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NOTES ON WICK CALCULUS ON PARAMETRIZED SPACES OF TEST FUNCTIONS OF MEIXNER WHITE NOISE

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Abstract. Using a general approach that covers the cases of Gaussian, Poissonian, Gamma, Pascal and Meixner measures, we construct elements of a Wick calculus on parametrized Kondratiev-type spaces of test functions; consider the interconnection between the extended stochastic integration and the Wick calculus; and give an example of a stochastic equation with a Wick-type nonlinearity. The main results consist in studying properties of a Wick product and Wick versions of holomorphic functions on the parametrized Kondratiev-type spaces of test functions. These results are necessary, in particular, in order to describe properties of solutions of stochastic equations with Wick type nonlinearities in the "Meixner analysis".

Introduction

A development of modern mathematical branches of science, in particular, mathematical physics, stochastic analysis, financial mathematics etc. requires to construct and study the theory of test and generalized functions of infinitely many variables with dual pairings generated by non-Gaussian measures. A class of such measures important for applications is the so-called *Meixner class of probability measures* (see [41] for the one-dimensional case, [39, 40, 48] for the infinite-dimensional case). This class contains, in particular, the Gaussian, Poissonian, Gamma, Pascal, and Meixner measures. During recent years an analysis connected with measures from the Meixner class became an object of investigation for many authors. In particular, in [1] Lévy processes on the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ were investigated, components of these processes are classical Lévy processes on $\mathbb{R}$ corresponding to Meixner classes; in [4] a stochastic integral was introduced and studied for a wide class of stochastic processes and in [2] it was proved that the results of [4] can be applied in the "Meixner analysis"; in [20, 21] a stochastic integration theory with applications was constructed for Meixner processes and its generalizations; in [16, 17, 18], in particular, a stochastic integration theory and elements of a Wick calculus with applications to financial mathematics were developed; in [5] all Meixner classes within a quantum white noise context were considered from a general point of view. In the papers [39, 40] E. W. Lytvynov proposed a natural generalization of the results [41] to the infinite-dimensional case and gave some applications to stochastic analysis. His approach is based on the so-called Jacobi fields theory (e.g., [9]). In the paper [48] I. V. Rodionova constructed an infinite-dimensional "Meixner analysis" that is based on generalization of results in [40], considering the Gaussian, Poissonian, Gamma, Pascal and Meixner measures as a one probability measure, the so-called *generalized Meixner measure*. It is worth noticing that the white noise in [48] is not a Lévy one, generally speaking (it is not time homogeneous). In the investigations of [39, 40, 48] an important role belongs to the so-called *extended Fock space* [33, 10], this space naturally arises in the "Meixner analysis" and, in fact, constitutes the interacting Fock space [3, 23].

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On the other hand, many specialists study a (non-Gaussian, generally speaking) analysis on the so-called Hida (e.g., [22, 18, 16]), Kondratiev (e.g., [35, 7, 6, 36, 28, 8, 29, 34, 11, 50]) and another similar spaces of test and generalized functions (and on the corresponding weighted Fock spaces). Such an analysis includes a stochastic integration theory, a Wick calculus and different applications (including a theory of normally ordered white noise equations or, in another terminology, stochastic equations with Wick-type nonlinearities). Thereupon we refer, in particular, to the papers [35, 7, 46, 14, 13, 37, 45, 47, 44, 43, 15, 24, 18, 16, 25]. One of tasks in these investigations consists in a study of properties of different operators (including stochastic integrals and stochastic derivatives) and operations (e.g., of a Wick multiplication) subject to the particular spaces under consideration. For example, in [18, 16, 17] stochastic integrals with respect to a wide class of Lévy processes on Hida spaces are studied and the corresponding Wick calculus is developed; the constructions in these works are based on the so-called power jump processes [42].

In the papers [32, 27] the author investigated the extended stochastic integral and elements of the Wick calculus on the Kondratiev spaces in the so-called Gamma-analysis (i.e., in the analysis connected with the Gamma-measure, — a particular case of the generalized Meixner measure [48]); the constructions of these papers are based on the structure of (the Gamma-version of) the extended Fock space. In the paper [31] the author introduced and studied the extended stochastic integral and elements of the Wick calculus on the "classical" Kondratiev-type spaces in the "Meixner analysis", this paper can be considered as an enhanced generalization of [32, 27]. In [26] the results of [31] are transferred to the case of parametrized Kondratiev-type spaces. In particular, the Wick calculus on the parametrized spaces of regular generalized functions is developed. The main aim of this paper is to study elements of the Wick calculus (properties of the Wick product and of Wick versions of holomorphic functions) on the *parametrized spaces of test functions* of Meixner white noise. These results are necessary, in particular, in order to describe properties of solutions of stochastic equations with Wick-type nonlinearities. Note that equations of this type were studied in different situations by many specialists (see, e.g., [38, 46, 45, 47, 44, 15, 24, 25]); but, as far as it is known to the author, in the present paper we first propose to consider such equations on parametrized spaces of test functions that are concerned with the structure of the extended Fock space.

Finally we remark that in [37] a wide class of Kondratiev-type spaces (including parametrized ones) was studied. But the authors in [37] considered *nonregular* spaces that can be associated with weighted symmetric Fock spaces; whereas we consider in the present paper *regular* spaces that can be associated with weighted extended Fock spaces. Therefore our results cannot be obtained from results of [37]. Moreover, in [37] Wick versions of holomorphic functions (the most non-trivial objects in the Wick calculus!) are not considered.

The paper is organized in the following manner. In the first section we recall necessary definitions and results: the generalized Meixner measure, the corresponding orthogonal polynomials, the extended stochastic integral, the parametrized spaces of test and generalized functions, elements of the Wick calculus on the parametrized spaces of generalized functions. The second section is devoted to the Wick calculus on the parametrized spaces of test functions.

1. Preliminaries

By $\mathcal{D}$ denote the set of all real-valued infinite differentiable functions on $\mathbb{R}_+ = [0, +\infty)$ with compact supports. This set can be naturally endowed with a (projective limit) topology of a nuclear space (e.g., [12]): $\mathcal{D} = \operatorname{pr}\lim_{\tau \in T} \mathcal{H}_\tau$, where T is the set of all pairs

$\tau=(\tau_1,\tau_2)$, $\tau_1\in\mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that $\tau_2(t)\geq 1$ for all $t\in\mathbb{R}_+$; $\mathcal{H}_\tau=\mathcal{H}_{(\tau_1,\tau_2)}$ is the Sobolev space of order τ_1 weighted by the function τ_2, i.e., the scalar product in $\mathcal{H}_\tau$ is given by the formula

$$(f,g)_\tau:=(f,g)_{\mathcal{H}_\tau}=\int_{\mathbb{R}_+}\Big(f(t)g(t)+\sum_{k=1}^{\tau_1}f^{(k)}(t)g^{(k)}(t)\Big)\tau_2(t)dt.$$

Hence in what follows, we understand $\mathcal{D}$ as the corresponding *topological space*.

Let us consider the (nuclear) chain (the rigging of $L^2(\mathbb{R}_+)$ – the space of square integrable with respect to the Lebesgue measure real-valued functions on $\mathbb{R}_+$)

$$\mathcal{D}'=\operatorname{ind}\lim_{\tau'\in T}\mathcal{H}_{-\tau'}\supset\mathcal{H}_{-\tau}\supset L^2(\mathbb{R}_+)=:\mathcal{H}\supset\mathcal{H}_\tau\supset\operatorname{pr}\lim_{\tau'\in T}\mathcal{H}_{\tau'}=\mathcal{D},\tag{1.1}$$

where $\mathcal{H}_{-\tau}$, $\mathcal{D}'$ are the dual of $\mathcal{H}_\tau$, $\mathcal{D}$ with respect to $\mathcal{H}$ spaces correspondingly. Let $\langle\cdot,\cdot\rangle$ be the dual pairing between elements of $\mathcal{D}'$ and $\mathcal{D}$ (and also $\mathcal{H}_{-\tau}$ and $\mathcal{H}_\tau$) that is generated by the scalar product in $\mathcal{H}$, this notation will be preserved for tensor powers and complexifications of spaces.

Remark 1.1. Note that in this paper by the term "scalar product" we understand *real* scalar products or, by another words, *bilinear forms*. So, a scalar product $(\cdot,\cdot)$ on a complex space is a bilinear form that is connected with the norm $\|\cdot\|$ in this space by the formula $\|\cdot\|=\sqrt{(\cdot,\bar{\cdot})}$.

Remark 1.2. One can use a "base chain" that is more general than chain (1.1). For example, instead of $L^2(\mathbb{R}_+)$ one can use the space $L^2(\mathbb{R}_+,\sigma)$, where σ is a measure on $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ (here and below $\mathcal{B}$ denotes the Borel σ-algebra) satisfying some assumptions (e.g., [31, 26]); but such a generalization is not essential in this paper.

Let $\alpha,\gamma:\mathbb{R}_+\to\mathbb{C}$ be smooth functions such that

$$\theta\stackrel{def}{=}\alpha-\gamma:\mathbb{R}_+\to\mathbb{R},\quad \eta\stackrel{def}{=}\alpha\gamma:\mathbb{R}_+\to\mathbb{R}_+$$

and, moreover, θ and η are bounded on $\mathbb{R}_+$. Further, let for each $u\in\mathbb{R}_+$ $v_{\alpha(u),\gamma(u)}(ds)$ be a probability measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ that is defined by its Fourier transform

$$\int_{\mathbb{R}}e^{i\lambda s}v_{\alpha(u),\gamma(u)}(ds)=\exp\Big\{-i\lambda(\alpha(u)+\gamma(u))$$
$$+2\sum_{m=1}^{\infty}\frac{(\alpha(u)\gamma(u))^m}{m}\Big[\sum_{n=2}^{\infty}\frac{(-i\lambda)^n}{n!}\big(\gamma^{n-2}(u)+\gamma^{n-3}(u)\alpha(u)+\cdots+\alpha^{n-2}(u)\big)\Big]^m\Big\}.$$

Definition 1.1. ([48]). A probability measure μ on the measurable space $(\mathcal{D}',\mathcal{F}(\mathcal{D}'))$ (here $\mathcal{F}$ is the σ-algebra on $\mathcal{D}'$ generated by cylinder sets) with the Fourier transform

$$\int_{\mathcal{D}'}e^{i\langle x,\xi\rangle}\mu(dx)=\exp\Big\{\int_{\mathbb{R}_+}du\int_{\mathbb{R}}v_{\alpha(u),\gamma(u)}(ds)\frac{1}{s^2}\big(e^{is\xi(u)}-1-is\xi(u)\big)\Big\}$$

(here $\xi\in\mathcal{D}$) is called the *generalized Meixner measure*.

Let us denote by a subindex $\mathbb{C}$ complexifications of spaces.

Theorem 1.1. ([48]). *The generalized Meixner measure μ is a generalized stochastic process with independent values in the sense of* [19]. *The Laplace transform of μ is a holomorphic at $0\in\mathcal{D}_{\mathbb{C}}$ function.*

Remark 1.3. Let α and γ be constants. Accordingly to the classical classification [41] (see also [40, 48]) for $\alpha=\gamma=0$ μ is the Gaussian measure; for $\alpha\neq 0$, $\gamma=0$ μ is the centered Poissonian measure; for $\alpha=\gamma\neq 0$ μ is the centered Gamma-measure; for $\alpha\neq\gamma$, $\alpha\gamma\neq 0$, $\alpha,\gamma:\mathbb{R}_+\to\mathbb{R}$ μ is the centered Pascal measure; for $\alpha=\overline{\gamma}$, $\mathrm{Im}(\alpha)\neq 0$ μ is the centered Meixner measure.

It was established in [31] that there exists $\widetilde{\tau} \in T$ such that the generalized Meixner measure is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$.

Now by $(L^2) = L^2(\mathcal{D}', \mu)$ denote the space of square integrable with respect to μ complex-valued functions on $\mathcal{D}'$. Let us construct orthogonal polynomials on (L^2). Denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the scalar product in (L^2), this notation will be preserved for dual pairings that are generated by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. For $n \in \mathbb{N}$ denote by $\overline{\mathcal{P}}_n$ the closure in (L^2) of the set of all continuous polynomials on $\mathcal{D}'$ of degree $\leq n$, $\overline{\mathcal{P}}_0 := \mathbb{C}$, let also $(L^2_n) := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ – the orthogonal difference in (L^2), $(L^2_0) := \mathbb{C}$. Since μ has a holomorphic at zero Laplace transform, the set of continuous polynomials on $\mathcal{D}'$ is dense in (L^2) [49], therefore $(L^2) = \bigoplus_{n=0}^{\infty} (L^2_n)$.

Denote by $\widehat{\otimes}$ the symmetric tensor product. For each $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, we define $:\langle x^{\otimes n}, f^{(n)} \rangle:$ as the orthogonal projection of $\langle x^{\otimes n}, f^{(n)} \rangle$ onto (L^2_n). It follows from results of [48] that $:\langle x^{\otimes n}, f^{(n)} \rangle: = \langle P_n(x), f^{(n)} \rangle$, where $P_n(x) \in \mathcal{D}'^{\widehat{\otimes} n}_{\mathbb{C}}$ ($n \in \mathbb{Z}_+$) are the kernels of generalized Appell polynomials that are associated with the generalized Meixner measure (see the detailed description in [48]).

Definition 1.2. We say that the polynomials $\{\langle P_n, f^{(n)} \rangle,\ f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n},\ n \in \mathbb{Z}_+\}$ are called the *generalized Meixner polynomials.*

Let us define a scalar product $\langle \cdot, \cdot \rangle_{\mathrm{ext}}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$

$$\langle f^{(n)}, g^{(n)} \rangle_{\mathrm{ext}} := \frac{1}{n!} \int_{\mathcal{D}'} \langle P_n, f^{(n)} \rangle \langle P_n, g^{(n)} \rangle \mu(dx).$$

It follows from results of [48] that

$$\begin{aligned}
\langle f^{(n)}, g^{(n)} \rangle_{\mathrm{ext}} = & \sum_{\substack{k, l_j, s_j \in \mathbb{N}:\ j=1,\dots,k,\ l_1 > l_2 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{l_1^{s_1} \dots l_k^{s_k} s_1! \dots s_k!} \\
& \times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} f^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k}) \\
& \times g^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k}) \\
& \times \eta^{l_1 - 1}(u_1) \dots \eta^{l_1 - 1}(u_{s_1}) \\
& \times \eta^{l_2 - 1}(u_{s_1+1}) \dots \eta^{l_2 - 1}(u_{s_1+s_2}) \dots \eta^{l_k - 1}(u_{s_1+\dots+s_{k-1}+1}) \dots \eta^{l_k - 1}(u_{s_1+\dots+s_k}) \\
& \times du_1 \dots du_{s_1+\dots+s_k}.
\end{aligned}$$

Let $|\cdot|_{\mathrm{ext}}$ denote the norm generated by the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{ext}}$, i.e., $|f^{(n)}|_{\mathrm{ext}} := \sqrt{\langle f^{(n)}, \overline{f^{(n)}} \rangle_{\mathrm{ext}}}$. Denote by $\mathcal{H}^{(n)}_{\mathrm{ext}}$ the closure of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ with respect to $|\cdot|_{\mathrm{ext}}$. The space $\mathcal{H}^{(n)}_{\mathrm{ext}}$ can be understood as an extension of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ in a generalized sense: let $F^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, $\dot{F}^{(n)} \in F^{(n)}$ be a representative (a function) from the equivalence class $F^{(n)}$ with a "zero diagonal", i.e., $\dot{F}^{(n)}(u_1, \dots, u_n) = 0$ if there exist $i, j \in \{1, \dots, n\}$ such that $i \neq j$ but $u_i = u_j$. The function $\dot{F}^{(n)}$ generates an equivalence class in $\mathcal{H}^{(n)}_{\mathrm{ext}}$ that can be identified with $F^{(n)}$ [31].

For $F^{(n)} \in \mathcal{H}^{(n)}_{\mathrm{ext}}$, $n \in \mathbb{Z}_+$, we define $\langle P_n, F^{(n)} \rangle \in (L^2)$ as an (L^2)-limit

$$\langle P_n, F^{(n)} \rangle := \lim_{k \to \infty} \langle P_n, f_k^{(n)} \rangle,$$

where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \ni f_k^{(n)} \to F^{(n)}$ in $\mathcal{H}^{(n)}_{\mathrm{ext}}$ as $k \to \infty$. The forthcoming statement easily follows from the construction of polynomials $\langle P_n, F^{(n)} \rangle$ (see also [48]).

Theorem 1.2. *A function $F \in (L^2)$ if and only if there exists a sequence of kernels*

$$\big(F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}\big)_{n=0}^{\infty} \tag{1.2}$$

such that F can be presented in the form

$$F = \sum_{n=0}^{\infty} \langle P_n, F^{(n)} \rangle, \tag{1.3}$$

where the series converges in (L^2), i.e., the (L^2)-norm of F

$$\|F\|^2_{(L^2)} = \sum_{n=0}^{\infty} n! |F^{(n)}|^2_{\mathrm{ext}} < \infty.$$

Moreover, the system $\{\langle P_n, F^{(n)} \rangle,\ F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)},\ n \in \mathbb{Z}_+\}$ is an orthogonal basis in (L^2) in the sense that for $F, G \in (L^2)$ of form (1.3)

$$\langle\!\langle F, G \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F^{(n)}, G^{(n)} \rangle_{\mathrm{ext}}.$$

Let us recall briefly the construction of an extended stochastic integral in the "Meixner analysis" (see [31] for details). By analogy with the Gaussian analysis, on the probability triplet $(\mathcal{D}', \mathcal{F}(\mathcal{D}'), \mu)$ we define the Meixner random process M by setting for each $u \in \mathbb{R}_+$ $M_u := \langle P_1, 1_{[0,u)} \rangle \in (L^2)$ (here and below $1_A(t)$ denotes the indicator of $\{t \in A\}$). Using results of [48] one can show that M *is a locally square integrable normal martingale (with respect to the flow of σ-algebras generated by M) with orthogonal independent increments*, therefore the Itô integral with respect to M is well-definite. Note that M is not a Lévy process, generally speaking (not time-homogeneous). Let $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$. It follows from Theorem 1.2 that F can be presented in the form

$$F(\cdot) = \sum_{n=0}^{\infty} \langle P_n, F_{\cdot}^{(n)} \rangle, \quad F_{\cdot}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}, \tag{1.4}$$

with

$$\|F\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} n! |F_{\cdot}^{(n)}|^2_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty.$$

If in addition F is such that the kernels $F_{\cdot}^{(n)}$ belong to $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\mathbb{C}} \subset \mathcal{H}_{\mathrm{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ (the embedding in the generalized sense described above) then one can show [31] that F can be presented in the form

$$F(\cdot) = \sum_{n=0}^{\infty} n! \int_0^{\infty} \int_0^{u_n} \dots \int_0^{u_2} F_{\cdot}^{(n)}(u_1, \dots, u_n)\, dM_{u_1} \dots dM_{u_n},$$

i.e., as a series of repeated Itô stochastic integrals with respect to the Meixner process. In this case for arbitrary $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, and under an additional condition

$$\sum_{n=0}^{\infty} (n+1)! |\widehat{F}_{[t_1,t_2)}^{(n)}|^2_{\mathrm{ext}} < \infty, \tag{1.5}$$

where $\widehat{F}_{[t_1,t_2)}^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1} \subset \mathcal{H}_{\mathrm{ext}}^{(n+1)}$ is the projection of $F_{\cdot}^{(n)} 1_{[t_1,t_2)}(\cdot)$ onto $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1}$, one can define the extended stochastic integral of F with respect to M on $[t_1, t_2)$ as

$$\begin{aligned}\int_{t_1}^{t_2} F(u)\, \widehat{d}M_u &:= \sum_{n=0}^{\infty} (n+1)! \int_0^{\infty} \int_0^{u} \dots \int_0^{u_2} \widehat{F}_{[t_1,t_2)}^{(n)}(u_1, \dots, u_n, u)\, dM_{u_1} \dots dM_{u_n} dM_u \\ &= \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{F}_{[t_1,t_2)}^{(n)} \rangle \in (L^2).\end{aligned}$$

In a general case such a definition cannot be accepted because it is impossible to project elements of $\mathcal{H}_{\mathrm{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ onto $\mathcal{H}_{\mathrm{ext}}^{(n+1)}$, generally speaking. Nevertheless, the following generalization is possible.

Lemma. ([31]). *For given $F_{\cdot}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{Z}_+$, and $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, we construct the element $\widehat{F}_{[t_1,t_2)}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n+1)}$ by the following way. Let $\dot{F}_{\cdot}^{(n)} \in F_{\cdot}^{(n)}$ be some representative (a function) from the equivalence class $F_{\cdot}^{(n)}$. We set*

$$\widetilde{\dot{F}}_{[t_1,t_2)}^{(n)}(u_1,\ldots,u_n,u) := \begin{cases} \dot{F}_u^{(n)}(u_1,\ldots,u_n)1_{[t_1,t_2)}(u), & \text{if} \quad u \neq u_1, \ldots, u \neq u_n, \\ 0, & \text{in other cases} \end{cases},$$

$\widehat{\dot{F}}_{[t_1,t_2)}^{(n)} := \mathrm{Pr}\widetilde{\dot{F}}_{[t_1,t_2)}^{(n)}$, where Pr is the symmetrization operator. Let $\widehat{F}_{[t_1,t_2)}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n+1)}$ be the equivalence class in $\mathcal{H}_{\mathrm{ext}}^{(n+1)}$ that is generated by $\widehat{\dot{F}}_{[t_1,t_2)}^{(n)}$. This class is well-defined, does not depend on the representative $\dot{F}_{\cdot}^{(n)}$, and

$$|\widehat{F}_{[t_1,t_2)}^{(n)}|_{\mathrm{ext}} \leq |F_{\cdot}^{(n)}1_{[t_1,t_2)}(\cdot)|_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} \leq |F_{\cdot}^{(n)}|_{\mathcal{H}_{\mathrm{ext}}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}}.$$

Definition 1.3. ([31]). For $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ of form (1.4) such that (1.5) is fulfilled, and $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, we define the extended stochastic integral on $[t_1, t_2)$ with respect to M by setting

$$\int_{t_1}^{t_2} F(u)\,\widehat{d}M_u := \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{F}_{[t_1,t_2)}^{(n)} \rangle \in (L^2).$$

Theorem 1.3. ([31]). *Let $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ be integrable on $\mathbb{R}_+$ by Itô with respect to M (i.e., be adapted with respect to the generated by M flow of σ-algebras). Then for any $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, F is integrable on $[t_1, t_2)$ by Itô and in the extended sense, and $\int_{t_1}^{t_2} F(u)\,\widehat{d}M_u = \int_{t_1}^{t_2} F(u)\,dM_u$ (the last integral is the Itô one).*

One of the main lacks of the extended stochastic integral consists in the dependence of its domain on the integration interval. In order to overcome this problem, one can use a rigging of (L^2) by some suitable spaces. Now we describe one possible rigging (the so-called *regular parametrized rigging*).

Let

$$\mathcal{P}(\mathcal{D}') = \Big\{ \sum_{n=0}^{N} \langle x^{\otimes n}, g^{(n)} \rangle \ : \ x \in \mathcal{D}',\ g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n},\ N \in \mathbb{Z}_+ \Big\}$$

be the set of continuous polynomials on $\mathcal{D}'$. One can show (e.g., [28]) that any element of $\mathcal{P}(\mathcal{D}')$ can be presented in the form

$$f = \sum_{n=0}^{N_f} \langle P_n, f^{(n)} \rangle, \quad f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad N_f \in \mathbb{Z}_+. \tag{1.6}$$

Now we introduce a family of Hilbert norms $\|\cdot\|_{q,\beta}$, $q \in \mathbb{Z}_+$, $\beta \in [0,1]$ (in what follows, we accept these conditions on default), by setting for f of form (1.6)

$$\|f\|_{q,\beta}^2 := \sum_{n=0}^{N_f} (n!)^{1+\beta} 2^{qn} |f^{(n)}|_{\mathrm{ext}}^2. \tag{1.7}$$

By $(L^2)_q^\beta$ denote a Hilbert space that is the closure of $\mathcal{P}(\mathcal{D}')$ with respect to norm (1.7). Let also $(L^2)^\beta := \operatorname*{pr\,lim}_{q \in \mathbb{Z}_+} (L^2)_q^\beta$.

Definition 1.4. The spaces $(L^2)_q^\beta$, $(L^2)^\beta$ are called the *parametrized Kondratiev-type spaces of test functions.*

It is easy to see that $f \in (L^2)^\beta_q$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} \langle P_n, f^{(n)}\rangle, \quad f^{(n)} \in \mathcal{H}^{(n)}_{\rm ext}, \tag{1.8}$$

with

$$\|f\|^2_{q,\beta} := \|f\|^2_{(L^2)^\beta_q} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |f^{(n)}|^2_{\rm ext} < \infty. \tag{1.9}$$

It is easy to show [31] that for arbitrary $q \in \mathbb{Z}_+$ and $\beta \in [0,1]$ the space $(L^2)^\beta_q$ is densely and continuously embedded into (L^2), therefore one can consider the chain

$$(L^2)^{-\beta} = \operatorname*{ind\,lim}_{q' \in \mathbb{Z}_+} (L^2)^{-\beta}_{-q'} \supset (L^2)^{-\beta}_{-q} \supset (L^2) \supset (L^2)^\beta_q \supset (L^2)^\beta,$$

where $(L^2)^{-\beta}_{-q}$, $(L^2)^{-\beta}$ are the spaces dual of $(L^2)^\beta_q$, $(L^2)^\beta$ with respect to (L^2) correspondingly.

Definition 1.5. The spaces $(L^2)^{-\beta}_{-q}$, $(L^2)^{-\beta}$ are called the *parametrized Kondratiev-type spaces of (regular) generalized functions.*

Note that for $\beta = q = 0$ $(L^2)^0_0 = (L^2)^{-0}_{-0} = (L^2)$.

Since the generalized Meixner polynomials are orthogonal in (L^2), these polynomials form orthogonal bases in $(L^2)^{-\beta}_{-q}$. More exactly, a function $F \in (L^2)^{-\beta}_{-q}$ if and only if there exists sequence (1.2) such that F can be presented in form (1.3) with

$$\|F\|^2_{-q,-\beta} := \|F\|^2_{(L^2)^{-\beta}_{-q}} = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |F^{(n)}|^2_{\rm ext} < \infty.$$

The extended stochastic integral on the spaces $(L^2)^{-\beta}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$, $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ is described in [26], on the spaces $(L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$, $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$ – in [30]. Here we note only that

$$\int_{t_1}^{t_2} \circ(u)\, \widehat{d}M_u : (L^2)^\beta_{q+1} \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^\beta_q,$$
$$\int_{t_1}^{t_2} \circ(u)\, \widehat{d}M_u : (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^\beta$$

are linear continuous operators, whereas

$$\int_{t_1}^{t_2} \circ(u)\, \widehat{d}M_u : (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^\beta_q$$

is a linear closed operator with the domain

$$\Big\{ f \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}} : \sum_{n=0}^{\infty} \big((n+1)!\big)^{1+\beta} 2^{q(n+1)} |\widehat{f}^{(n)}_{[t_1,t_2)}|^2_{\rm ext} < \infty \Big\}$$

(all these integrals are restrictions of the integral from Definition 1.3 on the corresponding spaces).

On the space $(L^2)^{-\beta}$ there is a natural multiplication that is closely connected with the extended stochastic integration and is convenient for different applications – the so-called Wick multiplication. Let us describe briefly this multiplication and the corresponding (Wick) calculus (more detailed information is given in [26], see also [31]).

Let at first $\beta = 1$.

Definition 1.6. For $F \in (L^2)^{-1}$ we define an S-transform as a *formal* series

$$(SF)(\lambda) = \sum_{n=0}^{\infty} \langle F^{(n)}, \lambda^{\otimes n} \rangle_{\mathrm{ext}},$$

where $F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$ ($n \in \mathbb{Z}_+$) are the kernels from decomposition (1.3) for F. In particular, $(SF)(0) = F^{(0)}$, $S1 \equiv 1$.

Definition 1.7. For $F, G \in (L^2)^{-1}$ and a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ we define the *Wick product* $F \lozenge G \in (L^2)^{-1}$ and the *Wick version of* h $h^{\lozenge}(F) \in (L^2)^{-1}$ by setting

$$F \lozenge G := S^{-1}(SF \cdot SG), \quad h^{\lozenge}(F) := S^{-1} h(SF).$$

The correctness of this definition and, moreover, the fact that the Wick multiplication is continuous in the topology of $(L^2)^{-1}$ are proved in [31].

Remark 1.4. It is easy to see that the Wick multiplication $\lozenge$ is commutative, associative and distributive (over the field $\mathbb{C}$). Further, if h from Definition 1.7 is presented in the form

$$h(u) = \sum_{m=0}^{\infty} h_m (u - (SF)(0))^m \tag{1.10}$$

then $h^{\lozenge}(F) = \sum_{m=0}^{\infty} h_m (F - (SF)(0))^{\lozenge m}$, where $F^{\lozenge m} := \underbrace{F \lozenge \ldots \lozenge F}_{m \ times}$.

Let us write out the "coordinate form" of $F \lozenge G$ and $h^{\lozenge}(F)$.

Lemma 1.1. ([31]). *Let $F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$, $G^{(m)} \in \mathcal{H}_{\mathrm{ext}}^{(m)}$, $n, m \in \mathbb{Z}_+$. We define the element $F^{(n)} \diamond G^{(m)} \in \mathcal{H}_{\mathrm{ext}}^{(n+m)}$ as follows. Let $\dot{F}^{(n)} \in F^{(n)}$, $\dot{G}^{(m)} \in G^{(m)}$ be some representatives (functions) from the equivalence classes $F^{(n)}$, $G^{(m)}$. Set*

$$\begin{aligned}
&(\widetilde{F^{(n)} \dot{G}^{(m)}})(t_1, \ldots, t_n; t_{n+1}, \ldots, t_{n+m}) \\
&\quad := \begin{cases} \dot{F}^{(n)}(t_1, \ldots, t_n) \dot{G}^{(m)}(t_{n+1}, \ldots, t_{n+m}), & \text{if} \quad {}_{\forall j \in \{n+1, \ldots, n+m\}}^{\forall i \in \{1, \ldots, n\},} \ t_i \neq t_j, \\ 0, & \text{in other cases} \end{cases}
\end{aligned}$$

$\widehat{F^{(n)} \dot{G}^{(m)}} := \mathrm{Pr} \widetilde{F^{(n)} \dot{G}^{(m)}}$, *where* Pr *is the symmetrization operator. Then $F^{(n)} \diamond G^{(m)}$ is the equivalence class in $\mathcal{H}_{\mathrm{ext}}^{(n+m)}$ that is generated by $\widehat{F^{(n)} \dot{G}^{(m)}}$, this class is well-defined and does not depend on a choice of the representatives $\dot{F}^{(n)}$, $\dot{G}^{(m)}$. Moreover,*

$$|F^{(n)} \diamond G^{(m)}|_{\mathrm{ext}} \le |F^{(n)}|_{\mathrm{ext}} |G^{(m)}|_{\mathrm{ext}}. \tag{1.11}$$

Remark 1.5. Note that, non-strictly speaking, $F^{(n)} \diamond G^{(m)}$ is the symmetrization of the "function"

$$\begin{aligned}
&\widetilde{F^{(n)} G^{(m)}}(t_1, \ldots, t_n; t_{n+1}, \ldots, t_{n+m}) \\
&\quad := \begin{cases} F^{(n)}(t_1, \ldots, t_n) G^{(m)}(t_{n+1}, \ldots, t_{n+m}), & \text{if} \quad {}_{\forall j \in \{n+1, \ldots, n+m\}}^{\forall i \in \{1, \ldots, n\},} \ t_i \neq t_j, \\ 0, & \text{in other cases} \end{cases}
\end{aligned}$$

with respect to $n + m$ "variables".

It is obvious that the "multiplication" $\diamond$ is commutative, associative and distributive (over the field $\mathbb{C}$).

Remark 1.6. Note that for $\eta = 0$ (the Gaussian and Poissonian cases) $F^{(n)} \diamond G^{(m)} = F^{(n)} \widehat{\otimes} G^{(m)}$ (we recall that in this case $\mathcal{H}_{\mathrm{ext}}^{(n)} = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ for each $n \in \mathbb{Z}_+$).

Proposition 1.1. ([31]). *For $F,G\in(L^2)^{-1}$ and a holomorphic at $(SF)(0)$ function $h:\mathbb{C}\to\mathbb{C}$*

$$F\Diamond G=\sum_{n=0}^{\infty}\Big\langle P_n,\sum_{k=0}^{n}F^{(k)}\diamond G^{(n-k)}\Big\rangle, \tag{1.12}$$

$$h^{\Diamond}(F)=h_0+\sum_{n=1}^{\infty}\Big\langle P_n,\sum_{m=1}^{n}h_m\sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}}F^{(k_1)}\diamond\dots\diamond F^{(k_m)}\Big\rangle, \tag{1.13}$$

where $F^{(k)},G^{(k)}\in\mathcal{H}_{\rm ext}^{(k)}$ are the kernels from decompositions (1.3) *for F and G correspondingly, $h_m\in\mathbb{C}$ ($m\in\mathbb{Z}_+$) are the coefficients from decomposition* (1.10) *for h.*

Remark 1.7. It follows from (1.12) that, in particular,

$$\langle P_n,F^{(n)}\rangle\Diamond\langle P_m,G^{(m)}\rangle=\langle P_{n+m},F^{(n)}\diamond G^{(m)}\rangle,$$

$$F\Diamond\langle P_m,G^{(m)}\rangle=\sum_{n=0}^{\infty}\langle P_{n+m},F^{(n)}\diamond G^{(m)}\rangle.$$

The first formula can be used in order to *define* the Wick product and the Wick version of a holomorphic function (as a series) without the S-transform. Formulas (1.12) and (1.13) can also be used as definitions. Finally we note that for $F_1,\dots,F_m\in(L^2)^{-1}$

$$F_1\Diamond\dots\Diamond F_m=\sum_{n=0}^{\infty}\Big\langle P_n,\sum_{k_1,\dots,k_m\in\mathbb{Z}_+,k_1+\dots+k_m=n}F_1^{(k_1)}\diamond\dots\diamond F_m^{(k_m)}\Big\rangle, \tag{1.14}$$

where $F_j^{(k_j)}\in\mathcal{H}_{\rm ext}^{(k_j)}$ ($j\in\{1,\dots,m\}$, $k_j\in\mathbb{Z}_+$) are the kernels from decompositions (1.3) for F_j.

In the case $\beta<1$ properties of the Wick product are analogous to the corresponding properties for the case $\beta=1$; but, unfortunately, one cannot say the same about properties of Wick versions of holomorphic functions. More exactly, the following statement is fulfilled.

Theorem 1.4. ([26]).

1. Let $F,G\in(L^2)^{-\beta}$. Then the Wick product $F\Diamond G\in(L^2)^{-\beta}$. Moreover, the Wick multiplication is continuous in the topology of $(L^2)^{-\beta}$: for $F_1,\dots,F_m\in(L^2)^{-\beta}$, $m\in\mathbb{N}$, there exist $q,q'\in\mathbb{N}$ and $c>0$ such that

$$\|F_1\Diamond\dots\Diamond F_m\|_{-q,-\beta}\le c\|F_1\|_{-q',-\beta}\dots\|F_m\|_{-q',-\beta}.$$

2. Let $h:\mathbb{C}\to\mathbb{C}$ be a holomorphic at $u_0\in\mathbb{C}$ not polynomial function such that all coefficients h_m, $m\in\mathbb{Z}_+$, from the decomposition $h(u)=\sum_{m=0}^{\infty}h_m(u-u_0)^m$ are nonnegative. Then for each $\beta\in[0,1)$ there exists $F\in(L^2)^{-\beta}$ such that $(SF)(0)=u_0$ and $h^{\Diamond}(F)\notin(L^2)^{-\beta}$.

3. Let $F=\sum_{k=0}^{N}\langle P_k,F^{(k)}\rangle\in\mathcal{P}$ and $h(u)=\sum_{m=0}^{\infty}h_m(u-F^{(0)})^m$ be such that $\exists K>0$: $\forall m\in\mathbb{N}$

$$|h_m|\le\frac{K^m}{m^{mN\frac{1-\beta}{2}}}. \tag{1.15}$$

Then $h^{\Diamond}(F)\in(L^2)^{-\beta}$.

4. Let $0\le\beta_1<\beta_2\le1$, $F\in(L^2)^{-\beta_1}$, $h:\mathbb{C}\to\mathbb{C}$ be a holomorphic at $(SF)(0)$ function. If there exists $K>0$ such that for each $m\in\mathbb{Z}_+$

$$|h_m|\le\frac{K^m}{\max\limits_{n\in\mathbb{N}:\ n\ge m}\left[\frac{m^{n\frac{1-\beta_2}{2}}}{([\frac{n}{m}]!)^{m\frac{\beta_2-\beta_1}{2}}}\right]},$$

where h_m ($m \in \mathbb{Z}_+$) are the coefficients from decomposition (1.10) *for h, then $h^{\Diamond}(F) \in (L^2)^{-\beta_2}$.*

The interconnection between the Wick calculus and the extended stochastic integration can be described as follows. Denote by M' the Meixner white noise (the generalized stochastic process from Theorem 1.1). *Formally* $M'_\cdot = \langle P_1, \delta_\cdot \rangle$, where $\delta_\cdot$ is the Dirac delta-function (see [31] for more details).

Theorem 1.5. ([26]). *For any $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, and $F \in (L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ formally defined $\int_{t_1}^{t_2} F(s) \Diamond M'_s ds$ generates a linear continuous functional on $(L^2)^{\beta}$ that coincides with $\int_{t_1}^{t_2} F(s) \, \widehat{d}M_s$, i.e.,*

$$\int_{t_1}^{t_2} F(s) \Diamond M'_s ds = \int_{t_1}^{t_2} F(s) \, \widehat{d}M_s \in (L^2)^{-\beta}. \tag{1.16}$$

By analogy with the "classical" Gaussian analysis one can apply mentioned results in order to study so-called stochastic equations with Wick-type nonlinearities.

Example 1.1. Let

$$X_t = X_0 + \int_0^t X_s \Diamond F ds + \int_0^t X_s \Diamond G \, \widehat{d}M_s, \tag{1.17}$$

where $X_0, F, G \in (L^2)^{-\beta}$. Applying the S-transform (taking into consideration (1.16)), solving the obtained *not stochastic* integral equation and applying the inverse S-transform we obtain the solution [31]

$$X_t = X_0 \Diamond \exp^{\Diamond} \left\{ Ft + G \Diamond M_t \right\} \in (L^2)^{-1}.$$

In order to have $X_t \in (L^2)^{-\beta}$ ($\beta < 1$) we must impose some addition conditions. For example, let F and G be polynomials and $N := \max\,[\mathrm{pow}\, F, \mathrm{pow}\, G + 1]$, where "pow" denotes the power of a polynomial. It was shown in [26] that if $N \leq \frac{2}{1-\beta}$ then $X_t \in (L^2)^{-\beta}$.

Unfortunately, the space of square integrable functions (L^2) is not invariant with respect to the Wick product. Nevertheless, it is natural to raise the question about properties of the Wick product and Wick versions of holomorphic functions on the spaces of test functions (for example, if in equation (1.17) $X_0 \in (L^2)^{\beta} \subset (L^2)^{-1}$, what can we say about properties of X_t?). In the next section we try to give a detailed answer on this question.

2. Wick calculus on spaces of test functions

We begin from the analog of the first statement of Theorem 1.4 on the space of test functions $(L^2)^{\beta}$.

Theorem 2.1. *Let $f, g \in (L^2)^{\beta}$. Then the Wick product $f \Diamond g \in (L^2)^{\beta}$. Moreover, the Wick multiplication is continuous in the topology of $(L^2)^{\beta}$: for $f_1, \dots, f_m \in (L^2)^{\beta}$, $m \in \mathbb{N}$, and any $q \in \mathbb{Z}_+$*

$$\|f_1 \Diamond \dots \Diamond f_m\|_{q,\beta} \leq c(m-1) \|f_1\|_{q',\beta} \dots \|f_m\|_{q',\beta}, \tag{2.1}$$

where $q' \geq q + (1+\beta)\log_2 m + 1$ and $c(m-1) := \sqrt{\max_{n \in \mathbb{Z}_+} [2^{-n}(n+1)^{m-1}]}$.

Proof. It is clear that it is sufficient to prove estimate (2.1). Using (1.8), (1.14), (1.9), (1.11) and well-known estimates

$$\Big|\sum_{l=1}^{p} a_l\Big|^2 \leq p \sum_{l=1}^{p} |a_l|^2, \quad \frac{n!}{k_1! \dots k_{m-1}!(n - k_1 - \dots - k_{m-1})!} \leq m^n,$$

we obtain

$$
\begin{aligned}
\|f_1\Diamond\dots\Diamond f_m\|^2_{q,\beta} &= \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\Big|\sum_{\substack{k_1,\dots,k_m\in\mathbb{Z}_+,\\ k_1+\dots+k_m=n}} f_1^{(k_1)}\diamond\dots\diamond f_m^{(k_m)}\Big|^2_{\mathrm{ext}}\\
&= \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\\
&\times\Big|\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\dots\sum_{k_{m-1}=0}^{n-k_1-\dots-k_{m-2}} f_1^{(k_1)}\diamond\dots\diamond f_{m-1}^{(k_{m-1})}\diamond f_m^{(n-k_1-\dots-k_{m-1})}\Big|^2_{\mathrm{ext}}\\
&\le \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}(n+1)\\
&\times\sum_{k_1=0}^{n}\Big|\sum_{k_2=0}^{n-k_1}\dots\sum_{k_{m-1}=0}^{n-k_1-\dots-k_{m-2}} f_1^{(k_1)}\diamond\dots\diamond f_{m-1}^{(k_{m-1})}\diamond f_m^{(n-k_1-\dots-k_{m-1})}\Big|^2_{\mathrm{ext}}\\
&\le\dots\le \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}(n+1)^{m-1}\\
&\times\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\dots\sum_{k_{m-1}=0}^{n-k_1-\dots-k_{m-2}} |f_1^{(k_1)}\diamond\dots\diamond f_{m-1}^{(k_{m-1})}\diamond f_m^{(n-k_1-\dots-k_{m-1})}|^2_{\mathrm{ext}}\\
&\le \sum_{n=0}^{\infty}[2^{-n}(n+1)^{m-1}]m^{(1+\beta)n}2^{(q+1)n}(k_1!)^{1+\beta}\dots((n-k_1-\dots-k_{m-1})!)^{1+\beta}\\
&\times\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\dots\sum_{k_{m-1}=0}^{n-k_1-\dots-k_{m-2}} |f_1^{(k_1)}|^2_{\mathrm{ext}}\dots|f_{m-1}^{(k_{m-1})}|^2_{\mathrm{ext}}|f_m^{(n-k_1-\dots-k_{m-1})}|^2_{\mathrm{ext}}\\
&\le \big(c(m-1)\big)^2\sum_{k_1=0}^{\infty}(k_1!)^{1+\beta}2^{q'k_1}|f_1^{(k_1)}|^2_{\mathrm{ext}}\\
&\times\sum_{n=k_1}^{\infty}\sum_{k_2=0}^{n-k_1}\dots\sum_{k_{m-1}=0}^{n-k_1-\dots-k_{m-2}}(k_2!)^{1+\beta}2^{q'k_2}|f_2^{(k_2)}|^2_{\mathrm{ext}}\\
&\dots((n-k_1-\dots-k_{m-1})!)^{1+\beta}2^{q'(n-k_1-\dots-k_{m-1})}|f_m^{(n-k_1-\dots-k_{m-1})}|^2_{\mathrm{ext}}\\
&= \big(c(m-1)\big)^2\|f_1\|^2_{q',\beta}\sum_{n=0}^{\infty}\sum_{k_2=0}^{n}\dots\sum_{k_{m-1}=0}^{n-k_2-\dots-k_{m-2}}(k_2!)^{1+\beta}2^{q'k_2}|f_2^{(k_2)}|^2_{\mathrm{ext}}\\
&\dots((n-k_2-\dots-k_{m-1})!)^{1+\beta}2^{q'(n-k_2-\dots-k_{m-1})}|f_m^{(n-k_2-\dots-k_{m-1})}|^2_{\mathrm{ext}}\\
&=\dots=\big(c(m-1)\big)^2\|f_1\|^2_{q',\beta}\dots\|f_m\|^2_{q',\beta}.
\end{aligned}
$$

□

In particular, it follows from the proved theorem that for a *polynomial* h and a test function $f\in(L^2)^\beta$ we have $h^{\Diamond}(f)\in(L^2)^\beta$. Unfortunately, a general (holomorphic at $(Sf)(0)$) function $h:\mathbb{C}\to\mathbb{C}$ has no such a property. More exactly, the following statements are fulfilled.

Proposition 2.1. *Let $h:\mathbb{C}\to\mathbb{C}$ be a holomorphic at $u_0\in\mathbb{C}$ function such that all coefficients h_n from the decomposition*

$$h(u)=\sum_{n=0}^{\infty}h_n(u-u_0)^n \tag{2.2}$$

are non-negative and for some $K > 0$ the series $\sum_{n=1}^{\infty}(n!)^{1+\beta}h_n^2K^n$ diverges. Then there exists $f \in (L^2)^\beta$ with $(Sf)(0) = u_0$ such that $h^\Diamond(f) \notin (L^2)_0^\beta$ (and, consequently, $h^\Diamond(f) \notin (L^2)^\beta$).

Proof. Let

$$f = u_0 + \sum_{n=1}^{\infty}\langle P_n, \frac{\varphi^{\diamond n}}{(n!)^{\frac{2+\beta}{2}}}\rangle, \quad 0 \neq \varphi \in \mathcal{H}_{\mathbb{C}}.$$

Then for each $q \in \mathbb{Z}_+$ (see (1.9))

$$\|f\|_{q,\beta}^2 = |u_0|^2 + \sum_{n=1}^{\infty}(n!)^{1+\beta}2^{qn}\frac{|\varphi|_{\text{ext}}^{2n}}{(n!)^{2+\beta}} < \infty,$$

therefore $f \in (L^2)^\beta$. It is clear that $(Sf)(0) = u_0$. Further (see (1.9), (1.13)),

$$\begin{aligned}
\|h^\Diamond(f)\|_{0,\beta}^2 &= |h_0|^2 + \sum_{n=1}^{\infty}(n!)^{1+\beta}\Big|\sum_{m=1}^{n}h_m\sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}}\frac{\varphi^{\diamond k_1}}{(k_1!)^{\frac{2+\beta}{2}}}\diamond\dots\diamond\frac{\varphi^{\diamond k_m}}{(k_m!)^{\frac{2+\beta}{2}}}\Big|_{\text{ext}}^2\\
&= |h_0|^2 + \sum_{n=1}^{\infty}(n!)^{1+\beta}\Big|\sum_{m=1}^{n}h_m\sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}}\frac{1}{(k_1!\dots k_m!)^{\frac{2+\beta}{2}}}\Big|^2|\varphi|_{\text{ext}}^{2n}\\
&\geq |h_0|^2 + \sum_{n=1}^{\infty}(n!)^{1+\beta}h_n^2|\varphi|_{\text{ext}}^{2n} = +\infty,
\end{aligned}$$

if $|\varphi|_{\text{ext}}$ is sufficiently large. □

Proposition 2.2. *Let $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic at $u_0 \in \mathbb{C}$ not polynomial function such that all coefficients h_n from decomposition (2.2) are non-negative. Then for each $q \in \mathbb{Z}_+$ there exists $f \in (L^2)_q^\beta$ with $(Sf)(0) = u_0$ such that $h^\Diamond(f) \notin (L^2)_0^\beta$.*

Proof. Let us fix arbitrary $q \in \mathbb{Z}_+$ and set

$$f = u_0 + \sum_{n=1}^{\infty}\langle P_n, \frac{\varphi^{\diamond n}}{(n!)^{\frac{1+\beta}{2}}K^{n/2}}\rangle,$$

where $0 \neq \varphi \in \mathcal{H}_{\mathbb{C}}$, $K > 2^q|\varphi|_{\text{ext}}^2$. Then (see (1.9))

$$\|f\|_{q,\beta}^2 = |u_0|^2 + \sum_{n=1}^{\infty}(n!)^{1+\beta}2^{qn}\frac{|\varphi|_{\text{ext}}^{2n}}{(n!)^{1+\beta}K^n} < \infty,$$

therefore $f \in (L^2)_q^\beta$ and $(Sf)(0) = u_0$. Further (see (1.16), (1.13)),

$$\begin{aligned}
\|h^\Diamond(f)\|_{0,\beta}^2 &= |h_0|^2 + \sum_{n=1}^{\infty}(n!)^{1+\beta}\Big|\sum_{m=1}^{n}h_m\sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}}\frac{\varphi^{\diamond n}}{(k_1!\dots k_m!)^{\frac{1+\beta}{2}}K^{n/2}}\Big|_{\text{ext}}^2\\
&= |h_0|^2 + \sum_{n=1}^{\infty}\frac{|\varphi|_{\text{ext}}^{2n}}{K^n}\Big(\sum_{m=1}^{n}h_m\sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}}\big(\frac{n!}{k_1!\dots k_m!}\big)^{\frac{1+\beta}{2}}\Big)^2.
\end{aligned}$$

If the last series converges then for all sufficiently large n

$$\sum_{m=1}^{n}h_m\sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}}\big(\frac{n!}{k_1!\dots k_m!}\big)^{\frac{1+\beta}{2}} < C^n, \tag{2.3}$$

where $C := \frac{K^{1/2}}{|\varphi|_{\mathrm{ext}}}$. Inequality (2.3) can be rewritten in the form

$$\sum_{m=1}^{n} h_m \sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}} \Big(\frac{n!}{k_1!\dots k_m! C^{\frac{2n}{1+\beta}}}\Big)^{\frac{1+\beta}{2}} < 1. \tag{2.4}$$

Denote $c := C^{\frac{2}{1+\beta}}$ and consider the ratio $\frac{n!}{k_1!\dots k_m! c^n}$. Let $n = lm$, where $m \in \mathbb{N}$ is such that $h_m > 0$ and $m > c$. Denote $a_l := \frac{(lm)!}{(l!c^l)^m}$ (this corresponds to the case $k_1 = \dots = k_m = l$). Now $\lim\limits_{l\to\infty} \frac{a_{l+1}}{a_l} = \big(\frac{m}{c}\big)^m > 1$, therefore $\lim\limits_{l\to\infty} a_l = +\infty$. But in this case

$$\lim_{n\to\infty} \sum_{m=1}^{n} h_m \sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}} \Big(\frac{n!}{k_1!\dots k_m! C^{\frac{2n}{1+\beta}}}\Big)^{\frac{1+\beta}{2}} = +\infty,$$

i.e., (2.4) is not fulfilled. □

Proposition 2.3. *Let $f \in (L^2)^\beta$. Then for each $q \in \mathbb{Z}_+$ there exists a holomorphic at $(Sf)(0)$ not polynomial function $h : \mathbb{C} \to \mathbb{C}$ such that $h^\Diamond(f) \in (L^2)^\beta_q$.*

Proof. Let $f = \sum_{n=0}^{\infty} \langle P_n, f^{(n)}\rangle \in (L^2)^\beta$, then for each $q \in \mathbb{Z}_+$ there exists a sequence of non-negative numbers $(\alpha_n)_{n=0}^{\infty}$ such that for each $n \in \mathbb{Z}_+$

$$|f^{(n)}|_{\mathrm{ext}} \le \frac{\alpha_n}{(n!)^{\frac{1+\beta}{2}} 2^{\frac{q+3}{2}n}}$$

and for each $K > 0$ $\lim\limits_{n\to\infty} \alpha_n K^n = 0$. Let h be presented in the form

$$h(u) = \sum_{m=0}^{\infty} h_m (u - (Sf)(0))^m. \tag{2.5}$$

Then using (1.9), (1.13), (1.11) and the equality

$$\sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}} 1 = C_{n-1}^{m-1} \tag{2.6}$$

we obtain

$$\begin{aligned}
\|h^\Diamond(f)\|^2_{q,\beta} &= |h_0|^2 + \sum_{n=1}^{\infty} (n!)^{1+\beta} 2^{qn} \Big| \sum_{m=1}^{n} h_m \sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}} f^{(k_1)} \diamond \dots \diamond f^{(k_m)} \Big|^2_{\mathrm{ext}} \\
&\le |h_0|^2 + \sum_{n=1}^{\infty} (n!)^{1+\beta} 2^{qn} \Big(\sum_{m=1}^{n} |h_m| \sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}} \frac{\alpha_{k_1}\dots\alpha_{k_m}}{(k_1!\dots k_m!)^{\frac{1+\beta}{2}} 2^{\frac{q+3}{2}n}} \Big)^2 \\
&= |h_0|^2 + \sum_{n=1}^{\infty} 2^{-3n} \Big(\sum_{m=1}^{n} |h_m| \sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}} \big(\frac{n!}{k_1!\dots k_m!}\big)^{\frac{1+\beta}{2}} \alpha_{k_1}\dots\alpha_{k_m} \Big)^2 \\
&\le |h_0|^2 + \sum_{n=1}^{\infty} 2^{-n-2} < \infty,
\end{aligned}$$

if

$$|h_m| \le \frac{1}{\max\limits_{\substack{n\ge m, k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}} \Big[\big(\frac{n!}{k_1!\dots k_m!}\big)^{\frac{1+\beta}{2}} \alpha_{k_1}\dots\alpha_{k_m} \Big]}. \tag{2.7}$$

□

Proposition 2.4. *Let* $f = \sum_{n=0}^{N}\langle P_n, f^{(n)}\rangle$, $f^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$, *and coefficients* h_m *from the decomposition* $h(u) = \sum_{m=0}^{\infty} h_m(u - f^{(0)})^m$ *for a holomorphic at* $f^{(0)}$ *function* $h : \mathbb{C} \to \mathbb{C}$ *satisfy estimates*

$$|h_m| \le \frac{K^m \min\limits_{n\in\{m,\dots,Nm\}} \alpha_n}{\big((Nm)!\big)^{\frac{1+\beta}{2}}} \tag{2.8}$$

with some $K > 0$, *where* $(\alpha_n > 0)_{n=0}^{\infty}$*—a numerical sequence such that for each* $C > 0$ $\sum_{n=0}^{\infty} C^n \alpha_n < \infty$. *Then* $h^{\Diamond}(f) \in (L^2)^{\beta}$.

Proof. The case $N = 0$ is trivial, therefore we consider the case $N \in \mathbb{N}$ only. Let $C := \max\{|f^{(1)}|_{\mathrm{ext}}, \dots, |f^{(N)}|_{\mathrm{ext}}\}$. Using (1.9), (1.13), (1.11) and (2.6), for each $q \in \mathbb{Z}_+$ we obtain

$$\begin{aligned}
\|h^{\Diamond}(f)\|_{q,\beta}^2 &= |h_0|^2 + \sum_{n=1}^{\infty} (n!)^{1+\beta} 2^{qn} \Big| \sum_{m=\{\frac{n}{N}\}}^{n} h_m \sum_{\substack{k_1,\dots,k_m\in\mathbb{N},\\ k_1+\dots+k_m=n}} f^{(k_1)} \Diamond \dots \Diamond f^{(k_m)} \Big|_{\mathrm{ext}}^2 \\
&\le |h_0|^2 + \sum_{n=1}^{\infty} (n!)^{1+\beta} 2^{qn} \Big(\sum_{m=\{\frac{n}{N}\}}^{n} |h_m| C^m 2^{n-1} \Big)^2 \\
&\le |h_0|^2 + \sum_{n=1}^{\infty} 2^{(q+2)n-2} \alpha_n^2 \Big(\sum_{m=\{\frac{n}{N}\}}^{n} |h_m| \frac{C^m \big((Nm)!\big)^{\frac{1+\beta}{2}}}{\alpha_n} \Big)^2 < \infty,
\end{aligned}$$

here $\{\frac{n}{N}\} = \frac{n}{N}$ if $\frac{n}{N} \in \mathbb{N}$, and $\{\frac{n}{N}\} = [\frac{n}{N}] + 1$ if $\frac{n}{N} \notin \mathbb{N}$, $[\cdot]$ denotes the entire part of a number (note that by construction $m \le n \le Nm$). $\square$

Remark 2.1. As an example of a sequence $(\alpha_n)_{n=0}^{\infty}$ from Proposition 2.4 one can consider $\alpha_n = \frac{1}{(n!)^{\varepsilon}}$, $\varepsilon > 0$. In this case formula (2.8) has a form

$$|h_m| \le \frac{K^m}{\big((Nm)!\big)^{\frac{1+\beta}{2}+\varepsilon}}. \tag{2.9}$$

Example 2.1. Let $h(u) = \exp(u)$, i.e., for each $m \in \mathbb{Z}_+$ $h_m = \frac{1}{m!}$. We clarify, under which conditions $\exp^{\Diamond}(f) \in (L^2)^{\beta}$, where $f = \sum_{n=0}^{N}\langle P_n, f^{(n)}\rangle$, $f^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$. For $\alpha_n = \frac{1}{(n!)^{\varepsilon}}$ estimate (2.8) has a form (see (2.9))

$$\frac{\big((Nm)!\big)^{\frac{1+\beta}{2}+\varepsilon}}{m! K^m} \le 1 \tag{2.10}$$

for each $m \in \mathbb{Z}_+$ with some $K > 0$. Set $a_m := \frac{\big((Nm)!\big)^{\frac{1+\beta}{2}+\varepsilon}}{m! K^m}$, then

$$\frac{a_{m+1}}{a_m} = \frac{[(Nm+1)\dots(Nm+N)]^{\frac{1+\beta}{2}+\varepsilon}}{(m+1)K}.$$

It follows from here that (2.10) is fulfilled if

$$N\Big(\frac{1+\beta}{2} + \varepsilon\Big) \le 1, \tag{2.11}$$

i.e., $\exp^{\Diamond}(f) \in (L^2)^{\beta}$ if $\beta < 1$ and $N \le 1$.

Remark 2.2. Estimate (2.11) is not rough. In fact, let $f = \langle P_N, f^{(N)}\rangle$, $f^{(N)} \in \mathcal{H}_{\mathrm{ext}}^{(N)}$. Then (see (1.13), (1.9), (1.11))

$$h^{\Diamond}(f) = h_0 + \sum_{m=1}^{\infty} \langle P_{Nm}, h_m {f^{(N)}}^{\Diamond m}\rangle,$$

$$\|h^{\Diamond}(f)\|_{q,\beta}^2 \le |h_0|^2 + \sum_{m=1}^{\infty} \big((Nm)!\big)^{1+\beta} 2^{qNm} |h_m|^2 |f^{(N)}|_{\mathrm{ext}}^{2m}.$$

The last series converges for each $q \in \mathbb{Z}_+$ if for each $m \in \mathbb{N}$

$$|h_m| \le \frac{K^m \alpha_m}{\big((Nm)!\big)^{\frac{1+\beta}{2}}}, \tag{2.12}$$

where $K > 0$ and the numerical sequence $(\alpha_m)_{m=1}^{\infty}$ as in Proposition 2.4. In the case $h_m = \frac{1}{m!}$ and $N \ge 2$ estimate (2.12) is not fulfilled.

Proposition 2.5. *Let $0 \le \beta_1 < \beta_2 \le 1$, $f \in (L^2)^{\beta_2}$ and a holomorphic at $(Sf)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ be such that coefficients h_m ($m \in \mathbb{N}$) from decomposition* (2.5) *satisfy estimates* (2.7) *with $\beta = \beta_1$. Then $h^{\Diamond}(f) \in (L^2)^{\beta_1}$.*

Proof. This result follows from the proof of Proposition 2.3 and embedding $(L^2)_0^{\beta_2} \subset (L^2)^{\beta_1}$. □

Proposition 2.6. *Let $0 \le \beta_1 < \beta_2 \le 1$, $f \in (L^2)^{\beta_2}$, $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic at $(Sf)(0)$ function such that coefficients h_m ($m \in \mathbb{N}$) from decomposition* (2.5) *satisfy estimates*

$$|h_m| \le \frac{1}{c^m \max\limits_{\substack{n \ge m, k_1,\dots,k_m \in \mathbb{N},\\ k_1+\dots+k_m = n}} \left[\frac{(n!)^{\frac{1+\beta_3}{2}} 2^{n(1+\varepsilon)}}{(k_1!\dots k_m!)^{\frac{1+\beta_2}{2}}}\right]}, \tag{2.13}$$

where $\beta_3 \in (\beta_1, \beta_2)$, $\varepsilon > 0$, $c \ge \|f\|_{0,\beta_2}$. Then $h^{\Diamond}(f) \in (L^2)_0^{\beta_3} \subset (L^2)^{\beta_1}$.

Proof. Since (see (1.9)) for each $n \in \mathbb{N}$ $|f^{(n)}|_{\mathrm{ext}} \le \frac{c}{(n!)^{\frac{1+\beta_2}{2}}}$, we have (see (1.13), (1.9), (1.11), (2.6))

$$\begin{aligned}
\|h^{\Diamond}(f)\|_{0,\beta_3}^2 &= |h_0|^2 + \sum_{n=1}^{\infty} (n!)^{1+\beta_3} \Big| \sum_{m=1}^{n} h_m \sum_{\substack{k_1,\dots,k_m \in \mathbb{N},\\ k_1+\dots+k_m = n}} f^{(k_1)} \Diamond \dots \Diamond f^{(k_m)} \Big|_{\mathrm{ext}}^2 \\
&\le |h_0|^2 + \sum_{n=1}^{\infty} (n!)^{1+\beta_3} \Big(\sum_{m=1}^{n} |h_m| \sum_{\substack{k_1,\dots,k_m \in \mathbb{N},\\ k_1+\dots+k_m = n}} \frac{c^m}{(k_1!\dots k_m!)^{\frac{1+\beta_2}{2}}} \Big)^2 \\
&= |h_0|^2 + \sum_{n=1}^{\infty} 2^{-2n(1+\varepsilon)} \Big(\sum_{m=1}^{n} |h_m| \sum_{\substack{k_1,\dots,k_m \in \mathbb{N},\\ k_1+\dots+k_m = n}} \frac{(n!)^{\frac{1+\beta_3}{2}} c^m 2^{n(1+\varepsilon)}}{(k_1!\dots k_m!)^{\frac{1+\beta_2}{2}}} \Big)^2 \\
&\le |h_0|^2 + \sum_{n=1}^{\infty} 2^{-2n(1+\varepsilon)} \Big(\sum_{m=1}^{n} C_{n-1}^{m-1} \Big)^2 = |h_0|^2 + \sum_{n=1}^{\infty} 2^{-2n\varepsilon - 2} < \infty. \quad \square
\end{aligned}$$

Remark 2.3. One can replace estimates (2.13) by the slightly more rough estimates

$$|h_m| \le \frac{1}{c^m \max\limits_{n \ge m} \left[\frac{(n!)^{\frac{1+\beta_3}{2}} 2^{n(1+\varepsilon)}}{\big([\frac{n}{m}]!\big)^{\frac{1+\beta_2}{2} m}}\right]},$$

where $[\cdot]$ denotes the entire part of a number.

Finally, let us consider an example of a stochastic equation with Wick-type nonlinearities.

Example 2.2. (a linear equation).

Let

$$X_t = X_0 + \int_0^t X_u \Diamond F du + g \int_0^t X_u \widehat{d} M_u,$$

where $X_0 \in (L^2)^\beta$, $f = \sum_{n=0}^N \langle P_n, f^{(n)} \rangle$, $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $g \in \mathbb{C}$. It was shown in [31] that the solution of this equation is

$$X_t = X_0 \Diamond \exp^\Diamond \{Ft + gM_t\} \in (L^2)^{-1}.$$

But if $N \leq 1$ and $\beta \in [0, 1)$ then we can conclude that $X_t \in (L^2)^\beta$, this result from estimate (2.11) and Theorem 2.1 follows.

Remark 2.4. It is easy to see that in the Gaussian and Poissonian analysis all results of this section can be naturally transferred to the spaces of test functions of the so-called *nonregular* rigging of (L^2) (this rigging is described in, e.g., [31]).

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