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ON THE GROUP OF LIE-ORTHOGONAL OPERATORS ON A LIE ALGEBRA

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Abstract. Finite dimensional Lie algebras over the field of complex numbers with a linear operator $T : L \to L$ such that $[T(x), T(y)] = [x, y]$ for all $x, y \in L$ are studied. The group of such non-degenerative linear operators on L is considered. Some properties of this group and its relations with the group $\mathrm{Aut}(L)$ in the general linear group $GL(L)$ are described.

1. Introduction

In the paper [1], abelian complex structures on real Lie algebras were studied (in connection with some problems in geometry of Kaehler manifolds, see [2]). Recall that a linear operator J on a real Lie algebra L is called a complex structure if $J^2 = -E$ (where E is the identity operator on L). In the above mentioned paper, a complex structure J on L is called abelian if $[J(x), J(y)] = [x, y]$ for any $x, y \in L$. This condition on the linear operator J is similar to the condition of orthogonality for a linear operator in Euclidean spaces and therefore it is interesting to study properties of linear operators on Lie algebras which preserve their Lie brackets in the above-mentioned sense. This work is devoted to studying of such linear operators on Lie algebras (mainly finite dimensional) over the field $\mathbb{C}$ of complex numbers. The group of such linear operators which is an analogue of the orthogonal group is also studied.

A linear operator $T : L \to L$ will be called for convenience *Lie-orthogonal* if $[J(x), J(y)] = [x, y]$ for all $x, y \in L$. The set of all non-degenerative Lie-orthogonal operators on a Lie algebra L is a group which we denote by $O(L)$. The group $O(L)$ has only few common features with the automorphism group $\mathrm{Aut}(L)$ of a Lie algebra L such that $L = [L, L]$ (accordingly to Theorem 3) and vice-verse it coincides with $\mathrm{Aut}(L)$ for abelian Lie algebras.

It is also proved that if L is a semi-simple Lie algebra over the field $\mathbb{C}$ then the group $O(L)$ has the non-trivial center, a quotient group of which is either trivial or a torsion-free group.

The notations in the paper are standard. We denote by $Z(L)$ the center of a Lie algebra L, if T is a linear operator on the algebra L then we denote by $L(\lambda)$ the root space which corresponds to an eigenvalue λ of the operator T. We put $L(\lambda) = 0$ if λ is not an eigenvalue of T.

2. On the structure of a Lie algebra which admits an orthogonal linear operator $\neq \pm E$

The next results can be found in [4], Lemma 1.

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Lemma 1. *Let L be a finite dimensional Lie algebra over the field $\mathbb{C}$, T be a linear Lie-orthogonal operator on L. Then*

(1) *if the operator T is non-degenerative, then $T(Z) \subseteq Z$, where $Z = Z(L)$ is the center of L;*
(2) *if $L(0)$ is the root subspace for T corresponding to the eigenvalue $\lambda = 0$ then $L(0) \subseteq Z(L)$;*
(3) *if $L(\lambda)$ and $L(\mu)$ are root subspaces with $\lambda\mu \neq 1$ then $[L(\lambda), L(\mu)] = 0$.*

The next theorem shows the structure of finite dimensional Lie algebra without center on which a Lie-orthogonal operator is defined.

Theorem 1. *Let L be a finite dimensional Lie algebra without center over $\mathbb{C}$, T be a linear Lie-orthogonal operator on L. Then*

$$L = L(1) \oplus L(-1) \oplus (L(\alpha_1) + L(-\alpha_1)) \oplus \cdots \oplus (L(\alpha_k) + L(-\alpha_k))$$

is a direct sum of ideals where the ideals $L(\alpha_i) + L(-\alpha_i)$ with $\alpha_i \neq \pm 1$ are solvable of derived length ≤ 2.

Proof. If $T = E$ or $T = -E$ then it is obviously $L = L(1)$ or $L = L(-1)$ and all is proved.

Let

$$L = L(1) \oplus L(-1) \oplus L(\alpha_1) \oplus L(-\alpha_1) \oplus \cdots \oplus L(\alpha_k) \oplus L(-\alpha_k) \tag{1}$$

be a decomposition of L into a direct sum of the root subspaces which correspond to the eigenvalues $1, -1, \alpha_1, -\alpha_1, \ldots, \alpha_k, -\alpha_k$ of the operator T. By Lemma 1 the relation $[L(1), L(\beta)] = 0$ holds for any eigenvalue $\beta \neq 1$. Therefore the subspace

$$V = L(-1) \oplus L(\alpha_1) \oplus L(-\alpha_1) \oplus \cdots \oplus L(\alpha_k) \oplus L(-\alpha_k)$$

lies in the centralizer $C_L(L(1))$. Denote by M_1 the subalgebra of the Lie algebra L which is generated by K-subspace $L(1)$ and by M_2 – the subalgebra which is generated by the subspace V. It can be easily proved that $M_1 \cap V = 0$ (because this intersection lies in the center $Z(L)$ and $Z(L) = 0$ by the condition of the theorem). It easily follows from here that $M_1 \cap M_2 = 0$. But then, taking into account the relations $L(1) \subseteq M_1$ and $V \subseteq M_2$ we have that $M_1 = L(1)$ and $M_2 = V$. By Lemma 1 $[M_1, M_2] = 0$ and therefore M_1 and M_2 are the ideals of the Lie algebra L. Repeating these considerations for subspace $L(-1)$ from subalgebra

$$M_2 = L(-1) \oplus L(\alpha_1) \oplus L(-\alpha_1) \oplus \cdots \oplus L(\alpha_k) \oplus L(-\alpha_k)$$

we get that $L(-1)$ is an ideal of the Lie algebra L and $L = L(1) \oplus L(-1) \oplus M_3$, where

$$M_3 = L(\alpha_1) \oplus L(-\alpha_1) \oplus \cdots \oplus L(\alpha_k) \oplus L(-\alpha_k).$$

Accordingly to Lemma 1 $L(\alpha_i) \oplus L(-\alpha_i)$ is an ideal of the Lie algebra L and therefore M_3 is a direct sum of the ideals $L(\alpha_i) \oplus L(-\alpha_i)$. By Lemma 3 from [4] each sum $L(\alpha_i) \oplus L(-\alpha_i)$ is a solvable ideal of derived length ≤ 2. So, we obtain the decomposition (1). The proof is complete. ☐

3. On the group of non-degenerative linear orthogonal operators on a Lie algebra L

Lemma 2. *Let L be an arbitrary complex Lie algebra. Then all bijective linear orthogonal operators on L form a group relatively to superposition.*

Proof. If S, T are arbitrary bijective Lie-orthogonal linear operators on the Lie algebra L then

$$[ST(x), ST(y)] = [T(x), T(y)] = [x, y]$$

for any $x, y \in L$, so the product ST is a Lie-orthogonal linear operator. Further, by the condition for T there exists the inverse linear operator T^{-1}. Then

$$[T^{-1}(x), T^{-1}(y)] = [T(T^{-1}(x)), T(T^{-1}(y))] = [x, y]$$

and therefore T^{-1} is also orthogonal. □

The group of all bijective linear Lie-orthogonal operators on a Lie algebra L will be denoted by $O(L)$. If $L \neq 0$ then $O(L)$ is non-trivial. Really, the identity operator E lies in $O(L)$ as well as the operator $-E$. If L is an abelian Lie algebra then obviously $O(L) = GL(L)$ is the general linear group of the vector space L.

The next example is not so obvious.

Example 1. Let $L = \langle e_1, e_2 \mid [e_1, e_2] = e_1 \rangle$ be the 2-dimensional nonabelian Lie algebra over the field $\mathbb{C}$. Then $O(L) = SL_2(\mathbb{C})$ is the group of all linear operators T on L with $\det T = 1$.

Really, let $T \in O(L)$ be an arbitrary element and $T(e_1) = \alpha_{11}e_1 + \alpha_{21}e_2$, $T(e_2) = \alpha_{12}e_1 + \alpha_{22}e_2$, $\alpha_{ij} \in \mathbb{C}$. Then

$$\begin{aligned}[T(e_1), T(e_2)] &= [e_1, e_2] = e_1 \\ &= [\alpha_{11}e_1 + \alpha_{21}e_2, \alpha_{12}e_1 + \alpha_{22}e_2] = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})e_1.\end{aligned}$$

It follows from this that $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1$, i.e. $T \in SL_2(\mathbb{C})$. And vice versa, for any linear operator $T \in SL_2(\mathbb{C})$ we have the relation

$$[T(e_1), T(e_2)] = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})e_1 = [e_1, e_2] = e_1,$$

so $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1$, i.e. $T \in O(L)$. Hence, $O(L) \simeq SL_2(\mathbb{C})$.

Lemma 3. *Let L be an arbitrary complex Lie algebra, I be an ideal of the algebra L such that the center of the quotient algebra L/I is zero. Then the ideal I is invariant under action of the Lie-orthogonal group $O(L)$, i.e. $T(I) \subseteq I$ for any $T \in O(L)$.*

Proof. Let us show at first that the $\mathbb{C}$-subspace $I + T(I)$ is an ideal of the Lie algebra L for any $T \in O(L)$. Really, for arbitrary $x \in I, y \in L$ we have

$$[T(x), y] = [T(x), T(T^{-1}(y))] = [x, T^{-1}(y)] \in I.$$

If the element $T(y)$ goes through the whole Lie algebra L then the element y goes also through the algebra L, so we have $[T(I), L] \subseteq I$. Therefore $I + (T(I)$ is an ideal of the Lie algebra L for any $T \in O(L)$.

Now assume that for some $T \in O(L)$ we have a strong inclusion $I \subsetneq I + T(I)$. Then it holds for the quotient algebra $[I + T(I)/I, L/I] = 0$ (because it has been just shown that $[T(I), L] \subseteq I$). The last equality is impossible because according to the condition the center of L/I is zero. The proof is complete. □

Corollary 1. *Let L be a finite dimensional complex Lie algebra. Then the solvable radical $S(L)$ is invariant under action of the group $O(L)$.*

Lemma 4. *Let L be an arbitrary Lie algebra over the field $\mathbb{C}$, $Z(L)$ be the center of L. Then $Z(L)$ is invariant under action of the group $O(L)$.*

Proof. Assume that for an element $z \in Z(L)$ its image $T(z) \notin Z(L)$ for some $T \in O(L)$. Then there exists $x \in L$ such that $[T(z), x] \neq 0$. So we have

$$[T(z), x] = [T(z), T(T^{-1}(x))] = [z, T^{-1}(x)] \neq 0.$$

But it is impossible since $z \in Z(L)$. This contradiction shows that $T(Z(L)) \subseteq Z(L)$. □

Corollary 2. *Let L be a nilpotent complex Lie algebra. Then the upper central series of the algebra L*

$$0 = Z_0(L) \subseteq Z_1(L) \subseteq \cdots \subseteq Z_n(L) = L$$

is invariant under action of the group $O(L)$.

Theorem 2. *Let L be a finite dimensional semi-simple Lie algebra over the field $\mathbb{C}$. Then the group $O(L)$ contains a central normal divisor N which is an elementary abelian group of order 2^k, where k is the number of simple ideals in the decomposition of L into a direct sum of simple ideals. The quotient group $O(L)/N$ is either identity or torsion-free.*

Proof. Let $L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$ be a decomposition of L into a direct sum of simple ideals $L_i, i = 1, \dots, k$. By Lemma 2 every ideal L_i is invariant under action of the group $O(L)$ and therefore

$$O(L) = O(L_1) \times O(L_2) \times \cdots \times O(L_k).$$

So, to prove the theorem we can consider only the case of simple Lie algebras. It is obvious that linear operators E and $-E$ on L are Lie-orthogonal and lie in the center of $O(L)$. Assume that $O(L) \neq \{E, -E\}$ and take an arbitrary element $T \in O(L)\setminus\{E, -E\}$. Consider the decomposition of the vector space L over $\mathbb{C}$ into a direct sum of root subspaces

$$L = L(1) \oplus L(-1) \oplus L(\alpha_1) \oplus L(-\alpha_1) \oplus \cdots \oplus L(\alpha_k) \oplus L(-\alpha_k),$$

that correspond to the eigenvalues $1,\ -1,\ \alpha_1, -\alpha_1, \dots, \alpha_k, -\alpha_k$ of the operator L (if λ is not an eigenvalue of T we put $L(\lambda) = 0$). Since the Lie algebra L is simple we obtain using Theorem 1 that either $L = L(1)$ or $L = L(-1)$. Let at first $L = L(1)$. Then the linear operator T has on L the only eigenvalue $\lambda = 1$. Choosing an appropriate Jordan basis we see that the matrix of T is a direct sum of cells $\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 1 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$.

Since $T \neq E$ at least one of the cells is of order > 1. But then $T^n \neq E$ for any n. The similar proposition is true in the case of $L = L(-1)$. It easily follows from this that $O(L)/\{E, -E\}$ does not have elements of finite order. The proof is complete. $\square$

Note that the group $O(L)$ as well as the group $\mathrm{Aut}(L)$ of all automorphisms of the Lie algebra L are subgroups of the general linear group $GL(L)$ of the vector space L. The following statement shows in general how the subgroups $O(L)$ and $\mathrm{Aut}(L)$ are embedded in the group $GL(L)$.

Theorem 3. *Let L be a Lie algebra over the field $\mathbb{C}$ (not necessarily finite dimensional). If $L = [L, L]$ then for the subgroups $O(L)$ and $\mathrm{Aut}(L)$ of the general linear group $GL(L)$ the following holds:*

$$\mathrm{Aut}(L) \cap O(L) = \{E\}, \quad [O(L), \mathrm{Aut}(L)] \subseteq O(L).$$

In particular, $O(L) \cdot \mathrm{Aut}(L) = O(L) \leftthreetimes \mathrm{Aut}(L)$ is a semi-direct product of two groups.

Proof. Let θ be an arbitrary automorphism of the Lie algebra L and T is an arbitrary orthogonal operator on L. Then for any elements $x, y \in L$ we have

$$[\theta^{-1}T\theta(x), \theta^{-1}T\theta(y)] = \theta^{-1}[T\theta(x), T\theta(y)] = \theta^{-1}[\theta(x), \theta(y)] = [x, y].$$

It means that $\theta^{-1}T\theta \in O(L)$, that is $[\mathrm{Aut}(L), O(L)] \subseteq O(L)$.

Let φ be an arbitrary element from the intersection $\mathrm{Aut}(L) \cap O(L)$. Than for any elements $x, y \in L$ we have

$$[\varphi(x), \varphi(y)] = [x, y] = \varphi([x, y]).$$

It means that φ acts as identity on all elements from the derived algebra $[L, L]$. Since by condition $L = [L, L]$ we obtain $\varphi = E$. Thus, $\mathrm{Aut}(L) \cap O(L) = \{E\}$. The proof is complete. $\square$

We show another construction of orthogonal linear operators on Lie algebras with nonzero center.

Proposition 1. *Let L be an arbitrary Lie algebra over the field $\mathbb{C}$ and S be a linear map $S : L \to Z(L)$ such that $T = E + S$ is a bijective linear operator on L. Then T is an orthogonal operator on L.*

Proof. Take arbitrary elements $x, y \in L$ and consider the equalities

$$\begin{aligned}[T(x), T(y)] &= [(E+S)(x), (E+S)(y)] = [x + S(x), y + S(y)] \\ &= [x, y] + [x, S(y)] + [S(x), y] + [S(x), S(y)] = [x, y],\end{aligned}$$

because $S(x), S(y) \in Z(L)$ We see that $E + S$ is an orthogonal linear operator on L. $\square$

Example 2. Let $L = \langle e, f, h \mid [e, f] = h, [h, e] = 2e, [h, f] = -2f\rangle$ be the simple 3-dimensional complex Lie algebra. Then $O(L) = \{\pm E\}$.

Really, according to Theorem 1 we can assume without loss of generality that every linear operator $T \in O(L)\backslash\{\pm E\}$ has only one eigenvalue $\lambda = 1$. Then the operator T has in a Jordan basis $\{e_1, e_2, e_3\}$ a matrix which is either Jordan cell of size 3×3 or a direct sum of two Jordan cells of order 1 and 2.

Consider at first the Jordan cell of order 3. We can write down in the above mentioned basis $T(e_1) = e_1, T(e_2) = e_2 + e_1, T(e_3) = e_3 + e_2$. Then we obtain

$$[T(e_1), T(e_3)] = [e_1, e_3] = [e_1, e_3 + e_2]$$

and therefore $[e_1, e_2] = 0$. Further,

$$[T(e_2), T(e_3)] = [e_2, e_3] = [e_2 + e_1, e_3 + e_2] = [e_2, e_3] + [e_1, e_3]$$

and therefore $[e_1, e_3] = 0$. This is impossible because then $e_1 \in Z(L)$ which contradicts to that L is a simple Lie algebra.

Let now the matrix of T in the basis $\{e_1, e_2, e_3\}$ be a direct sum of Jordan cells of orders 1 and 2. Then we can write down the following:

$$T(e_1) = e_1, T(e_2) = e_2, T(e_3) = e_3 + e_2.$$

So, we have $[T(e_1), T(e_3)] = [e_1, e_3] = [e_1, e_3 + e_2]$. Thus, $[e_1, e_2] = 0$ and therefore the simple Lie algebra L contains an abelian subalgebra of dimension 2. The latter is impossible. This contradiction shows that $O(L) = \{\pm E\}$.

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