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ON $(\bigwedge, \mu)$-CLOSED SETS IN GENERALIZED TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study $(\bigwedge, \mu)$-open sets and $(\bigwedge, \mu)$-closed sets via μ-open and μ-closed sets in generalized topological spaces. Moreover, we introduce some generalized separation axioms in generalized topological spaces.

1. Introduction

In the past few years, different forms of open sets have been studied. Recently, a significant contribution to the theory of generalized open sets, was extended by A. Császár. Especially, the author defined some basic operators on generalized topological spaces. It is observed that a large number of papers is devoted to the study of generalized open like sets of a topological space, containing the class of open sets and possessing properties more or less similar to those of open sets.

We recall some notions defined in [1]. Let X be a non-empty set, $\exp X$ denotes the power set of X. We call a class $\mu \subseteq \exp X$ a generalized topology [1], (briefly, GT) if $\varnothing \in \mu$ and union of elements of μ belongs to μ. A set X with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ).

For a GTS (X, μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [1, 2]).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : expX \to expX$ is said to be idempotent if $A \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if $A \subseteq B \subseteq X$ implies $\gamma(A) \subseteq \gamma(B)$. It is also well known from [2, 3] that if μ is a GT on X, $x \in X$ and $A \subseteq X$, then $x \in c_\mu(A)$ iff for $x \in M \in \mu$, $M \cap A \neq \varnothing$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

2. $\bigwedge_\mu$-sets, $(\bigwedge, \mu)$-closed sets and some separation axioms

Definition 2.1. [4]. Let (X, μ) be a GTS and $A \subseteq X$. Then the subset $\bigwedge_\mu(A)$ is defined by

$$\bigwedge_\mu(A) = \begin{cases} \bigcap\{G : A \subseteq G, G \in \mu\}, & \text{if there exists } G \in \mu \text{ such that } A \subseteq G, \\ X, & \text{otherwise.} \end{cases}$$

Proposition 2.2. [4]. Let A, B and $\{B_\alpha : \alpha \in \Omega\}$ be subsets of a GTS (X, μ). Then the following properties hold :
(a) $B \subseteq \bigwedge_\mu(B)$;
(b) If $A \subseteq B$, then $\bigwedge_\mu(A) \subseteq \bigwedge_\mu(B)$;
(c) $\bigwedge_\mu(\bigwedge_\mu(B)) = \bigwedge_\mu(B)$;

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(d) If $A \in \mu$, then $A = \bigwedge_\mu(A)$;
(e) $\bigwedge_\mu[\bigcup_{\alpha\in\Omega} B_\alpha] = \bigcup_{\alpha\in\Omega}[\bigwedge_\mu(B_\alpha)]$;
(f) $\bigwedge_\mu[\bigcap_{\alpha\in\Omega} B_\alpha] \subseteq \bigcap_{\alpha\in\Omega}[\bigwedge_\mu(B_\alpha)]$.

Definition 2.3. [4]. In a GTS (X, μ) a subset B is called a $\bigwedge_\mu$-set if $B = \bigwedge_\mu(B)$.

It follows from Proposition 2.2 and Definition 2.3 that

Observation 2.4. For subsets A and A_α $(\alpha \in \Omega)$ of a GTS (X, μ), the followings hold:
(i) $\bigwedge_\mu(A)$ is a $\bigwedge_\mu$-set.
(ii) If $A \in \mu$, then A is a $\bigwedge_\mu$-set.
(iii) If A_α is a $\bigwedge_\mu$-set for each $\alpha \in \Omega$, then $\bigcap_{\alpha\in\Omega} A_\alpha$ is a $\bigwedge_\mu$-set.
(iv) If A_α is a $\bigwedge_\mu$-set for each $\alpha \in \Omega$, then $\bigcup_{\alpha\in\Omega} A_\alpha$ is a $\bigwedge_\mu$-set.

Definition 2.5. A subset A of a GTS (X, μ) is said to be $(\bigwedge, \mu)$-closed if $A = T \cap C$, where T is a $\bigwedge_\mu$-set and C is a μ-closed set. The complement of a $(\bigwedge, \mu)$-closed set is called a $(\bigwedge, \mu)$-open set.

We shall denote the collection of all $(\bigwedge, \mu)$-open sets (resp. $(\bigwedge, \mu)$-closed sets) in a GTS (X, μ) by $\bigwedge_\mu$-$O(X, \mu)$ (resp. $\bigwedge_\mu$-$C(X, \mu)$).

Theorem 2.6. Let A be a subset of a GTS (X, μ). Then the followings are equivalent:
(i) A is $(\bigwedge, \mu)$-closed;
(ii) $A = T \cap c_\mu(A)$, where T is a $\bigwedge_\mu$-set;
(iii) $A = \bigwedge_\mu(A) \cap c_\mu(A)$.

Proof. **(i) $\Rightarrow$ (ii)** : Let $A = T \cap F$, where T is a $\bigwedge_\mu$-set and F is a μ-closed set in (X, μ). Since $A \subseteq F$, $c_\mu(A) \subseteq c_\mu(F) = F$. Thus $A = T \cap F \supseteq T \cap c_\mu(A) \supseteq A$. Therefore we have $A = T \cap c_\mu(A)$.
(ii) $\Rightarrow$ (iii) : Let $A = T \cap c_\mu(A)$, where T is a $\bigwedge_\mu$-set. Since $A \subseteq T$, then we have $\bigwedge_\mu(A) \subseteq \bigwedge_\mu(T) = T$ and hence $A \subseteq \bigwedge_\mu(A) \cap c_\mu(A) \subseteq T \cap c_\mu(A) = A$. Therefore $A = \bigwedge_\mu(A) \cap c_\mu(A)$.
(iii) $\Rightarrow$ (i) : By Observation 2.4(i), $\bigwedge_\mu(A)$ is a $\bigwedge_\mu$-set and $c_\mu(A)$ is μ-closed. By (iii), $A = \bigwedge_\mu(A) \cap c_\mu(A)$ and hence by Definition 2.5, A is a $(\bigwedge, \mu)$-closed set. □

It thus follows from Theorem 2.6 that

Observation 2.7. For a GTS (X, μ)
(i) every $\bigwedge_\mu$-set (every μ-closed set) is a $(\bigwedge, \mu)$-closed set.
(ii) $\bigwedge_\mu$-$C(X, \mu)$ (resp. $\bigwedge_\mu$-$O(X, \mu)$) is closed under arbitrary intersection (resp. union).

Example 2.8.
(a) Let $X = \{a, b, c, d\}$ and $\mu = \{\varnothing, \{a\}, \{a, b\}\}, \{a, b, c\}\}$. Then (X, μ) is a GTS. It is easy to see that $\{b, c\}$ is a $(\bigwedge, \mu)$-closed but neither μ-closed nor $\bigwedge_\mu$-set.
(b) Consider $X = \{a, b, c\}$ and $\mu = \{\varnothing, \{a\}, \{a, c\}\}$. Then (X, μ) is a GTS. In view of Theorem 2.6, it is easy to see that $A = \{a\}$ and $B = \{b\}$ are two $(\bigwedge, \mu)$-closed subsets of (X, μ) but $A \cup B = \{a, b\}$ is not a $(\bigwedge, \mu)$-closed sets in (X, μ).

Definition 2.9. A GTS (X, μ) is said to be
(a) μ-T_0 [6] if for any pair of distinct points in X there exists a μ-open set containing one of the points but not the other.
(b) μ-$T_{1/2}$ [4, 6] if for each $x \in X$, $\{x\}$ is either μ-open or μ-closed.
(c) μ-T_1 [6] if for each pair of distinct points x and y of X, there exist a μ-open set U_x

containing x but not y and a μ-open set U_y containing y but not x.
(d) μ-R_0 [5] if for each μ-open set U and each $x \in U$, $c_\mu(\{x\}) \subseteq U$.

Theorem 2.10. For a GTS (X, μ) the following conditions are equivalent :
(a) X is a μ-T_0 space.
(b) Every singleton is $(\bigwedge, \mu)$-closed.

Proof. **(a) ⇒ (b)** : Let $x \in X$. Since X is μ-T_0, then for every point $x \neq y$ there exists a set A_y containing x and is disjoint from $\{y\}$ such that A_y is either μ-open or μ-closed. Let L be the intersection of all μ-open sets A_y and F be the intersection of all μ-closed sets A_y. Clearly L is a $\bigwedge_\mu$-set and F is μ-closed. Note that $\{x\} = L \cap F$. This shows that $\{x\}$ is $(\bigwedge, \mu)$-closed.
(b) ⇒ (a) : Let x and y be two different points of X. Then by (ii), $\{x\} = L \cap F$, where L is a $\bigwedge_\mu$-set and F is μ-closed. If F does not contain y, then $X \setminus F$ is a μ-open set containing y and we are done. If F contains y, then $y \notin L$ and thus for some μ-open set U containing L, we have $y \notin U$. Hence X is μ-T_0. □

Theorem 2.11. For a GTS (X, μ) the following conditions are equivalent :
(a) X is a μ-$T_{1/2}$.
(b) Every subset of X is $(\bigwedge, \mu)$-closed.

Proof. **(a) ⇒ (b)** : Let $A \subseteq X$. Let A_1 be the set of all μ-open singletons of $X \setminus A$ and $A_2 = X \setminus (A \cup A_1)$. Set $F = \bigcap_{x \in A_1} (X \setminus \{x\})$ and $L = \bigcap_{x \in A_2} (X \setminus \{x\})$. Note that F is μ-closed and L is $\bigwedge_\mu$-set. Moreover, $A = F \cap L$. Thus A is $(\bigwedge, \mu)$-closed.
(b) ⇒ (a) : Let $x \in X$. Assume that $\{x\}$ is not μ-open. Then $A = X \setminus \{x\}$ is not μ-closed and since A is a $(\bigwedge, \mu)$-closed $A = T \cap F$, where T is a $\bigwedge_\mu$-set and F is μ-closed. Then the only possibility for $F = X$ and $T = X \setminus \{x\}$, then A is $\bigwedge_\mu$-set, i.e., $A = \bigwedge_\mu(A)$. Since X is the only superset of A, then A is μ-open. Hence $\{x\}$ is μ-closed. □

Definition 2.12. A GTS (X, μ) is said to be weak μ-R_0 if every $(\bigwedge, \mu)$-closed singleton is a $\bigwedge_\mu$-set.

Theorem 2.13. Every μ-R_0 GTS (X, μ) is a weak μ-R_0 space.

Proof. Suppose that (X, μ) is a μ-R_0 GTS. Let $x \in X$ with $\{x\} = L \cap F$, where L is a $\bigwedge_\mu$-set and F is μ-closed. Let $y \in \bigwedge_\mu(\{x\})$ be such that $y \neq x$. Then clearly $y \in L$. Thus $y \notin F$ and since X is μ-R_0, then $c_\mu(\{y\}) \subseteq X \setminus F$. This shows that $x \notin c_\mu(\{y\})$. Thus there exists a μ-open set containing x and disjoint from y. Thus $y \notin \bigwedge_\mu(\{x\})$ and this is a contradiction. Hence, (X, μ) is a weak μ-R_0 space. □

Example 2.14. Consider $X = \{a, b, c\}$ and $\mu = \{\varnothing, \{a\}\}$. Then (X, μ) is a GTS which is weak μ-R_0 space but not μ-R_0.

Theorem 2.15. For a GTS (X, μ) the following conditions are equivalent:
(a) X is μ-T_1.
(b) X is μ-T_0 and μ-R_0.
(c) X is μ-T_0 and weak μ-R_0.

Proof. **(a) ⇒ (b)** : If X is μ-T_1 then it is μ-T_0. Let U be a μ-open set such that $x \in U$. Let $y \notin U$. Then x and y are distinct points of X. So by (i), there exist a μ-open set G such that $y \in G$ but $x \notin G$ and so $y \notin c_\mu(\{x\})$. Thus, (X, μ) is μ-R_0.
(b) ⇒ (c) : Follows from Theorem 2.13.
(c) ⇒ (a) : In view of Theorem 2.10 and Definition 2.12 it follows that every singleton subset of X is a $\bigwedge_\mu$-set and the rest follows from Definition 2.1. □

3. $\bigwedge_\mu$-D SETS

Definition 3.1. A subset A of a GTS (X, μ) is called a $\bigwedge_\mu$-D set if there are two $(\bigwedge, \mu)$-open sets U and V in X such that $U \neq X$ and $A = U \setminus V$.

Definition 3.2. A GTS (X, μ) is called
(i) $\bigwedge_\mu$-D_0 if for any distinct pair of points x and y of X there exists a $\bigwedge_\mu$-D set of X containing x but not y or a $\bigwedge_\mu$-D set of X containing y but not x.
(ii) $\bigwedge_\mu$-D_1 if for any distinct pair of points x and y of X there exist a $\bigwedge_\mu$-D set of X containing x but not y and a $\bigwedge_\mu$-D set of X containing y but not x.
(iii) $\bigwedge_\mu$-D_2 if for any distinct pair of points x and y of X there exist disjoint $\bigwedge_\mu$-D sets G and H of X containing x and y respectively.

A GTS (X, μ) satisfies $(\bigwedge, \mu)$-property if for any distinct pair of points x and y of X there exist disjoint $(\bigwedge, \mu)$-open sets containing one but not the other.

Remark 3.3. The following hold for a GTS (X, μ):
(i) If (X, μ) satisfies $(\bigwedge, \mu)$-property, then it is $\bigwedge_\mu$-D_0.
(ii) If (X, μ) is $\bigwedge_\mu$-D_i, then it is $\bigwedge_\mu$-$D_{i-1}, i = 1, 2$.

Theorem 3.4. A GTS (X, μ) is
(i) $\bigwedge_\mu$-D_0 if and only if it satisfies $(\bigwedge, \mu)$-property.
(ii) (X, μ) is $\bigwedge_\mu$-D_1 if and only if $\bigwedge_\mu$-D_2.

Proof. **(i)** By Remark 3.3, one part is trivial. Let (X, μ) be $\bigwedge_\mu$-D_0. So for any pair of distinct points x and y of X at least one belongs to a $\bigwedge_\mu$-D set U. Suppose $U = P \setminus Q$ for which $P \neq X$ and P and Q are $(\bigwedge, \mu)$-open sets in X. Without loss of generality let $x \in U$ and $y \notin U$. This implies that $x \in P$. For the case $y \notin U$ we have (i) $y \notin P$ (ii) $y \in P$ and $y \in Q$. For (i), the space X satisfies the $(\bigwedge, \mu)$-property since $x \in P$ and $y \notin P$. For (ii), the space X also satisfies $(\bigwedge, \mu)$-property since $y \in Q$, but $x \notin Q$.
(ii) One part is trivial from Remark 3.3. Suppose X is $\bigwedge_\mu$-D_1. It follows from the definition that for any two distinct points x and y in X there exist $\bigwedge_\mu$-D sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. Let $G = U \setminus V$ and $H = W \setminus D$, where U, V, W and D are $(\bigwedge, \mu)$-open sets in X. By the fact that $x \notin H$, we have two cases, i.e., either $x \notin W$ or both W and D contains x. If $x \notin W$, then from $y \notin G$ either (i) $y \notin U$ or (ii) $y \in U$ and $y \in V$. If (i) is the case, then it follows from $x \in U \setminus V$ that $x \in U \setminus V \cup W$ and also it follows from $y \in W \setminus D$ that $y \in W \setminus U \cup D$. Thus we have $U \setminus V \cup W$ and $W \setminus U \cup D$ are disjoint. If (ii) is the case, it follows that $x \in U \setminus V$, $y \in V$ and $(U \setminus V) \cap V = \varnothing$. If $x \in W$ and $x \in D$ we have $y \in W \setminus D$, $x \in D$ and $(W \setminus D) \cap D = \varnothing$. Thus X is $\bigwedge_\mu$-D_2. □

Definition 3.5. Let (X, μ) be a GTS. A point $x \in X$ is said to be a $\bigwedge_\mu$-neat point if there does not exist any $(\bigwedge, \mu)$-open set containing x other than X.

Theorem 3.6. For a GTS (X, μ) that satisfies $(\bigwedge, \mu)$-property the followings are equivalent:
(i) (X, μ) is $\bigwedge_\mu$-D_1;
(ii) (X, μ) has no $\bigwedge_\mu$-neat point.

Proof. **(i) ⇒ (ii)** : Since (X, μ) is $\bigwedge_\mu$-D_1, so each point x of X is contained in a $\bigwedge_\mu$-D set $O = U \setminus V$ and thus $x \in U$. By definition $U \neq X$ and U is $(\bigwedge, \mu)$-open. This implies that x is not a $\bigwedge_\mu$-neat point.
(ii) ⇒ (i) : Since (X, μ) satisfies $(\bigwedge, \mu)$-property, then for each distinct pair of points $x, y \in X$, at least one of them, choose x for example is contained in a $(\bigwedge, \mu)$-open set U not containing the point y. Thus U is a $\bigwedge_\mu$-D set different from X (as every $(\bigwedge, \mu)$-open

set is a $\bigwedge_\mu$-D set). Since by (ii) X has no $\bigwedge_\mu$-neat point, then y is not a $\bigwedge_\mu$-neat point. So there exist a $(\bigwedge, \mu)$-open set V $(\neq X)$ containing y. Thus $y \in V \setminus U$ but $x \notin V \setminus U$ and $V \setminus U$ is a $\bigwedge_\mu$-D set. Hence X is $\bigwedge_\mu$-D_1. □

Remark 3.7. It is clear that a GTS (X, μ) that satisfies $(\bigwedge, \mu)$-property is not $\bigwedge_\mu$-D_1 if and only if there is a unique $\bigwedge_\mu$-neat point in X. It is unique because if x and y are both $\bigwedge_\mu$-neat point in X, then at least one of them say x is contained in a $(\bigwedge, \mu)$-open set U but not y. But this is a contradiction as $U \neq X$.

Definition 3.8. Let (X, μ) be a GTS. A point $x \in X$ is called a $(\bigwedge, \mu)$-cluster point of A if for every $(\bigwedge, \mu)$-open set U of X containing x we have $A \cap U \neq \varnothing$. The set of all $(\bigwedge, \mu)$-cluster points of A is called the $(\bigwedge, \mu)$-closure of A and is denoted by $A^{(\wedge,\mu)}$.

Definition 3.9. A GTS (X, μ) is called $(\bigwedge, \mu)$-symmetric if for x and y in X, $x \in \{y\}^{(\wedge,\mu)}$ implies $y \in \{x\}^{(\wedge,\mu)}$.

Lemma 3.10. Let (X, μ) be a GTS. Then the following properties hold :
(i) $A \subseteq A^{(\wedge,\mu)}$;
(ii) $A^{(\wedge,\mu)} = \cap\{F : A \subseteq F$ and F is $(\bigwedge, \mu)$-closed$\}$;
(iii) $A \subseteq B \Rightarrow A^{(\wedge,\mu)} \subseteq B^{(\wedge,\mu)}$;
(iv) A is $(\bigwedge, \mu)$-closed iff $A = A^{(\wedge,\mu)}$;
(v) $A^{(\wedge,\mu)}$ is $(\bigwedge, \mu)$-closed.

Theorem 3.11. A GTS (X, μ) is $(\bigwedge, \mu)$-symmetric if and only if for $x \in X$, $\{x\}^{(\wedge,\mu)} \subseteq E$ whenever $x \in E$ and E is $(\bigwedge, \mu)$-open in (X, μ).

Proof. Assume that $x \in \{y\}^{(\wedge,\mu)}$ but $y \notin \{x\}^{(\wedge,\mu)}$. This means that the complement of $\{x\}^{(\wedge,\mu)}$ contains y. Now $\{y\}$ is a subset of the complement of $\{x\}^{(\wedge,\mu)}$. This implies that $\{y\}^{(\wedge,\mu)}$ is a subset of the complement of $\{x\}^{(\wedge,\mu)}$. Now the complement of $\{x\}^{(\wedge,\mu)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subseteq E$ and E is $(\bigwedge, \mu)$-open in (X, μ) but $\{x\}^{(\wedge,\mu)} \nsubseteq E$. Then $\{x\}^{(\wedge,\mu)}$ intersects the complement of E. Let y be a member of this intersection. Now we have $x \in \{y\}^{(\wedge,\mu)}$ which is a subset of the complement of E and hence $x \notin E$. But this is a contradiction. □

Theorem 3.12. For a $(\bigwedge, \mu)$-symmetric GTS (X, μ) the following are equivalent :
(a) (X, μ) satisfies the $(\bigwedge, \mu)$-property;
(b) (X, μ) is $\bigwedge_\mu$-D_0;
(c) (X, μ) is $\bigwedge_\mu$-D_1.

Proof. **(a)** $\Leftrightarrow$ **(b)** : Follows from Theorem 3.4.
(c) $\Rightarrow$ **(b)** : Follows from Remark 3.3.
(a) $\Rightarrow$ **(c)** : Let $x \neq y$ and by (a), we may assume that $x \in E \subseteq X \setminus \{y\}$ for some $(\bigwedge, \mu)$-open set E in (X, μ). Then $x \notin \{y\}^{(\wedge,\mu)}$ and $y \notin \{x\}^{(\wedge,\mu)}$. Hence there exist $(\bigwedge, \mu)$-open sets G and H such that $y \in G \subseteq \{x\}^c$ and $x \in H \subseteq \{y\}^c$. Since every $(\bigwedge, \mu)$-open set is a $\bigwedge_\mu$-D set, we have that (X, μ) is a $\bigwedge_\mu$-D_1 space. □

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