de Brange matrix.

It is not difficult to adduce the examples of the operators of the class $K$. For this let us denote by $\mathcal{K}_0$ the class of nondissipative completely continious operators with two-dimensional imaginary parts, which don't have real eigenvalues. In work [5] it is proved the inclusion $\mathcal{K}_0 \subset \mathcal{K}$.

**2.** In this subsection we formulate the main facts about functional models of operators of the class $\mathcal{K}$ in de Branges spaces.

Completely continious self-adjoint operator $\operatorname{Re} K := \frac{1}{2}(K+K^*)$, $K \in \mathcal{K}$ has a spectral decomposition

$$(\operatorname{Re} K)h = \sum_k s_k(h; u_k)u_k, \quad s_k \neq 0, \tag{3}$$

where $\{u_k\}$ is an orthonormalized system of vectors. Let us examine the matrix-valued function $q(z)$ with elements

$$q_{\alpha\beta}(z) = \frac{z}{2}((I - z\operatorname{Re} K)^{-1}e_\beta, e_\alpha), \quad 1 \leq \alpha, \quad \beta \leq 2.$$

Let us denote by $Q$ the Weierstrass product with the sequence of roots $w_k := s_k^{-1}$, where numbers $s_k$ are a part of the formula (3). It is not difficult to prove that the function

$$S(z) := iQ^2(z)\det(E - q(z)j), \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

is entire. Recall that an entire function is called real, if all the coefficients of its Maclaurin-series expansion are real.

**Proposition 1.** *Let $w$ be a characteristic matrix-valued function of the operator $K$, which is determined by equalities (1), (2). Then the following represantation is true*

$$w(z) = \frac{1}{S(z)}\begin{pmatrix} d(z) & c(z) \\ -b(z) & a(z) \end{pmatrix}, \tag{5}$$

*where $a, b, c, d$ — are entire real functions, $S$ is determined by formula (4).*

From the analytical properties of a matrix-valued function $w$ [3] it follows that an entire function

$$E(z) := a(z) - ib(z)$$

satisfies the condition $|E(\bar{z})| < |E(z)|$, $\operatorname{Im} z > 0$. That is why it generates de Branges space $\mathcal{H}(E)$, and also the function $S$ associates to it [4].

Let us examine the following operator in space $\mathcal{H}(E)$

$$(K_m F)(z) = \frac{F(z)S(0) - S(z)F(0)}{zS(0)}, \quad F \in \mathcal{H}(E).$$

In work [5] it is proved the following result.

**Proposition 2.** *Let $K$ be completely nonselfadjoint operator of the class $\mathcal{K}$, $w$ — its charscteristic matrix-valued function. Let the function $S$ also be determined by formula (4), functions $a, b$ are taken from the representation (5), $E(z) = a(z) - ib(z)$. Then the operator $K$ is unitary equivalent to the operator $K_m$ in the space $\mathcal{H}(E)$.*

Let us denote by $\Lambda = \{\lambda_k\}$ a Fredholm spectrum $F(K)$ of the operator $K$. Let $\mathfrak{N}_k$ be a root subspace, corresponding to the eigenvalue $\lambda_k^{-1}, \lambda_k \in F(K)$, $n_k := \dim \mathfrak{N}_k$. The Fredholm spectrum of the model operator $K_m$ coincides with a set of roots of the function $S$, moreover a root subspace, corresponding to the eigenvalue $\lambda_k^{-1}$, is stretched on the functions

$$(-1)^j j! S(z)/(z - \lambda_k)^{j+1}, \quad 0 \leq j < n_k, \tag{6}$$

where $n_k$ — is multiplicity of the root $\lambda_k$. Let us consider the functions

$$\varphi_{\pm}(z) := w_{22}(z) \pm i w_{21}(z), \tag{7}$$

which are built, using the elements of a lower line of a matrix-valued function $w(z)$ of the operator $K$ of the class $\mathcal{K}$.

Let us suppose that function $\varphi_+$ in every point $\lambda_k \in F(K)$ has a pole of the order $n_k = \dim \mathfrak{N}_k$. Then, under unitary equivalence, mentioned in proposition 2, root vectors (6) of the model operator turn into root vectors $K$ in the form

$$\begin{gathered} h_{kj} = \frac{\partial^j}{\partial z^j} \left\{ \varphi_+^{-1}(z)(I - zK)^{-1} u \right\} \Big|_{z=\lambda_k}, \quad 0 \le j < n_k, \\ u := e_1 - i e_2, \quad \lambda_k \in F(K). \end{gathered} \tag{8}$$

The criteria of completeness and unconditional basis property of the families (6) in de Branges spaces were received in work [5]. Taking into account these results we obtain the criteria of unconditional basis property of the families of the root vectors in the form (8) in Hilbert space $\mathfrak{H}$. To formulate the corresponding result let us recall some definitions.

Let an infinite sequence $\{\mu_k\}_1^\infty$ from the upper half-plane $\mathbb{C}_+$ satisfy the Blashke condition, i.e. the product converges

$$B(z) = \prod_{k=1}^{\infty} b_k(z),$$

where $b_k$ — an elementary Blashke multiplier, such that $b_k(\mu_k) = 0$. Let us assume $B_k(z) = B(z)/b_k(z)$. They say that sequence $\{\mu_k\}_1^\infty$ satisfies the Carleson condition [6], if

$$\inf_k |B_k(\mu_k)| > 0.$$

The Carleson condition is formulated similarly for the sequences from the lower half-plane $\mathbb{C}_-$.

Recall also that positive weight $w^2(x)$ on $\mathbb{R}$ satisfies the Muckenhoupt condition $(A_2)$, if

$$\sup_{\Delta} \left\{ M(w^2) M(w^{-2}) \right\} < \infty, \quad M(w^{\pm 2}) := \frac{1}{|\Delta|} \int_{\Delta} w^{\pm 2}(x)\, dx,$$

where $\Delta$ is an arbitrary interval on $\mathbb{R}$, $|\Delta|$ — its length.

The results about unconditional basis property of the root vectors of the operator $K$ are formulated in terms of its characteristic matrix-valued function. Let us particularly mention that operators of the $\mathcal{K}_0$ class are completely nonselfadjoint. Let us also remark that if operator $K \in \mathcal{K}_0$, then $F(K) = \Lambda_+ \cup \Lambda_-$, $\Lambda_\pm := \mathbb{C}_\pm \cap F(K)$.

**Theorem 1.** *Let $K \in \mathcal{K}_0$, $w(z)$ — its' characteristic matrix-valued function, $\{\mathfrak{N}_k : \lambda_k \in F(K)\}$ — the family of its root subspaces. If the function $\varphi_+$ in every point $\lambda_k \in F(K)$ has a pole of the order $n_k = \dim \mathfrak{N}_k$ and weight $|\varphi_+(x)|^2$ satisfies the condition $(A_2)$ on $\mathbb{R}$, then the family of the root vectors (8) forms unconditional basis of the space $\mathfrak{H}$ if and only if the following conditions are fulfilled:*
*1)* $\liminf_{y\to+\infty}(y^{-1}\log|\varphi_+(iy)|) = \liminf_{y\to-\infty}(|y|^{-1}\log|\varphi_-(iy)|) = 0$;
*2)* $\sup_k \dim \mathfrak{N}_k < \infty$;
*3) the sequences $\Lambda_+, \Lambda_-$ satisfy the Carleson condition in half-planes $\mathbb{C}_\pm$.*
*If only conditions 1), 2) are fulfilled, then the family of root vectors is complete in space $\mathfrak{H}$.*

In case, when all the root subspaces of the operator $K$ are one-dimensional and Fredholm spectrum satisfies some conditions near the real line, Theorem 1 can be ampliphied. Let us examine it more explicitly.

It follows from the properties of matrix-valued function $w$ [3], that the function

$$\varphi(z) := \varphi_-(z)/\varphi_+(z), \quad z \in \mathbb{C}_+$$

is analytic in an upper half-plane and satisfies the condition

$$|\varphi(z)| < 1, \quad z \in \mathbb{C}_+.$$

**Theorem 2.** *Let $K \in \mathcal{K}_0$, $w$ — its characteristic matrix-valued function, root subspaces are one-dimensional and $\Lambda_\pm = F(K) \cap \mathbb{C}_\pm$. In order that sequences $\Lambda_+, \Lambda_-$ satisfy the condition*

$$\sup_{\lambda_k \in \Lambda_+} |\varphi(\lambda_k)| < 1, \qquad \sup_{\lambda_k \in \Lambda_-} |\varphi(\bar{\lambda}_k)| < 1 \tag{9}$$

*and the family of the eigenvectors of the operator $K$ is unconditional basis of the space $\mathfrak{H}$ it is necessary and sufficient the fulfillment of the following conditions:*
*1)* $\liminf_{y\to+\infty}(y^{-1}\log|\varphi_+(iy)|) = \liminf_{y\to-\infty}(|y|^{-1}\log|\varphi_-(iy)|) = 0$;
*2) the weight $|\varphi_+(x)|^2$ satisfies the condition ($A_2$) on $\mathbb{R}$;*
*3) the sequences $\Lambda_+, \Lambda_-$ satisfy the Carleson condition in half-planes $\mathbb{C}_+, \mathbb{C}_-$.*

As it was mentioned above, the formulated theorems are derived from the corresponding results for the model operators [5]. In addition to this the following theorem, which is of interest, plays an important role. Recall that vectors $e_1, e_2$ form the representation (1), $\varphi_+$ is determined by equality (7).

**Theorem 3.** *Let $K \in \mathcal{K}_0$, $w$ — its characteristic matrix-valued function, $u := e_1 - ie_2$. The following conditions are equivalent:*
*1) the weight $|\varphi_+(x)|^2$ satisfies the condition ($A_2$) on $\mathbb{R}$;*
*2) for all $h \in \mathfrak{H}$ the following estimate takes place*

$$\int_{\mathbb{R}} |((I - xK)^{-1}u, h)|^2 dx \le M\|h\|^2$$

*with some constant $M > 0$.*

**3.** Let us examine now the problem of unconditional basis property of the root vectors of an arbitrary operator $K$ of the class $\mathcal{K}$. Again we start from the same representation (5) of a characteristic matrix-valued function $w$. Since entire function $E(z) = a(z) - ib(z)$ satisfies the condition $|E(\bar{z})| < |E(z)|$, $z \in \mathbb{C}_+$, then the roots of the function

$$S_\eta(z) := E^*(z) - \eta E(z), \quad |\eta| > 1, \quad E^*(z) := \overline{E(\bar{z})} \tag{10}$$

lie in a half-plane $\mathbb{C}_-$. Let

$$(\eta_0)^{-1} := \lim_{y\to+\infty} \frac{E^*(iy)}{E(iy)}$$

under the condition that this limit exists. If it is equal to 0, then let us assume that $\eta_0 = \infty$ and in this case let us add the following assumption to the formula (10):

$$S_\infty(z) = E(z).$$

In the sequel, such operators $K$ of the class $\mathcal{K}$, for which one of the following conditions is held, are considered:
*A*) $E$ — entire function of exponential type, moreover the weight $|E(x)|^2$ satisfies the condition ($A_2$) on a real line;
*B*) at some $\eta \ne \eta_0$, $|\eta| > 1$ the sequence $M = \{\mu_k\}$ of the roots of a function $S_\eta(z) = E^*(z) - \eta E(z)$ is such that:
$1_B$) $\sup_{\mu_k \in M} \operatorname{Im}\mu_k < 0$;
$2_B$) $M$ satisfies the Carleson condition in $\mathbb{C}_-$;
$3_B$) $\sup_{\mu_k} m_k < \infty$, where $m_k$ is multiplicity of root $\mu_k \in M$.

If Fredholm spectrum of the operator $K$ does not intersect some straight line $\mathbb{R} + i\varepsilon = \{x + i\varepsilon,\, x \in \mathbb{R}\}$, $\varepsilon > 0$, then let us adduce the following notations:

$$\Lambda_+^\varepsilon := \{\lambda_k - i\varepsilon,\, \operatorname{Im}\lambda_k > \varepsilon,\, \lambda_k \in F(K)\}, \quad \Lambda_-^\varepsilon := \{\lambda_k - i\varepsilon,\, \operatorname{Im}\lambda_k < \varepsilon,\, \lambda_k \in F(K)\}.$$

Also recall that the function is called the function of bounded type in $\mathbb{C}_-$, if it is representable there in the form of relation of analytical bounded functions.

**Theorem 4.** *Let the Fredholm spectrum of completely nonselfadjoint operator $K \in \mathcal{K}$ not intersect some straight line $\mathbb{R} + i\varepsilon$, $\varepsilon > 0$. Also let for the operator $K$ one of the conditions A), B) take place. The family of root vectors (8) forms unconditional basis in closure of its linear span, if the following holds:*
*1)* $\liminf_{y\to+\infty}(y^{-1}\log|\varphi_+(iy)|) = \liminf_{y\to-\infty}(|y|^{-1}\log|\varphi_-(iy)|) = 0$;
*2) the weight $|\varphi_+(x+i\varepsilon)|^2$ satisfies the condition $(A_2)$ on $\mathbb{R}$;*
*3)* $\sup_{\lambda_k\in F(K)}\dim\mathfrak{N}_k < \infty$*;*
*4) the sequences $\Lambda_+^\varepsilon, \Lambda_-^\varepsilon$ satisfy the Carleson condition in $\mathbb{C}_+, \mathbb{C}_-$ respectively.*
*If conditions 1)–4) are fulfilled, the family (8) forms unconditional basis of the space $\mathfrak{H}$ if and only if, $(z+i\varepsilon)^{-1}(\varphi_+(z+i\varepsilon)-1)$ is the function of bounded type in a half-plane $\mathbb{C}_-$.*

On account of proposition 2 it is sufficient to prove the theorem for the model operator $K_m$ in corresponding space $\mathcal{H}(E)$. Let us remark that every condition $A$), or $B$) implies a double-sided estimate

$$\int\limits_{\mathbb{R}+i\varepsilon} |F(z)/E(z)|^2 dz \asymp \int_{\mathbb{R}} |F(x)/E(x)|^2 dx$$

for all $F \in \mathcal{H}(E)$. That is why Theorem 4 is also proved by means of approach, stated in work [5].

**4.** Let us denote by $\Omega$ a class of entire functions $Q$ of an exponential type, which satisfies the following conditions:
1) $\limsup_{y\to+\infty} y^{-1}\log|Q(iy)| = \limsup_{y\to-\infty}|y|^{-1}\log|Q(-iy)| = \sigma > 0$;
2) $Q$ has only simple real roots $\{w_k\}_{-\infty}^{+\infty}$ and is the function of a sinus type [7];
3) $\inf_{k\neq j}|w_k - w_j| > 0$;
4) $\inf_{x\in\mathbb{R}}(|Q(x)|^2 + |Q'(x)|^2) > 0$;
5) $Q(0) \neq 0, \quad Q'(0) = 0$.

Let $f$ be an arbitrary real-valued Riemann integrable function on the segment $[0,\sigma]$ such that

$$\int_0^\sigma |f(t)|\,dt \le \frac{1}{2}.$$

Then every function $Q$ in the form

$$Q(z) = \cos\sigma z + \int_0^\sigma f(t)\sin(z+\frac{1}{2})t\,dt \tag{11}$$

satisfies the enumerated conditions 1)–4). Therefore it is easy to formulate additional conditions on $f$ in order that condition of norming 5) was fulfilled.

In separable Hilbert space $\mathfrak{H}$ let us consider the operator

$$Kh = Bh + (h, e_2)e_1, \quad h \in \mathfrak{H}, \tag{12}$$

where $e_1, e_2$ are linearly independent and $e_1, e_2 \in \mathfrak{H}$, self-adjoint operator $B$ acts according to the formula

$$Bh = \sum_{k=-\infty}^{\infty} w_k^{-1}(h, u_k)u_k,$$

where $\{w_k\}_{-\infty}^{+\infty}$ — the sequence of roots of some function $Q \in \Omega$, $\{u_k\}_{-\infty}^{+\infty}$ — orthonormalized basis of the space $\mathfrak{H}$. It is not difficult to verify that in case

$$|(u_k, e_k)| > 0, \quad |(u_k, e_1)| > 0, \quad k \in \mathbb{Z} \tag{13}$$

operator $K$ is completely nonselfadjoint.

If one calculates the elements of the matrix-valued function $w$ by the formulas (1), (2) we receive

$$w_{21}(z) = -z\varphi^{-1}(z)((I - zB)^{-1}e_2, e_2),$$
$$w_{22}(z) = \varphi^{-1}(z), \quad \varphi(z) := 1 - z((I - zB)^{-1}e_1, e_2).$$

Hence, it is not difficult to derive that in representation (5)

$$S(z) = \varphi(z)Q(z) = Q(z) - zQ(z)((I - zB)^{-1}e_1, e_2)m,$$
$$a(z) = Q(z), \quad b(z) = zQ(z)((I - zB)^{-1}e_2, e_2)$$

and, thus

$$E(z) = Q(z) - izQ(z)((I - zB)^{-1}e_2, e_2).$$

Let us introduce the notations for the Fourier coefficients:

$$(e_1, u_k) = \beta_k, \quad (e_2, u_k) = \alpha_k, \quad k \in \mathbb{Z}. \tag{14}$$

Then

$$E(z) = Q(z) + iz\sum_{k=-\infty}^{\infty} \frac{Q(z)w_k|\alpha_k|^2}{z - w_k}. \tag{15}$$

Let us denote by $W_2^\sigma$ the space of entire functions $f(z)$, with an exponential type not exceeding $\sigma$ with a norm

$$\|f\|^2 = \int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

It follows from the conditions 1)–3) (definition of the class $\Omega$) that the family $\{Q(z)(z - w_k)^{-1} : k \in \mathbb{Z}\}$ forms unconditional basis of the space $W_2^\sigma$ [7].

It follows from the Bernstein theorem about boundedness of a derivative on $\mathbb{R}$ and the condition $Q'(0) = 0$ that $Q'(z)/z \in W_2^\sigma$, i.e. the following representation is true

$$\frac{Q'(z)}{z} = \sum_{k=-\infty}^{\infty} \frac{Q(z)w_k|\alpha_k|^2}{z - w_k}, \quad z \in \mathbb{C}.$$

From this representation it follows that:

$$|\alpha_k|^2 = w_k^{-2}, \quad k \in \mathbb{Z}. \tag{16}$$

Thus, if one chooses Fourier coefficients $\alpha_k$ of the vector $e_2$ in order that the equalities (16) are true, then it follows the formula from (15)

$$E(z) = Q(z) + iQ'(z).$$

From the Bernstein theorem and from the condition 4) (see the definition of a class $\Omega$) it follows a double-sided estimate

$$|E(x)|^2 \asymp 1, \quad x \in \mathbb{R}.$$

Thus, if one chooses the vector $e_2$ in formula (12) in the mentioned way, then operator $K$ satisfies the condition $A$) (subsection 3).

Let us suppose now that the numbers $\beta_k$ (see (14)) are representable in the form:

$$\beta_k = w_k^{-1}\gamma_k, \quad \{\gamma_k\}_{-\infty}^{+\infty} \in l_2.$$

Then we receive the formula for the function $S$

$$\begin{gathered} S(z) = Q(z)(1 - z((I - zB)^{-1}e_1, e_2)) = \delta Q(z) + \sum_{k=-\infty}^{+\infty} \frac{Q(z)w_k\bar{\alpha}_k\gamma_k}{z - w_k}, \\ \delta := 1 + \sum_{k=-\infty}^{+\infty} \gamma_k\bar{\alpha}_k. \end{gathered} \tag{17}$$

On account of (16) the sequence $\{w_k\bar{\alpha}_k\gamma_k\}$ belongs to $l_2$ and since $Q \in \Omega$, the following equality takes place

$$S(z) = \delta Q(z) + g(z), \quad g \in W_2^\sigma. \tag{18}$$

From the Rouché's theorem it follows that if $\delta \neq 0$, then the roots $S$ (Fredholm spectrum of the operator $K$) come arbitrarily close to the real line, and that is why an attempt to apply the theorems 1, 2 to the operator $K$ most probably will not be successful.

Let us suppose that the coefficients $\alpha_k, \beta_k$ satisfy the conditions:

$$\operatorname{Re}(\bar{\alpha}_k\beta_k) = 0, \quad k \in \mathbb{Z}. \tag{19}$$

Then function $S$ has no real roots. In fact, if $S(z_0) = 0$, $z \in \mathbb{R}$, then with regard to (13), it follows from the formula (17) that

$$1 + \sum_{k=-\infty}^{+\infty} \gamma_k\bar{\alpha}_k + \sum_{k=-\infty}^{\infty} \frac{w_k\bar{\alpha}_k\gamma_k}{z - w_k} = 0.$$

Since $\operatorname{Re}(\bar{\alpha}_k\gamma_k) = 0$, this equality is impossible. Thus, if equalities (19) are held, operator $K \in \mathcal{K}$ and we apply the theorem 4 to it.

It follows from (18) that at every $\varepsilon > \varepsilon_0 > 0$

$$|S(x + i\varepsilon)|^2 \asymp |Q(x + i\varepsilon)|^2 \asymp 1, \quad x \in \mathbb{R}$$

and $F(K)$ lies under the straight line $\mathbb{R} + i\varepsilon$. Besides,

$$|E(x + i\varepsilon)|^2 = |Q(x + i\varepsilon) + iQ'(x + i\varepsilon)|^2 \asymp \left|1 + i\frac{Q'(x + i\varepsilon)}{Q(x + i\varepsilon)}\right|^2.$$

Since [8]

$$Q'(x + i\varepsilon)/Q(x + i\varepsilon) \to -i\sigma$$

at $\varepsilon \to +\infty$ uniformly on $x$, then $|E(x + i\varepsilon)|^2 \asymp 1$ at sufficiently great $\varepsilon$. Thus,

$$|\varphi_+(x + i\varepsilon)|^2 \asymp |E(x + i\varepsilon)/S(x + i\varepsilon)|^2 \asymp 1, \quad x \in \mathbb{R},$$

i.e. the condition 2) of the theorem 4) takes place. Similar reasonings show that the following limits exist

$$\lim_{y\to+\infty} \varphi_+(iy) \neq 0, \quad \lim_{y\to-\infty} \varphi_-(iy) \neq 0$$

and, consequently, the condition 1) of the theorem 4 is held. It follows from the formula (18) that since $\inf_{k\neq j} |w_k - w_j| > 0$, then the conditions 3), 4) of the Theorem 4 are also held.

And, finally, the following equality takes place:

$$z^{-1}(\varphi_+(z) - 1) = \frac{1}{z}\frac{E(z) - S(z)}{S(z)} = \frac{G(z)}{S(z)},$$

where the notation is introduced $G(z) = Q(z)((I - zB)^{-1}(e_1 - ie_2), e_2)$. It is not difficult to see that $G \in W_2^\sigma$, and with regard to the formula (18) let us introduce the function

$$\frac{G(z)}{S(z)} = \frac{e^{-i\sigma z}G(z)}{\gamma e^{-i\sigma z}Q(z) + e^{-i\sigma z}g(z)}$$

in the form of a relation of a bounded analytical functions in the area $\operatorname{Im} z < \varepsilon$. Thus, all the conditions of the theorem 4 are fulfilled. Let us formulate the received result in the form of a theorem.

**Theorem 5.** *Let in the formula (12)*

$$e_1 = \sum_{k=-\infty}^{\infty} \beta_k u_k, \quad e_2 = \sum_{k=-\infty}^{\infty} \alpha_k u_k, \quad \alpha_k\beta_k \neq 0, \quad k \in \mathbb{Z},$$

$$|\alpha_k|^2 = w_k^{-2}, \quad \beta_k = w_k^{-1}\gamma_k, \quad \mathrm{Re}\,(\bar{\alpha}_k\beta_k) = 0, \quad k \in \mathbb{Z},$$

*where* $\{\gamma_k\}_{-\infty}^{+\infty} \in l_2$, $\{w_k\}_{-\infty}^{+\infty}$ *— the set of roots of a some function of the* $\Omega$ *class. Then the family of root vectors in the form (8) of the operator* $K$ *forms an unconditional basis of the space* $\mathfrak{H}$.

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