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ON GENERALIZED SELFADJOINT OPERATORS ON SCALES OF HILBERT SPACES

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Abstract. We consider examples of generalized selfadjoint operators that act from a positive Hilbert space to a negative space. Such operators were introduced and studied in [1]. We give examples of selfadjoint operators on the principal Hilbert space H_0 that, being considered as operators from the positive space $H_+ \subset H_0$ into the negative space $H_- \supset H_0$, are not essentially selfadjoint in the generalized sense.

1. Introduction

Let us first give a definition of a generalized selfadjoint operator and some results from [1]. Consider a chain of Hilbert spaces (rigging)

$$H_- \supset H_0 \supset H_+, \quad ||\cdot||_{H_-} \leq ||\cdot||_{H_0} \leq ||\cdot||_{H_+}. \tag{1}$$

We will assume that the positive Hilbert space H_+ is dense in the principal space H_0. The space H_- is negative and consists of antilinear functionals defined on H_+. Here, the scalar product $(\cdot,\cdot)_{H_0}$ on H_0 is a continuous sesquilinear form on $H_- \times H_+$ that, being extended, defines a pairing $< \cdot, \cdot >$ between elements of H_- and H_+ [2, 3],

$$< \alpha, \varphi > = \overline{< \varphi, \alpha >}, \quad \alpha \in H_-, \quad \varphi \in H_+. \tag{2}$$

There is a standard procedure [2, 3] that relates chain (1) to an operator $\mathbb{I} : H_- \to H_+$ that is an isometry from H_- onto H_+,

$$\forall \alpha, \beta \in H_- \quad (\alpha, \beta)_{H_-} = < \alpha, \mathbb{I}\beta > = < \mathbb{I}\alpha, \beta > . \tag{3}$$

Let A be an operator that acts from H_+ into H_- with dense domain $\mathfrak{D}(A)$ in H_+. The operator A^+ that acts from H_+ into H_- is called adjoint to the operator A in the generalized sense (in the sense of chain (1)) if its domain $\mathfrak{D}(A^+)$ consists of all elements $\psi \in H_+$ such that there exists $\psi_- \in H_-$ satisfying $< A\varphi, \psi > = < \varphi, \psi_- >$ $\forall \varphi \in \mathfrak{D}(A)$ and $A^+\psi = \psi_-$ [1].

In the case when $H_+ = H_0 = H_-$, this definition coincides with the usual definition of an adjoint operator on a Hilbert space [4, 3]. By using the operator $\mathbb{I}$ defined in (3), it is easy to see that $\mathbb{I}A$ acts from H_+ into H_+ and its adjoint on H_+ is $(\mathbb{I}A)^*$ that coincides with $\mathbb{I}A^+$.

If the operator A^+ coincides with A, then such an operator will be called generalized selfadjoint [1]. In this case, the operator $A_+ = \mathbb{I}A$ is selfadjoint on the space H_+. Conversely, every operator A_+ that is selfadjoint on the space H_+ gives rise to a generalized selfadjoint operator, $A = (\mathbb{I})^{-1}A_+$, from the space H_+ into the space H_-. Note that there is a number of problems in physics connected with selfadjoint operators on the Sobolev spaces $W_2^s(\mathbb{R}^n)$ [5, 6].

If $< A\varphi, \psi > = < \varphi, A\psi >$ for all $\varphi, \psi \in \mathfrak{D}(A) \subset H_+$, then the operator A is called generalized Hermitian. The closure $\bar{A}$ of a generalized Hermitian operator A always exists and is a generalized Hermitian operator. If the closure $\bar{A}$ of an operator A that

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acts from H_+ into H_- is generalized selfadjoint, $\bar{A}^+ = \bar{A}$, then the operator A is called generalized essentially selfadjoint. It is clear that a bounded generalized Hermitian operator A acting from H_+ into H_- and having dense domain $\mathfrak{D}(A)$ in H_+ is essentially selfadjoint. Hence, it is also true that a generalized Hermitian operator defined on the whole space H_+ is a bounded selfadjoint operator acting from H_+ into H_-.

2. Selfadjoint differential operators with constant coefficients

Consider rigging (1), where $H_+ = W_2^s(\mathbb{R}^n)$ is the Sobolev space of functions $\varphi(x)$ defined on $x \in \mathbb{R}^n$ by the following norm for the integer s:

$$||\varphi||^2_{W_2^s(\mathbb{R}^n)} = \sum_{|\alpha|\le s} \int_{\mathbb{R}^n} |\varphi^{(\alpha)}(x)|^2\,dx, \quad \varphi^{(\alpha)}(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\varphi(x). \tag{4}$$

The principal space H_0 is the space $L_2(\mathbb{R}^n)$. The Fourier transform of a function φ

$$\tilde{\varphi}(\xi) = \int e^{i\xi\cdot x}\varphi(x)\,dx \tag{5}$$

belongs to the space $L_2(\mathbb{R}^n, \rho_s(\xi))$, where the weight is $\rho_s(\xi) = (1 + |\xi|^2)^s$ and the norm is defined by

$$||\tilde{\varphi}||^2_{L_2(\mathbb{R}^n,\rho_s(\xi))} = \int_{\mathbb{R}^n} |\tilde{\varphi}(\xi)(x)|^2\rho_s(\xi)\,d\xi. \tag{6}$$

By the Parceval identity, the W_2^s-norm of functions φ is equivalent to the $L_2(\mathbb{R}^n, \rho_s(\xi))$-norm of $\tilde{\varphi}$, the Fourier transform of the functions φ. This permits to define the Sobolev spaces $W_2^s(\mathbb{R}^n)$ for arbitrary real s. Note that, for $s > 0$, $W_2^{-s}(\mathbb{R}^n)$ is a negative space relatively to the positive space $W_2^s(\mathbb{R}^n)$ and the space $H_0 = L(\mathbb{R}^n)$.

Consider a differential expression with constant coefficients,

$$\mathcal{P}(\mathcal{D}) = \sum_{|\alpha|\le m} a_\alpha \mathcal{D}^\alpha, \tag{7}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an integer multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_n^{\alpha_n}$, $\mathcal{D}_k = \frac{1}{i}\frac{\partial}{\partial x_k}$, and a_α are constant complex coefficients. Restricted to compactly supported infinitely differentiable functions in $C_0^\infty(\mathbb{R}^n)$, the differential expression $\mathcal{P}(\mathcal{D})$ defines a minimal operator P, $P\varphi = \mathcal{P}(\mathcal{D})\varphi(x)$, on the space $L_2(\mathbb{R}^n)$. The operator P is Hermitian if and only if all the coefficients a_α in the differential expression (7) are real.

Theorem 1. *Let $\mathcal{P}(\xi) = \sum_{|\alpha|\le m} a_\alpha\xi^\alpha$ be a polynomial of n variables $\xi = (\xi_1, \dots, \xi_n)$ with real coefficients. Then the minimal operator P defined on $C_0^\infty(\mathbb{R}^n)$ by the differential expression $\mathcal{P}(\mathcal{D}) = \sum_{|\alpha|\le m} a_\alpha\mathcal{D}^\alpha$ is a generalized essentially selfadjoint operator from the Sobolev space $W_2^s(\mathbb{R}^n)$ into $W_2^{-s}(\mathbb{R}^n)$ for arbitrary $s \ge 0$.*

Proof. Passing to the Fourier transform, we need to prove that the operator of multiplication by the polynomial $\mathcal{P}(\xi)$ is generalized essentially selfadjoint acting from the space $L_2(\mathbb{R}^n, \rho_s)$ to the space $L_2(\mathbb{R}^n, \rho_{-s})$. Since the multiplication operator commutes with a weight, the claim follows, because the operator of multiplication by $\mathcal{P}(\xi)$ in the space $L_2(\mathbb{R}^n)$ is essentially selfadjoint if its domain is $\mathfrak{D}$, the space of all functions that are images of the Fourier transforms of functions in $C_0^\infty(\mathbb{R}^n)$, and the set $\mathfrak{D}$ is dense in $L_2(\mathbb{R}^n)$. □

Remark 1. Theorem 1 shows that example 3.4 in [1] is erroneous (the error of this example was indicated by the authors of [1] earlier).

Remark 2. If $s \geq \frac{m}{2}$, where m is the order of the differential operator, then the closure of the differential operator P is defined on the whole space $W_2^s(\mathbb{R}^n)$, and is a bounded selfadjoint operator from $W_2^s(\mathbb{R}^n)$ into $W_2^{-s}(\mathbb{R}^n)$.

Remark 3. If $\mathcal{P}(\mathcal{D})$ is an elliptic differential expression, then the closure $\bar{P}$ of the differential operator from $W_2^s(\mathbb{R}^n)$ into $W_2^{-s}(\mathbb{R}^n)$ is a generalized essentially selfadjoint operator with the domain $\mathfrak{D}(\bar{P}) = W_2^{m-s}$ for $s \leq \frac{m}{2}$.

3. A multiplication operator in a chain of Sobolev spaces

If an operator A acts on a space H_0, then it can be considered, with respect to scale (1), as an operator $\hat{A}$ from the space H_+ into H_- by setting $\mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \cap H_+$ and $\hat{A}\varphi = A\varphi \in H_0 \subset H_-$.

It was claimed in [1] that an operator A, which is selfadjoint on H_0, being considered as an operator from H_+ into H_-, i.e., the operator $\hat{A}$, may or may not be essentially selfadjoint. Consider an example.

Example 1. Let $H_0 = L_2(\mathbb{R}^1)$, $H_+ = W_2^1(\mathbb{R}^1)$, $H_- = W_2^{-1}(\mathbb{R}^1)$. The operator A of multiplication by the function $\frac{1}{x}$ will be selfadjoint on $L_2(\mathbb{R}^1)$, with the domain $\mathfrak{D}(A) = \{u : u \in L_2, \frac{1}{x}u(x) \in L_2\}$. It is easy to see that domain of the operator $\hat{A}$ is

$$\mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \cap W_2^1(\mathbb{R}^1) = \{u : u \in W_2^1(\mathbb{R}^1), u(0) = 0\}. \tag{8}$$

The set $\mathfrak{D}(\hat{A})$ is not dense in the space $W_2^1(\mathbb{R}^1)$. Indeed, we fix $\varphi_0 = \frac{1}{2}e^{-|x|} \in W_2^1(\mathbb{R}^1)$ and $(u, \varphi_0)_{W_2^1(\mathbb{R}^1)} = u(0)$, $\forall u \in W_2^1(\mathbb{R}^1)$. Hence, φ_0 is orthogonal to $\mathfrak{D}(\hat{A})$ in the space $W_2^1(\mathbb{R}^1)$. Since $\mathfrak{D}(\hat{A})$ is not dense in the space $H_+ = W_2^1(\mathbb{R}^1)$, the operator $\hat{A}$ can not be selfadjoint, if considered as an operator from H_+ into H_-.

Theorem 2. *An operator $\hat{A}$ defined by $\hat{A}u = \frac{1}{x}u(x)$ on functions $u \in W_2^1(\mathbb{R}^1)$ such that $u(0) = 0$ is a generalized Hermitian operator from $H_+ = W_2^1(\mathbb{R}^1)$ into $H_- = W_2^{-1}(\mathbb{R}^1)$ and admits a selfadjoint extension $\tilde{A}$ to the whole space $H_+ = W_2^1(\mathbb{R}^1)$,*

$$\tilde{A}u(x) = (\mathcal{P})\frac{u(x)}{x}, \tag{9}$$

where the functional $(\mathcal{P})\frac{u(x)}{x} \in H_- = W_2^{-1}(\mathbb{R}^1)$ is the Cauchy principal value, that is,

$$\begin{aligned} < (\mathcal{P})\frac{u(x)}{x}, \varphi(x) > &= (\mathcal{P})\int \frac{u(x)\overline{\varphi(x)}}{x}\,dx = \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^1\setminus[-\varepsilon,\varepsilon]} \frac{u(x)\overline{\varphi(x)}}{x}\,dx \\ &= \int\limits_{\mathbb{R}^1\setminus[-1,1]} \frac{u(x)\overline{\varphi(x)}}{x}\,dx + \int_{-1}^{1} \frac{u(x)\overline{\varphi(x)} - u(0)\overline{\varphi(0)}}{x}\,dx, \quad \varphi \in W_2^1(\mathbb{R}^1). \end{aligned} \tag{10}$$

All the generalized selfadjoint extensions of the operator $\hat{A}$, which act from $W_2^1(\mathbb{R}^1)$ into $W_2^{-1}(\mathbb{R}^1)$ are given by the identity

$$\tilde{A}_\varepsilon = \tilde{A} + \varepsilon\delta < \cdot, \delta > = \tilde{A} + \varepsilon\delta(\cdot, \varphi_0)_{W_2^1(\mathbb{R}^1)}, \tag{11}$$

where ε is an arbitrary real number and $\delta \in W_2^{-1}(\mathbb{R}^1)$ is the Dirac δ-functional, $< \varphi, \delta > = \varphi(0)$.

Proof. It is clear that the operator $\tilde{A}$ is an extension of the bounded operator $\hat{A}$ if considered as an operator from $H_+ = W_2^1(\mathbb{R}^1)$ to $H_- = W_2^{-1}(\mathbb{R}^1)$. It immediately follows from definition (10) of the operator $\tilde{A}$ that

$$| < \hat{A}u, \varphi > | \leq c||u||_{H_+}||\varphi||_{H_+},$$

this operator $\hat{A}$ is generalized Hermitian and defined on the whole space $H_+ = W_2^1(\mathbb{R}^1)$. Hence, $\tilde{A}$ is generalized essentially selfadjoint.

If $\tilde{\tilde{A}}$ is an arbitrary selfadjoint extension of the operator $\hat{A}$ from $H_+ = W_2^1(\mathbb{R}^1)$ into $H_- = W_2^{-1}(\mathbb{R}^1)$, then the operator $\tilde{\tilde{A}} - \tilde{A}$ is distinct from zero only on a one-dimensional subspace that contains $\varphi_0 = \frac{1}{2}e^{-|x|} \in W_2^1(\mathbb{R}^1)$, and it is generalized Hermitian. But then there exists a real number ε such that $\mathbb{I}(\tilde{\tilde{A}} - \tilde{A}) = \varepsilon\varphi_0(\cdot, \varphi_0)_{W_2^1}$. This implies (11), since $\varphi_0 = \mathbb{I}\delta$. □

4. Differential operators with boundary-value conditions

Example 2. On the space $L_2(\mathbb{R}^1)$, consider a selfadjoint differential operator A given by the differential expression $-\frac{d^2}{dx^2}$ on functions such that their restrictions to the positive and the negative semiaxes belong to the Sobolev spaces $W_2^2(0, \infty)$ and $W_2^2(-\infty, 0)$, correspondingly. The functions in the domain of the operator A satisfy, in the point $x = 0$, the adjacency conditions

$$u(+0) = u(-0) \equiv u(0), \quad u'(+0) - u'(-0) = \alpha u(0), \tag{12}$$

where $\alpha \neq 0$ is a real number. We have $Au(x) = -u''(x)$ for $x \neq 0$. Such an operator is used in quantum mechanics [7] and corresponds to a one-dimensional Schrödinger operator with the intensity α point interaction in the point $x = 0$.

Consider chain (1) with $H_+ = W_2^2(\mathbb{R}^1)$, $H_- = W_2^{-2}(\mathbb{R}^1)$, $H_0 = L_2(\mathbb{R}^1)$. Let the operator A be considered as acting from H_+ in H_-, that is, the operator $\hat{A}$ is defined on the set

$$\mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \cap H_+ = \{u : u \in W_2^2(\mathbb{R}^1),\ u(0) = 0\}. \tag{13}$$

It is easy to see that $\mathfrak{D}(\hat{A})$ is not dense in the space $H_+ = W_2^2(\mathbb{R}^1)$. This follows, in particular, from the fact proved in Section 3 that $\{u : u \in W_2^1(\mathbb{R}^1),\ u(0) = 0\}$ is not dense in $W_2^1(\mathbb{R}^1)$ since $W_2^2(\mathbb{R}^1)$ is dense in $W_2^1(\mathbb{R}^1)$. Hence, the operator $\hat{A}$ is not essentially selfadjoint.

Theorem 3. *Let A be the selfadjoint operator on $H_0 = L_2(\mathbb{R}^1)$ considered in Example 2. Let $H_+ = W_2^1(\mathbb{R}^1)$ and $H_- = W_2^{-1}(\mathbb{R}^1)$. Then the operator $\hat{A}$ is a generalized essentially selfadjoint bounded operator from H_+ into H_-. The closure $\bar{A}$ of this operator $\hat{A}$ admits the representation*

$$\bar{A} = -\frac{d^2}{dx^2} + \alpha\delta < \cdot, \delta >,$$

where δ is the Dirac delta-function.

Proof. This operator $\bar{A}$ is a sum of two bounded operators from H_+ into H_-. Let us show that it is an extension of the operator $\hat{A}$. Let $u \in \mathfrak{D}(\hat{A})$. Then $\hat{A}u = -u''(x)$ for $x \neq 0$. However, for an arbitrary function $\varphi \in \mathbb{C}_0^\infty$, we have $< \bar{A}u, \varphi > = < u, \bar{A}\varphi > = < u, -\varphi'' > + \alpha u(0)\bar{\varphi}(0) = (u, -\varphi'') + \alpha u(0)\bar{\varphi}(0) = (-u'', \varphi)$. Here we have used the integration by parts formula and the condition $u'(+0) - u'(-0) = \alpha u(0)$. Hence, $\bar{A}u = \hat{A}u$. Since $\mathfrak{D}(\hat{A})$ is a dense subset of H_+, the operator $\hat{A}$ coincides on $\mathfrak{D}(\hat{A})$ with the bounded operator $\bar{A}$. □

In Example 1, the operator $\hat{A}$ was not generalized essentially selfadjoint, since its domain $\mathfrak{D}(\hat{A})$ is not dense in H_+. However, such an operator has selfadjoint extensions given in Theorem 2. There is a question of whether the operator $\hat{A}$ would always have selfadjoint extensions if the set $\mathfrak{D}(\hat{A})$ is dense in H_+ [1]. A negative answer to this question is given by the example of [8].

Theorem 4. *Let A be the selfadjoint differential operator $i\frac{d}{dx}$ defined in the space $H_0 = L_2(0,1)$ on the set $W_2^1(0,1)$ of functions that satisfy the boundary condition $u(0) = u(1)$. Let $H_+ = L_2((0,1),\rho)$ be the space of functions that are square integrable on the interval $(0,1)$ with the weight $\rho \geq 1$, where $\rho(x)$ is a continuous unbounded function on the interval $(0,1)$, and $H_- = L_2((0,1),\rho^{-1})$. Then the operator $\hat{A}$ is densely defined on the space H_+. We have the following statements:*

1) *If $\int_0^1 \rho(x)\,dx = \infty$ then $\mathfrak{D}(\hat{A}) \equiv \mathfrak{D}(A) \cap H_+ \subset \overset{\circ}{W}{}_2^1 \equiv \{u : u \in W_2^1(0,1), u(0) = u(1) = 0\}$.*
2) *If $\rho = [x(1-x)]^{-1}$ then operator $\hat{A}$ is generalized essentially selfadjoint.*
3) *If $\rho = x^{-1}$ then operator $\hat{A}$ is not generalized essentially selfadjoint and, moreover, does not have generalized selfadjoint extensions as the operator from H_+ into H_-.*

Proof. For $\varphi_+ \in H_+$, it is necessary and sufficient that there would exist a function $\varphi \in L_2(0,1)$ and $\varphi_+(x) = \rho^{-\frac{1}{2}}(x)\varphi(x)$. For functions $\varphi_- \in H_-$, we have the representation $\varphi_- = \rho^{\frac{1}{2}}(x)\varphi(x)$. Here, $||\varphi_+||_{H_+} = ||\varphi||_{L_2}$, $||\varphi_-||_{H_-} = ||\varphi||_{L_2}$. The domain of the operator $\hat{A}$ is $\mathfrak{D}(A) \cap H_+$. Since $\mathfrak{D}(\hat{A}) \supset C_0^\infty(0,1)$ and the space $C_0^\infty(0,1)$ is dense in H_+, the set $\mathfrak{D}(\hat{A})$ is dense in H_+.

Statement 1) is obvious since, from the condition $u \in \mathfrak{D}(\hat{A})$, it follows that function $u(x)$ is continuous on the interval $[0,1]$ and $u(0) = u(1)$. However, if this function belongs to H_+ then $u(0) = u(1) = 0$ that is $\mathfrak{D}(\hat{A}) \subset \overset{\circ}{W}{}_2^1$. It is easy to prove that in the cases 2) and 3) we have $\mathfrak{D}(\hat{A}) = \overset{\circ}{W}{}_2^1$.

Existence of generalized selfadjoint extensions of the operator $\hat{A}$ is equivalent to existence of selfadjoint extensions of the operator $\mathbb{I}\hat{A}$ on H_+. Let us find deficiency indices of this symmetric operator. To this end, find functions $\eta_\pm(x) \in H_+$ that satisfy the identities

$$(\mathbb{I}\hat{A}u \pm iu, \eta_\pm)_{H_+} = 0 \tag{14}$$

for arbitrary $u \in \mathfrak{D}(\hat{A})$. Note that the operator $\mathbb{I}$ is a multiplication operator, $\mathbb{I}\varphi_-(x) = \rho^{-1}(x)\varphi_-(x)$. The identity (14) can be written in an equivalent form,

$$\int [i\frac{d}{dx} \pm i\rho(x)]u(x)\bar{\eta}_\pm(x)\,dx = 0. \tag{15}$$

Since $u \in \mathfrak{D}(\hat{A}) \supset C_0^\infty(0,1)$, it follows from (15) that the functions $\eta_\pm$ are generalized solutions of the equations

$$[i\frac{d}{dx} \mp i\rho(x)]\eta_\pm(x) = 0. \tag{16}$$

In the case where $\rho = [x(1-x)]^{-1}$, equation (16) has solutions $\eta_+ = x(1-x)^{-1}$ and $\eta_- = x^{-1}(1-x)$ that do not belong to the space H_+. Hence, the deficiency indices of the symmetric operator $\mathbb{I}\hat{A}$ are zero then the operator is essentially selfadjoint [4, 3].

If $\rho = x^{-1}$, equation (16) has solutions $\eta_+ = x$ and $\eta_- = x^{-1}$. The function $\eta_+ \in H_+$, and the function η_- does not belong to the space H_+. Hence, the deficiency indices of the operator $\mathbb{I}\hat{A}$ are $(1,0)$, and since they are different, the operator $\mathbb{I}\hat{A}$ does not admit selfadjoint extensions on H_+. □

Remark 4. As it has been shown in examples of selfadjoint operators A on the principal Hilbert space H_0 of theorems 1–4, the operator $\hat{A}$ may be essentially selfadjoint, if considered as an operator from H_+ into H_-, or may be not. This fact does not depend on essential selfadjointness of $A_+ = A \restriction_{\mathfrak{D}(A)\cap H_+}$ in H_0. All four cases are possible.

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