ON THE MULTI-DIMENSIONAL SCHRÖDINGER OPERATORS
WITH POINT INTERACTIONS

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ABSTRACT. We study two- and three-dimensional matrix Schrödinger operators with $m \in \mathbb{N}$ point interactions. Using the technique of boundary triplets and the corresponding Weyl functions, we complete and generalize the results obtained by the other authors in this field.

For instance, we parametrize all self-adjoint extensions of the initial minimal symmetric Schrödinger operator by abstract boundary conditions and characterize their spectra. Particularly, we find a sufficient condition in terms of distances and intensities for the self-adjoint extension $H_{\alpha,X}^{(3)}$ to have m' negative eigenvalues, i.e., $\kappa_-(H_{\alpha,X}^{(3)}) = m' \leq m$. We also give an explicit description of self-adjoint nonnegative extensions.

1. INTRODUCTION

Multi-dimensional Schrödinger operators with point interactions have been intensively studied in the three last decades (see [1, 3, 4, 5, 6, 7, 8, 16, 18, 21, 26]). Starting with the fundamental paper [7] by Berezin and Faddeev, operators associated in $L^2(\mathbb{R}^3)$ with the differential expression

$$(1.1) \qquad -\Delta + \sum_{j=1}^{m} \alpha_j \delta(\cdot - x_j), \quad \alpha_j \in \mathbb{R}, \quad m \in \mathbb{N}$$

have been treated in the framework of the extension theory. Namely, the authors proposed, in the case of one point interaction, to consider all self-adjoint extensions of the following minimal Schrödinger operator

$$(1.2) \quad H = -\Delta \restriction \mathrm{dom}(H), \quad \mathrm{dom}(H) := \left\{ f \in W_2^2(\mathbb{R}^3) : f(x_j) = 0, \ j \in \{1, \ldots, m\} \right\}$$

as a realizations of the expression (1.1).

It is well known that H is a closed nonnegative symmetric operator with deficiency indices $n_\pm(H) = m$ (cf. [3]). In [3], the authors proposed to associate with the Hamiltonian (1.1) a certain m-parametric family $H_{\alpha,X}^{(3)}$ of self-adjoint extensions of the operator H. They parametrized the extensions $H_{\alpha,X}^{(3)}$ in terms of the resolvents. The latter enabled them to describe the spectrum of the $H_{\alpha,X}^{(3)}$.

In the recent publications [8, 16], boundary triplets and the corresponding Weyl functions concept (see [10, 15] and also Section 2 was involved to investigate multi-dimensional Schrödinger operators with point interactions. In [6, 8, 16], two- and three-dimensional Schrödinger operators with one point interaction were studied.

In the present paper, we apply boundary triplets and the corresponding Weyl function approach to study the matrix multi-dimensional Schrödinger operators with point interactions. Namely, in $L^2(\mathbb{R}^d, \mathbb{C}^n)$ $(d \in \{2, 3\})$, we consider the following matrix Schrödinger

differential expression with singular potential localized on the set $X := \{x_j\}_{j=1}^m \subset \mathbb{R}^d$

$$(1.3) \qquad -\Delta \otimes I_n + \sum_{j=1}^m \Lambda_j \delta(\cdot - x_j), \quad \Lambda_j \in \mathbb{R}^{n \times n}, \quad j \in \{1, \dots, m\}.$$

The minimal symmetric operator associated with this expression in $L^2(\mathbb{R}^d, \mathbb{C}^n)$ is defined by

$$(1.4) \qquad H := -\Delta \otimes I_n, \quad \mathrm{dom}(H) := \big\{ f \in W_2^2(\mathbb{R}^d, \mathbb{C}^n) : f(x_j) = 0, \ x_j \in X \big\}.$$

The matrix three-dimensional Schrödinger operator with one point interaction was studied in [8]. We generalize the results of [8] to the case of m point interactions and $d = 2, 3$. Namely, we construct a boundary triplet Π for H^*. Moreover, we compute the corresponding Weyl function and the γ-field for Π, as well as the scattering matrix for a pair $\{H_0, H_\Theta\}$. It is worth to mention that the Weyl function coincides with the matrix-valued function appearing in the formulas for the resolvents of $H_{\alpha,X}^{(d)}$, $d = 2, 3$, in [3, Chapters II.1, II.4].

In addition, we describe proper, self-adjoint, and nonnegative self-adjoint extensions of the initial minimal symmetric operator H and characterize their spectra. In particular, we show that the family $H_{\alpha,X}^{(d)}$ might be parametrized by means of diagonal matrices (see Remark 4.8). In the case $n = 1$, we establish numerous links between our results and the results obtained in the previous publications mentioned above.

In Theorem 3.1 we establish a connection between the result on uniqueness of non-negative self-adjoint extension of an arbitrary nonnegative symmetric operator A in [10, Proposition 10] and the recent result of V. Adamyan [1, Theorem 2.4]. Particularly, we reproved the result on the uniqueness of nonnegative self-adjoint extension of the minimal symmetric operator H in the case $n = 1$ and $d = 2$.

Let us briefly review the structure of the paper. Section 2 is introductory. It contains definitions and facts necessary for further exposition. In Section 3, we establish the uniqueness criterion mentioned above. In Sections 4 and 5, we investigate the matrix Schrödinger operators with point interactions in the cases $d = 3$ and $d = 2$, respectively. Namely, in Subsection 4.1 (resp., 5.1), we define boundary triplet for the H^* and also compute the corresponding Weyl function and the γ-field. The description of the extensions of H is provided in Subsection 4.2 (5.2). Finally, Subsection 4.3 (5.3) is devoted to the spectral analysis of the self-adjoint extensions of H.

Notation. Let $\mathfrak{H}$ and $\mathcal{H}$ stand for separable Hilbert spaces; $[\mathfrak{H}, \mathcal{H}]$ stands for the space of bounded linear operators from $\mathfrak{H}$ to $\mathcal{H}$, $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; the set of closed operators in $\mathcal{H}$ is denoted by $\mathcal{C}(\mathcal{H})$. Let A be a linear operator in a Hilbert space $\mathfrak{H}$. In what follows, $R_z(A)$ denotes the resolvent $(A - z)^{-1}$ of the operator A; $\mathrm{dom}(A)$, $\ker(A)$, $\mathrm{ran}(A)$ are the domain, the kernel, and the range of A, respectively; $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A; $\mathcal{N}_z$ stands for the defect subspace of A corresponding to eigenvalue z. Denote by $C_0^\infty(\mathbb{R}^d \backslash X)$ the space of infinitely differentiable functions with compact support.

2. Preliminaries

2.1. **Boundary triplets and Weyl functions.** In this subsection, we recall basic notions and facts of the theory of boundary triplets (we refer to [10, 15] for a detailed exposition).

2.1.1. *Linear relations, boundary triplets and proper extensions.* **1.** The set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$ is the set of closed linear subspaces of $\mathcal{H} \oplus \mathcal{H}$. Recall that $\mathrm{dom}(\Theta) = \big\{ f : \big(\begin{smallmatrix} f \\ f' \end{smallmatrix}\big) \in \Theta \big\}$, $\mathrm{ran}(\Theta) = \big\{ f' : \big(\begin{smallmatrix} f \\ f' \end{smallmatrix}\big) \in \Theta \big\}$, and $\mathrm{mul}(\Theta) = \big\{ f' : \big(\begin{smallmatrix} 0 \\ f' \end{smallmatrix}\big) \in \Theta \big\}$ are the domain, the range, and the multivalued part of Θ. A closed linear operator in $\mathcal{H}$

is identified with its graph, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators in $\mathcal{H}$ is viewed as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$. In particular, a linear relation Θ is an operator if and only if the multivalued part $\mathrm{mul}\,(\Theta)$ is trivial. We recall that the adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of a linear relation Θ in $\mathcal{H}$ is defined by

$$\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h',k) = (h,k') \ \text{ for all } \ \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\}.$$

The linear relation Θ is said to be *symmetric* if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$. The linear relation Θ is said to be *nonnegative* if $(k',k) \geq 0$ for all $\begin{pmatrix} k \\ k' \end{pmatrix} \in \Theta$. For the symmetric relation $\Theta \subseteq \Theta^*$ in $\mathcal{H}$ the multivalued part $\mathrm{mul}\,(\Theta)$ is the orthogonal complement of $\mathrm{dom}(\Theta)$ in $\mathcal{H}$. Setting $\mathcal{H}_{\mathrm{op}} := \overline{\mathrm{dom}(\Theta)}$ and $\mathcal{H}_\infty = \mathrm{mul}\,(\Theta)$, one verifies that Θ can be written as the direct orthogonal sum of a self-adjoint operator Θ_{op} in the subspace $\mathcal{H}_{\mathrm{op}}$ and a "pure" relation $\Theta_\infty = \left\{ \begin{pmatrix} 0 \\ f' \end{pmatrix} : f' \in \mathrm{mul}\,(\Theta) \right\}$ in the subspace $\mathcal{H}_\infty$.

Any closed linear relation admits the following representation (see, for instance, [22])

$$(2.1) \qquad \Theta = \{(h,h')^\top \in \mathcal{H} \oplus \mathcal{H} : Ch - Dh' = 0\}, \quad C, D \in [\mathcal{H}].$$

Note that representation (2.1) is not unique.

2. Let A be a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) = \dim \ker(A^* \pm i) \leq \infty$.

Definition 2.1. ([15]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for the adjoint operator A^* of A if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}(A^*) \to \mathcal{H}$ are linear mappings such that
(i) the second Green identity

$$(2.2) \qquad (A^* f, g)_{\mathfrak{H}} - (f, A^* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}$$

holds for all $f, g \in \mathrm{dom}(A^*)$, and
(ii) the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \mathrm{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Since $n_+(A) = n_-(A)$, a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* exists and is not unique [15]. Moreover, $\dim \mathcal{H} = n_\pm(A)$ and $\mathrm{dom}(A) = \mathrm{dom}(A^*) \upharpoonright \ker(\Gamma_0) \cap \ker(\Gamma_1)$.

A closed extension $\widetilde{A}$ of A is called *proper* if $A \subseteq \widetilde{A} \subseteq A^*$. Two proper extensions $\widetilde{A}_1$ and $\widetilde{A}_2$ of A are called *disjoint* if $\mathrm{dom}(\widetilde{A}_1) \cap \mathrm{dom}(\widetilde{A}_2) = \mathrm{dom}(A)$ and *transversal* if, in addition, $\mathrm{dom}(\widetilde{A}_1) \dotplus \mathrm{dom}(\widetilde{A}_2) = \mathrm{dom}(A^*)$. The set of all proper extensions of A, $\mathrm{Ext}\,A$, may be described in the following way.

Proposition 2.2. ([10, 15]). *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$(2.3) \qquad \mathrm{Ext}_A \ni A_\Theta \to \Theta := \Gamma(\mathrm{dom}(\widetilde{A})) = \{\{\Gamma_0 f, \Gamma_1 f\} : \ f \in \mathrm{dom}(\widetilde{A})\}$$

establishes a bijective correspondence between the set $\widetilde{\mathcal{C}}(\mathcal{H})$ and the set of closed proper extensions $A_\Theta \subseteq A^*$ of A. Furthermore,

$$(A_\Theta)^* = A_{\Theta^*}$$

holds for any $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$. The extension A_Θ in (2.3) is symmetric (self-adjoint) if and only if Θ is symmetric (self-adjoint).

Proposition 2.2 and representation (2.1) yield the following corollary.

Corollary 2.3. (i) *The extensions $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and $A_1 := A^* \upharpoonright \ker(\Gamma_1)$ are self-adjoint.* (ii) *Any proper extension A_Θ of the operator A admits the representation*

$$(2.4) \qquad \begin{aligned} A_\Theta &= A_{C,D} = A^* \upharpoonright \mathrm{dom}(A_{C,D}), \\ \mathrm{dom}(A_{C,D}) &= \mathrm{dom}(A^*) \upharpoonright \ker(D\Gamma_1 - C\Gamma_0), \quad C, D \in [H]. \end{aligned}$$

(*iii*) *If, in addition, the closed extensions A_Θ and A_0 are disjoint, then* (2.4) *takes the form*

(2.5) $A_\Theta = A_B = A^* \upharpoonright \operatorname{dom}(A_B), \quad \operatorname{dom}(A_B) = \operatorname{dom}(A^*) \upharpoonright \ker(\Gamma_1 - B\Gamma_0), \quad B \in \mathcal{C}(\mathcal{H}).$

Remark 2.4. In the case $\dim(\mathcal{H}) < \infty$, it follows from the result of Rofe-Beketov [28] that the extension A_Θ defined by (2.4) is self-adjoint if and only if the following conditions hold

(2.6) $$CD^* = DC^*, \quad 0 \in \rho(CC^* + DD^*).$$

2.1.2. Weyl functions, γ-fields, and Krein type formula for resolvents.

Definition 2.5. ([10]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator valued functions $\gamma(\cdot) : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot) : \rho(A_0) \to [\mathcal{H}]$ defined by

(2.7) $\gamma(z) := (\Gamma_0 \upharpoonright \mathcal{N}_z)^{-1} \quad \text{and} \quad M(z) := \Gamma_1 \gamma(z), \quad z \in \rho(A_0),$

are called the γ-field and the *Weyl function*, respectively, corresponding to the boundary triplet Π.

The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.7) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$.

The spectrum of the closed (not necessarily self-adjoint) extensions of A can be described with the help of the function $M(\cdot)$.

Proposition 2.6. *Let $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, $A_\Theta \in \operatorname{Ext}_A$, and $z \in \rho(A_0)$. Then*

$$z \in \sigma_i(A_\Theta) \quad \Leftrightarrow \quad 0 \in \sigma_i(\Theta - M(z)), \quad i \in \{p, \ c, \ r\}.$$

Moreover, for $z \in \rho(A_0) \cap \rho(A_\Theta)$ the resolvent formula

(2.8) $R_z(A_\Theta) = R_z(A_0) + \gamma(z)(\Theta - M(z))^{-1}\gamma(\bar{z})^*, \quad z \in \rho(A_\Theta) \cap \rho(A_0)$

holds, cf. [10]. Formula (2.8) is a generalization of the well-known Krein formula for canonical resolvents. We emphasize that it is valid for any closed extension $A_\Theta \subseteq A^*$ of A with nonempty resolvent set.

According to the representation (2.4), it reads (see [22])

(2.9) $R_z(A_\Theta) = R_z(A_0) + \gamma(z)(C - DM(z))^{-1}D\gamma(\bar{z})^*, \quad z \in \rho(A_{C,D}) \cap \rho(A_0).$

Let now A be a closed densely defined nonnegative symmetric operator in the Hilbert space $\mathfrak{H}$. Among its nonnegative self-adjoint extensions two extremal extension A_F and A_K are laid special emphasis on. They are called Friedrichs and Krein extension, respectively, (see [19]). Operator $\widetilde{A}$ is nonnegative self-adjoint extension of A if and only if $A_K \leq \widetilde{A} \leq A_F$ in the sense of the corresponding quadratic forms.

Proposition 2.7. ([10, 11]). *Let A be a densely defined nonnegative symmetric operator with finite deficiency indices in $\mathfrak{H}$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 \geq 0$. Let also $M(\cdot)$ be the corresponding Weyl function. Then the following assertions hold.*
(*i*) *There exists a strong resolvent limit*

$$M(0) := s - R - \lim_{x\uparrow 0} M(x), \quad (M(-\infty) := s - R - \lim_{x\downarrow -\infty} M(x)).$$

(*ii*) *$M(0)$ $(M(-\infty))$ is a self-adjoint linear relation in $\mathcal{H}$ associated with the semibounded below (above) quadratic form $\mathrm{t}_0[f] = \lim_{x\uparrow 0}(M(x)f, f) \geq \beta$ (resp. $\mathrm{t}_{-\infty}[f] =$*

$\lim_{x\downarrow-\infty}(M(x)f,f) \leq \alpha)$ *with the domain*

(2.10) $\mathrm{dom}(\mathrm{t}_0) = \{f \in \mathcal{H} : \lim_{x\uparrow 0}|(M(x)f,f)| < \infty\} = \mathrm{dom}((M(0)_{\mathrm{op}} - \beta)^{1/2}),$

(2.11) $\mathrm{dom}(\mathrm{t}_{-\infty}) = \{f \in \mathcal{H} : \lim_{x\downarrow-\infty}|(M(x)f,f)| < \infty\} = \mathrm{dom}((\alpha - M(-\infty)_{\mathrm{op}}))^{1/2}).$

Moreover,

$$\mathrm{dom}(A_K) = \{f \in \mathrm{dom}(A^*) : (\Gamma_0 f, \Gamma_1 f)^\top \in M(0)\}$$

$$(\textit{resp.}\ \ \mathrm{dom}(A_F) = \{f \in \mathrm{dom}(A^*) : (\Gamma_0 f, \Gamma_1 f)^\top \in M(-\infty)\}).$$

(iii) Extensions A_0 and A_K are disjoint (A_0 and A_F are disjoint) if and only if $M(0) \in \mathcal{C}(\mathcal{H})$ ($M(-\infty) \in \mathcal{C}(\mathcal{H})$ resp.) Moreover,

(2.12)
$$\mathrm{dom}(A_K) = \mathrm{dom}(A^*) \upharpoonright \ker(\Gamma_1 - M(0)\Gamma_0)$$
$$(\mathrm{dom}(A_F) = \mathrm{dom}(A^*) \upharpoonright \ker(\Gamma_1 - M(-\infty)\Gamma_0)).$$

(iv) $A_F = A_0$ ($A_K = A_0$) if and only if

(2.13) $\lim_{x\downarrow-\infty}(M(x)f,f) = -\infty \quad (\lim_{x\uparrow 0}(M(x)f,f) = +\infty), \quad f \in \mathcal{H} \setminus \{0\}.$

(v) If $A_0 = A_F$ and $\mathrm{dom}(\mathrm{t}_{\Theta_{\mathrm{op}}}) \subset \mathrm{dom}(\mathrm{t}_0)$, then the number of negative eigenvalues of self-adjoint extension A_Θ of A equals the number of negative eigenvalues of the quadratic form $\mathrm{t}_{\Theta_{\mathrm{op}}} - \mathrm{t}_0$, i.e.,

(2.14) $\kappa_-(A_\Theta) = \kappa_-(\mathrm{t}_{\Theta_{\mathrm{op}}} - \mathrm{t}_0).$

Moreover, if $M(0) \in [\mathcal{H}]$, then $\kappa_-(A_\Theta) = \kappa_-(\Theta - M(0))$.
   (vi) In particular, the A_Θ is nonnegative self-adjoint if and only if

(2.15) $\mathrm{dom}(\mathrm{t}_{\Theta_{\mathrm{op}}}) \subset \mathrm{dom}(\mathrm{t}_0) \quad and \quad \mathrm{t}_{\Theta_{\mathrm{op}}} - \mathrm{t}_0 \geq 0.$

   If $M(0) \in [\mathcal{H}]$, the inequality in (2.15) takes the form $\Theta - M(0) \geq 0$.

2.2. **Scattering matrices.** Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal finite deficiency indices and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Assume that A_Θ is a self-adjoint extension of A with $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$. Since here $\dim \mathcal{H}$ is finite, by (2.8),

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1}, \quad z \in \rho(A_\Theta) \cap \rho(A_0),$$

is a finite rank operator and therefore the pair $\{A_\Theta, A_0\}$ performs a so-called *complete scattering system*, that is, the *wave operators*

$$W_\pm(A_\Theta, A_0) := \text{s-}\lim_{t\to\pm\infty} e^{itA_\Theta} e^{-itA_0} P^{ac}(A_0),$$

exist and their ranges coincide with the absolutely continuous subspace $\mathfrak{H}^{ac}(A_\Theta)$ of A_Θ, cf. [17, 30]. $P^{ac}(A_0)$ denotes the orthogonal projection onto the absolutely continuous subspace $\mathfrak{H}^{ac}(A_0)$ of A_0. The *scattering operator* $S(A_\Theta, A_0)$ of the *scattering system* $\{A_\Theta, A_0\}$ is then defined by

$$S(A_\Theta, A_0) := W_+(A_\Theta, A_0)^* W_-(A_\Theta, A_0).$$

If we regard the scattering operator as an operator in $\mathfrak{H}^{ac}(A_0)$, then $S(A_\Theta, A_0)$ is unitary, commutes with the absolutely continuous part $A_0^{ac} := A_0 \upharpoonright \mathrm{dom}(A_0) \cap \mathfrak{H}^{ac}(A_0)$ of A_0. It follows that $S(A_\Theta, A_0)$ is unitarily equivalent to multiplication operator induced by a family $\{S_\Theta(z)\}$ of unitary operators in a spectral representation of A_0^{ac} (for details, see [30, Section 2.4]). Define a family of Hilbert spaces $\{\mathcal{H}_z\}_{z\in\Lambda^M}$ by

$$\mathcal{H}_z := \mathrm{ran}\left(\mathrm{Im}(M(z+i0))\right) \subseteq \mathcal{H}, \quad z \in \Lambda^M,$$

where $M(z+i0) = s - \lim_{\epsilon\to 0} M(z+i\epsilon)$ and $\Lambda^M := \{z \in \mathbb{R} : M(z+i0) \text{ exists}\}$.

In the following theorem the scattering matrix is calculated in the case of a simple operator A. Recall that symmetric operator A densely defined in $\mathfrak{H}$ is said to be *simple* if there is no nontrivial subspace which reduces it to a self-adjoint operator.

Theorem 2.8. ([8]). *Let A be as above, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with the corresponding Weyl function $M(\cdot)$. Assume also that $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ and A_Θ is a self-adjoint extension of A. Then the scattering matrix $\{S_\Theta(z)\}_{z\in\mathbb{R}}$ of the scattering system $\{A_\Theta, A_0\}$ admits the representation*

(2.16)
$$S_\Theta(z) = I_{\mathcal{H}_z} + 2i\sqrt{\mathrm{Im}(M(z))}\big(\Theta - M(z)\big)^{-1}\sqrt{\mathrm{Im}(M(z))} \in [\mathcal{H}_z],$$
$$\textit{for a.e.} \quad z \in \Lambda^M.$$

3. Abstract description of nonnegative self-adjoint extensions

Let A be a densely defined nonnegative closed symmetric operator in $\mathfrak{H}$. A complete description of all nonnegative self-adjoint extensions of A, as well as uniqueness criterion for nonnegative self-adjoint extension, has originally been obtained by Krein in [19] (see also [2]). His results were generalized in numerous works (see for instance [1, 5, 10] and reference therein). Particularly, a description in terms of boundary triplets and the corresponding Weyl functions was obtained in [10, Theorem 4, Proposition 5] (cf. Proposition 2.7 in Section 2).

One more uniqueness criterion has recently been presented by V. Adamyan [1, Theorem 2.4]. In this section, we show that this criterion might be obtained in the framework of boundary triplets approach. We also find the description of all nonnegative self-adjoint extensions of A similar to that of Adamyan in the particular case $A > \mu I > 0$.

Theorem 3.1. *Let $\widetilde{A}_0$ be a nonnegative self-adjoint extension of a nonnegative closed symmetric operator A in $\mathfrak{H}$, and let P_{-1} be an orthogonal projector from $\mathfrak{H}$ onto $\mathcal{N}_{-1}$. Then $\widetilde{A}_0$ is a unique nonnegative self-adjoint extension of A if and only if*

(3.1)
$$\lim_{\varepsilon\downarrow 0}(P_{-1}(\widetilde{A}_0 + 1)(\widetilde{A}_0 + \varepsilon)^{-1} \upharpoonright \mathcal{N}_{-1})^{-1} = 0,$$

(3.2)
$$\lim_{\varepsilon\downarrow 0}(P_{-1}(\widetilde{A}_0 + 1)(\varepsilon\widetilde{A}_0 + I)^{-1} \upharpoonright \mathcal{N}_{-1})^{-1} = 0.$$

Proof. It is well known (see, for instance, [10]) that for each pair of transversal extensions $\widetilde{A}_1$ and $\widetilde{A}_0$ there exists boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $\ker\Gamma_i = \mathrm{dom}(\widetilde{A}_i)$, $i \in \{0, 1\}$. In particular, such boundary triplet may be constructed for the pair $\widetilde{A}_0 \geq 0$ and $\widetilde{A}_1$, where $\mathrm{dom}(\widetilde{A}_1) = \mathrm{dom}(A) \dotplus \mathcal{N}_{-a}$, $a > 0$. In this case setting

(3.3)
$$\mathcal{H} = \mathcal{N}_{-a}, \quad \Gamma_1 = P_{-a}(\widetilde{A}_0 + a)P_1, \quad \Gamma_0 = P_0,$$

where P_{-a} is the orthogonal projector from $\mathfrak{H}$ onto $\mathcal{N}_{-a}$ and P_1, P_0 are the projectors from $\mathrm{dom}(A^*) = \mathrm{dom}(\widetilde{A}_0) \dotplus \mathcal{N}_{-a}$ onto $\mathrm{dom}(\widetilde{A}_0)$ and $\mathcal{N}_{-a}$, respectively we obtain a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ (see [10]). The corresponding Weyl function is

(3.4) $\quad M_a(z) = (z + a)P_{-a}[I + (z + a)(\widetilde{A}_0 - z)^{-1}] = (z + a)P_{-a}(\widetilde{A}_0 + I)(\widetilde{A}_0 - z)^{-1}.$

Put $a = 1$. Then conditions (2.13) take the form

(3.5)
$$\mathsf{t}_0[f] = \lim_{\varepsilon\downarrow 0}\mathsf{t}_0^\varepsilon[f] := \lim_{\varepsilon\downarrow 0}((1 - \varepsilon)(\widetilde{A}_0 + 1)(\widetilde{A}_0 + \varepsilon)^{-1}f, f) = +\infty,$$

(3.6) $\mathsf{t}_{-\infty}[f] = \lim_{\varepsilon\downarrow 0}\mathsf{t}_{-\infty}^\varepsilon[f] := \lim_{\varepsilon\downarrow 0}((\varepsilon - 1))(\widetilde{A}_0 + 1)(\varepsilon\widetilde{A}_0 + I)^{-1}f, f) = -\infty, \quad f \in \mathcal{N}_{-1}.$

Since $\mathsf{t}_0^\varepsilon[f]$ is non-decreasing semi-bounded from below ($0 < \varepsilon < 1$) family of the closed symmetric forms, (3.5) is equivalent to (3.1) (cf. [17]). Analogously, since $\mathsf{t}_\infty^\varepsilon[f]$ is non-increasing semi-bounded from above family of the closed symmetric forms, (3.6)

is equivalent to (3.2). Therefore, by Proposition 2.7(iv), the equality $A_K = A_F$ and, consequently, the uniqueness of nonnegative self-adjoint extension of A is equivalent to the conditions (3.1)–(3.2) (see [19]). $\qquad\square$

Assume now that $A > \mu I > 0$ and $\widetilde{A}_0 = A_F$ in (3.3). Let also $a = 1$. According to Proposition 2.7(vi), the following description of all nonnegative self-adjoint extensions of A is valid.

Proposition 3.2. *Let A and $\widetilde{A}_0$ be as above. Then the set of all nonnegative self-adjoint extensions A_Y of A might be described as follows*

$$(3.7) \qquad\qquad \mathrm{dom}(A_Y) = \mathrm{dom}(A^*) \upharpoonright \ker\{Y\Gamma_1 - \Gamma_0\},$$

where Γ_0, Γ_1 are defined by (3.3) and Y runs over the set of all nonnegative contractions in $\mathcal{N}_{-1}$ satisfying the inequality $0 \leq Y \leq M_1^{-1}(0)$ with $M_1(\cdot)$ defined by (3.4).

Proof. It is easily seen that $M_1(0) \in [\mathcal{H}]$ since $\widetilde{A}_0 = A_F > \mu I > 0$. Thus, by Proposition 2.7, any nonnegative self-adjoint extension A_Θ is described by the condition $\Theta - M_1(0) \geq 0$. By (3.4), $\Theta \geq M_1(0) > 1$. Therefore $\Theta^{-1} \in \mathcal{C}(\mathcal{H})$ and $0 \leq \Theta^{-1} < 1$, i.e., in (2.4) C^{-1} exists and $\Theta^{-1} = C^{-1}D < 1$. Putting $Y := C^{-1}D$, we obtain the desired result. $\qquad\square$

4. Three-dimensional Schrödinger operator with point interactions

Consider in $L^2(\mathbb{R}^3, \mathbb{C}^n)$ matrix Schrödinger differential expression (1.3) (see [1, 3, 4, 5, 7, 8, 16, 26]). Minimal symmetric operator H associated with (1.3) is defined by (1.4).

Notice that H is closed since for any $x \in \mathbb{R}^3$ the linear functional $\delta_x : f \to f(x)$ is continuous in $W_2^2(\mathbb{R}^3, \mathbb{C}^n)$ due to the Sobolev embedding theorem. From the scalar case it is might be easily derived that deficiency indices of H are $n_\pm(H) = mn$.

4.1. **Boundary triplet and Weyl function.** In the following proposition we define a boundary triplet for the adjoint H^*. For $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ we agree to write

$$(4.1) \qquad r_j := |x - x_j| = \sqrt{(x^1 - x_j^1)^2 + (x^2 - x_j^2)^2 + (x^3 - x_j^3)^2}.$$

Proposition 4.1. *Let H be the minimal Schrödinger operator (1.4). Then the following assertions hold.*
(i) The domain of H^* is given by

$$(4.2) \quad \mathrm{dom}(H^*) = \Big\{ f = \sum_{j=1}^m \big(\xi_{0j}\frac{e^{-r_j}}{r_j} + \xi_{1j}e^{-r_j}\big) + f_H : \ \xi_{0j}, \xi_{1j} \in \mathbb{C}^n, \ f_H \in \mathrm{dom}(H) \Big\}.$$

(ii) A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for H^* is defined by

$$(4.3) \quad \mathcal{H} = \oplus_{j=1}^m \mathbb{C}^n, \quad \Gamma_0 f := \{\Gamma_{0j}f\}_{j=1}^m = 4\pi \{\lim_{x \to x_j} f(x)|x - x_j|\}_{j=1}^m = 4\pi\{\xi_{0j}\}_{j=1}^m,$$

$$(4.4) \qquad\qquad \Gamma_1 f := \{\Gamma_{1j}f\}_{j=1}^m = \{\lim_{x \to x_j}\big(f(x) - \tfrac{\xi_{0j}}{|x-x_j|}\big)\}_{j=1}^m.$$

(iii) The operator $H_0 = H^* \upharpoonright \ker(\Gamma_0)$ is self-adjoint with $\mathrm{dom}(H_0) = W_2^2(\mathbb{R}^3, \mathbb{C}^n)$.

Proof. (i) Without loss of generality, it can be assumed that $n = 1$.

Let us show that the functions $f_j = e^{-r_j}/r_j$ and $g_j = e^{-r_j}$ ($j \in \{1, \ldots, m\}$) belong to $\mathrm{dom}(H^*)$, i.e.,

$$(4.5) \qquad \begin{aligned} (H\varphi, e^{-r_j}/r_j) = (\varphi, H^*(e^{-r_j}/r_j)) \quad\text{and}\quad (H\varphi, e^{-r_j}) = (\varphi, H^*(e^{-r_j})), \\ \varphi \in C_0^\infty(\mathbb{R}^3 \setminus X). \end{aligned}$$

Let $u(\cdot), v(\cdot) \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then the second Green formula reads as follows:

$$(4.6) \qquad \int_\Omega \Big(\Delta u(\mathrm{x})\overline{v(\mathrm{x})} - u(\mathrm{x})\overline{\Delta v(\mathrm{x})}\Big)d\mathrm{x} = \int_{\partial\Omega} \Big(\frac{\partial u(\mathrm{s})}{\partial n}\overline{v(\mathrm{s})} - u(\mathrm{s})\frac{\overline{\partial v(\mathrm{s})}}{\partial n}\Big)d\mathrm{s}.$$

By (4.6), we obtain

$$(H\varphi, e^{-r_j}/r_j) - (\varphi, H^*(e^{-r_j}/r_j))$$

$$= \lim_{r \to \infty} \int_{B_r(x_j) \setminus B_{\frac{1}{r}}(x_j)} \left(-\Delta\varphi \frac{e^{-r_j}}{r_j} + \varphi\Delta(\frac{e^{-r_j}}{r_j}) \right) dx$$

(4.7)

$$= \lim_{r \to \infty} \int_{S_r(x_j)} \left(-\frac{\partial\varphi}{\partial n} \frac{e^{-r_j}}{r_j} + \varphi\frac{\partial}{\partial n}\left(\frac{e^{-r_j}}{r_j}\right) \right) ds$$

$$+ \lim_{r \to \infty} \int_{S_{\frac{1}{r}}(x_j)} \left(\frac{\partial\varphi}{\partial n} \frac{e^{-r_j}}{r_j} - \varphi\frac{\partial}{\partial n}\left(\frac{e^{-r_j}}{r_j}\right) \right) ds.$$

It is easily seen that $\frac{\partial}{\partial n}\left(\frac{e^{-r_j}}{r_j}\right) = -\frac{e^{-r_j}}{r_j}(1 + \frac{1}{r_j})$. Therefore the first integral in the right-hand side of (4.7) tends to 0 as $r \to \infty$ since $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus X)$. Further,

$$\lim_{r \to \infty} \int_{S_{\frac{1}{r}}(x_j)} \frac{\partial\varphi}{\partial n} \frac{e^{-r_j}}{r_j} ds = \lim_{r \to \infty} 4\pi \frac{\partial\varphi}{\partial n}(x^*) \frac{e^{-1/r}}{r} = 0, \quad x^* \in S_{\frac{1}{r}}(x_j),$$

$$-\frac{1}{4\pi} \lim_{r \to \infty} \int_{S_{\frac{1}{r}}(x_j)} \varphi\frac{\partial}{\partial n}\left(\frac{e^{-r_j}}{r_j}\right) ds = \lim_{r \to \infty} \left[\frac{e^{-1/r}}{r}(1 + r)\varphi(x') \right] = \lim_{x' \to x_j} \varphi(x') = \varphi(x_j) = 0,$$

$$x' \in S_{\frac{1}{r}}(x_j).$$

Thus, the first equality of (4.5) holds.

The second one can be proved analogously. It is not difficult to show that the functions f_j and g_j are linearly independent and $\dim(\text{span}\{f_j, g_j\}_{j=1}^m) = 2mn$. Since $\text{span}\{f_j, g_j\}_{j=1}^m \cap \text{dom}(H) = 0$ and $\dim(\text{dom}(H^*)/\text{dom}(H)) = 2mn$, the domain $\text{dom}(H^*)$ takes the form (4.2).

(ii) Let $f, g \in \text{dom}(H^*)$. By (4.2), we have

$$f = \sum_{k=1}^m f_k + f_H, \quad f_k = \xi_{0k}\frac{e^{-r_k}}{r_k} + \xi_{1k}e^{-r_k}, \quad g = \sum_{k=1}^m g_k + g_H, \quad g_k = \eta_{0k}\frac{e^{-r_k}}{r_k} + \eta_{1k}e^{-r_k},$$

where $f_H, g_H \in \text{dom}(H)$, and $\xi_{0k}, \xi_{1k}, \eta_{0k}, \eta_{1k} \in \mathbb{C}^n, k \in \{1, \ldots, m\}$.

Applying (4.3)–(4.4) to $f, g \in \text{dom}(H^*)$, we obtain

(4.8)

$$\Gamma_0 f = 4\pi\{\xi_{0j}\}_{j=1}^m, \quad \Gamma_1 f = \left\{ -\xi_{0j} + \sum_{k \neq j} \xi_{0k}\frac{e^{-|x_j-x_k|}}{|x_j - x_k|} + \sum_{k=1}^m \xi_{1k}e^{-|x_j-x_k|} \right\}_{j=1}^m,$$

$$\Gamma_0 g = 4\pi\{\eta_{0j}\}_{j=1}^m, \quad \Gamma_1 g = \left\{ -\eta_{0j} + \sum_{k \neq j} \eta_{0k}\frac{e^{-|x_j-x_k|}}{|x_j - x_k|} + \sum_{k=1}^m \eta_{1k}e^{-|x_j-x_k|} \right\}_{j=1}^m.$$

It is easily seen that

$$(H^*f, g) - (f, H^*g) = \sum_{j=1}^m \sum_{k=1}^m \left((\xi_{0j}H^*(\frac{e^{-r_j}}{r_j}), \eta_{1k}e^{-r_k}) \right.$$

$$- (\xi_{0j}\frac{e^{-r_j}}{r_j}, \eta_{1k}H^*(e^{-r_k}))$$

$$\left. + (\xi_{1j}H^*(e^{-r_j}), \eta_{0k}\frac{e^{-r_k}}{r_k}) - (\xi_{1j}e^{-r_j}, \eta_{0k}H^*(\frac{e^{-r_k}}{r_k})) \right).$$

Using the second Green formula (4.6), we get

$$(H^*(\frac{e^{-r_j}}{r_j}), e^{-r_k}) - (\frac{e^{-r_j}}{r_j}, H^*(e^{-r_k}))$$

$$(4.9) \qquad = \lim_{r \to \infty} \left(\int_{B_r(x_j) \setminus B_{\frac{1}{r}}(x_j)} -\Delta(\frac{e^{-r_j}}{r_j}) e^{-r_k} \mathrm{dx} + \int_{B_r(x_j) \setminus B_{\frac{1}{r}}(x_j)} \frac{e^{-r_j}}{r_j} \Delta(e^{-r_k}) \mathrm{dx} \right)$$

$$= -4\pi e^{-|x_k - x_j|}.$$

Finally, by (4.8) and (4.9),

$$(H^* f, g) - (f, H^* g) = 4\pi \sum_{j=1}^{m} \sum_{k=1}^{m} \left(-\xi_{0j} \eta_{1k} e^{-|x_j - x_k|} + \xi_{1j} \eta_{0k} e^{-|x_j - x_k|} \right)$$

$$(4.10)$$

$$= \sum_{j=1}^{m} (\Gamma_{1j} f, \Gamma_{0j} g) - (\Gamma_{0j} f, \Gamma_{1j} g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g).$$

Thus, the Green identity is satisfied. It follows from (4.2) that the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective. Namely, let $(h_0, h_1)^\top \in \mathcal{H} \oplus \mathcal{H}$, where $h_0 = \{h_{0j}\}_{j=1}^m, h_1 = \{h_{1j}\}_{j=1}^m$ are vectors from $\oplus_{j=1}^m \mathbb{C}^n$. If $f \in \mathrm{dom}(H^*)$, then, by (4.2), $f = f_H + \sum_{j=1}^m \left(\xi_{0j} \frac{e^{-r_j}}{r_j} + \xi_{1j} e^{-r_j} \right)$. Let us put

$$\xi_0 := \{\xi_{0j}\}_{j=1}^m, \quad \xi_1 := \{\xi_{1j}\}_{j=1}^m,$$

$$(4.11) \qquad E_0 := \left(-\frac{e^{-|x_k - x_j|}}{|x_k - x_j| - \delta_{kj}} \right)_{j,k=1}^m, \quad E_1 := \left(e^{-|x_k - x_j|} \right)_{k,j=1}^m,$$

where δ_{kj} stands for the Kronecker symbol. Therefore if $\xi_0 = \frac{1}{4\pi} h_0$ and $\xi_1 = (E_1 \otimes I_n)^{-1} \times (h_1 + \frac{1}{4\pi}(E_0 \otimes I_n) h_0)$, then $\Gamma_0 f = h_0$ and $\Gamma_1 f = h_1$. Hence assertion (ii) is proved.

(iii) Combining (1.4) with (4.2), we obtain that any $f \in W_2^2(\mathbb{R}^3, \mathbb{C}^n)$ admits the representation $f = \sum_{j=1}^{m} \xi_{1j} e^{-r_j} + f_H$ with $\sum_{k=1}^{m} \xi_{k1} e^{-|x_k - x_j|} = f(x_j)$ which proves (iii). $\square$

In what follows $\sqrt{\cdot}$ stands for the branch of the corresponding multifunction defined on $\mathbb{C} \setminus \mathbb{R}_+$ by the condition $\sqrt{1} = 1$.

Proposition 4.2. *Let H be the minimal Schrödinger operator (1.4) and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for H^* defined by (4.3)–(4.4). Then*
(i) the Weyl function $M(\cdot)$ corresponding to Π has the form

$$(4.12) \qquad M(z) = \bigoplus_{s=1}^{n} M_s(z), \quad M_s(z) = \left(\frac{i\sqrt{z}}{4\pi} \delta_{jk} + \widetilde{G}_{\sqrt{z}}(x_j - x_k) \right)_{j,k=1}^m, \quad z \in \mathbb{C}_+,$$

where $\widetilde{G}_{\sqrt{z}}(x) = \begin{cases} \frac{e^{i\sqrt{z}|x|}}{4\pi|x|}, & x \neq 0, \\ 0, & x = 0 \end{cases}$ and δ_{kj} stands for the Kronecker symbol;
(ii) the corresponding $\gamma(\cdot)$-field is

$$(4.13) \qquad \gamma(z)\bar{\xi} = \sum_{j=1}^{m} \xi_j \frac{e^{i\sqrt{z}r_j}}{4\pi r_j}, \quad \bar{\xi} = \{\xi_j\}_{j=1}^m, \quad \xi_j \in \mathbb{C}^n, \quad z \in \mathbb{C}_+.$$

Proof. Let $f_z \in \mathcal{N}_z$, $z \in \mathbb{C}_+$. Then $f_z = \sum_{j=1}^{m} a_j \frac{e^{i\sqrt{z}r_j}}{4\pi r_j}$, $a_j \in \mathbb{C}^n$ (see [3, chapter II.1]).

Applying Γ_0 and Γ_1 to f_z, we get

$$(4.14) \qquad \Gamma_0 f_z = \{a_j\}_{j=1}^m, \quad \Gamma_1 f_z = \left\{ a_j \frac{i\sqrt{z}}{4\pi} + \sum_{k \neq j} a_k \frac{e^{i\sqrt{z}|x_j - x_k|}}{4\pi|x_j - x_k|} \right\}_{j=1}^m.$$

Therefore (4.12) is proved (see Definition 2.5). Finally, combining (4.14) with (2.7), we arrive at (4.13). □

Remark 4.3. (*i*) The first construction of the boundary triplet, in the case $m = n = 1$, apparently goes back to the paper by Lyantse and Majorga [21, Theorem 2.1]. They also obtained the description of the spectrum of an arbitrary proper extension H_Θ of H [21, Theorem 4.5]. Their description of $(H_\Theta - z)^{-1}$ coincides with the Krein formula for canonical resolvents in Theorem 4.4. Another construction of the boundary triplet in the situation of general elliptic operator with the boundary conditions on the set of zero Lebesgue measure was obtained in [18]. However this construction is not suitable for our purpose. In the case $m = 1$, slightly different boundary triplet was obtained in [8, section 5.4].

(*ii*) Note also that the Weyl function in the form (4.12) appears in the paper by A. Posilicano [26, Example 5.3] and in the book [3] (see Theorem 1.1.1 in chapter II.1) without connection with boundary triplets.

4.2. Proper extensions of the minimal Schrödinger operator H. Proposition 2.2 gives a description of all proper extensions of H in terms of boundary triplets. The following theorem is its reformulation in more precise form.

Theorem 4.4. *Let H be the minimal Schrödinger operator* (1.4), *let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be the boundary triplet for H^* defined by* (4.3)–(4.4), *and $M(\cdot)$ the corresponding Weyl function. Assume that ξ_0, ξ_1, E_0, E_1 are defined by* (4.11) *and $H_{C,D}$ is a proper extension of H. Then the following assertions hold.*
(*i*) *The set of all proper extensions $H_{C,D}$ of H is described as follows:*

$$(4.15) \qquad \mathrm{dom}(H_{C,D}) = \{f \in \mathrm{dom}(H^*): \ D(E_1 \otimes I_n)\xi_1 = (4\pi C + D(E_0 \otimes I_n))\xi_0\},$$
$$C, D \in [\mathcal{H}].$$

(*ii*) *Moreover, $H_{C,D}$ is a self-adjoint extension of H if and only if* (2.6) *holds.*
(*iii*) *Friedrichs extension H_F of H coincides with H_0*

$$(4.16) \qquad \mathrm{dom}(H_F) = \mathrm{dom}(H_0) = W_2^2(\mathbb{R}^3, \mathbb{C}^n).$$

(*iv*) *The domain of Krein extension H_K is*

$$(4.17) \quad \mathrm{dom}(H_K) = \left\{f = \sum_{j=1}^m \xi_{0j}\frac{e^{-r_j}}{r_j} + \sum_{k,j=1}^m \boldsymbol{k}_{jk}\xi_{0k}e^{-r_j} + f_H : \ \xi_{0j} \in \mathbb{C}^n, \ f_H \in \mathrm{dom}(H)\right\},$$

with

$$(4.18) \qquad \boldsymbol{K} = (\boldsymbol{k}_{kj})_{k,j=1}^m = (E_1 \otimes I_n)^{-1}(4\pi M(0) + E_0 \otimes I_n),$$

$$(4.19) \qquad M(0) = I_n \otimes \left(\widetilde{G}_0(x_j - x_k)\right)_{j,k=1}^m = I_n \otimes \left(\frac{1 - \delta_{jk}}{4\pi|x_k - x_j| + \delta_{jk}}\right)_{j,k=1}^m.$$

(*v*) *Proper extension $H_{C,D}$ of the form* (4.15) *is self-adjoint and nonnegative if and only if* (2.6) *holds and*

$$(4.20) \qquad ((CD^* - DM(0)D^*)h, h) \geq 0, \quad h \in \mathcal{H} \setminus \{0\}.$$

(*vi*) *Krein formula for canonical resolvents takes the form*

$$(4.21) \qquad R_z(H_{C,D}) = R_z(H_0) + \gamma(z)\big(C - DM(z)\big)^{-1}D\gamma(\bar{z})^*, \quad z \in \rho(H_{C,D}) \setminus \mathbb{R}_+,$$

where $\gamma(\cdot)$-field is defined by (4.13) *and $R_z(H_0)$ is an integral operator with the kernel* $G_{\sqrt{z}}(x, x') = \frac{e^{i\sqrt{z}|x-x'|}}{4\pi|x-x'|} \otimes I_n$.

Proof. (i) and (ii) follow from the representation (2.4) and (4.8).

(iii) It is easily seen that (4.12) implies $s - R - \lim_{x\downarrow-\infty} M(x) = -\infty I_{nm}$. Then, by Proposition 2.7(iv), $H_F = H_0$. Finally, by Proposition 4.1(iii), $\mathrm{dom}(H_F) = \mathrm{dom}(H_0) = W_2^2(\mathbb{R}^3, \mathbb{C}^n)$.

(iv) Note that strong resolvent limit

$$s - R - \lim_{x\uparrow 0} M(x) = M(0) = I_n \otimes \left(\widetilde{G}_0(x_j - x_k)\right)_{j,k=1}^m = I_n \otimes \left(\frac{1-\delta_{jk}}{4\pi|x_k - x_j|+\delta_{jk}}\right)_{j,k=1}^m \text{ is}$$

an operator. Therefore operators H_0 and H_K are disjoint and, by Proposition 2.7(iii), formula (4.17) is valid.

(v) follows from Proposition 2.7(vi).

Finally, (2.9) and formula for the kernel of $(H_0 - z)^{-1}$ (see [3, chapter I.1]) yield (vi). $\qquad\square$

In [3], it is noted that, in the case $n = 1$, according to the extension theory, there are m^2-parametric family of self-adjoint extensions of the minimal operator H defined by (1.2). However, in [3], only certain m-parametric family $H_{\alpha,X}^{(3)}$ associated with the differential expression (1.1) is described [3, chapter II.1, Theorem 1.1.3]. The family $H_{\alpha,X}^{(3)}$ might be parametrized in the framework of boundary triplet approach.

Proposition 4.5. *Let Π be the boundary triplet for H^* defined by (4.3)–(4.4). Then the domain of the Schrödinger operator $H_{\alpha,X}^{(3)}$ is*

$$(4.22) \qquad \mathrm{dom}(H_{\alpha,X}^{(3)}) = \mathrm{dom}(H^*) \restriction \ker(\Gamma_1 - B_\alpha\Gamma_0), \quad B_\alpha = \mathrm{diag}(\alpha_1,\ldots,\alpha_m),$$
$$\alpha_k \in \mathbb{R}, \quad k \in \{1,\ldots,m\}.$$

Note also that the description of the $H_{\alpha,X}^{(3)}$ in terms of the resolvents [3, chapter II.1] coincides with the Krein formula for canonical resolvents (4.21) with $C = B_\alpha = \mathrm{diag}(\alpha_1,\ldots,\alpha_m)$ and $D = I_m$.

Remark 4.6. In the case $n = m = 1$, formulas (4.15) and (4.17) are essentially simplified. Namely,

$$(4.23) \qquad \mathrm{dom}(H_{C,D}) = \left\{ f = \xi_0 \frac{e^{-r_1}}{r_1} + \xi_1 e^{-r_1} + f_H : d\xi_1 = (4\pi c + d)\xi_0, \right.$$
$$\left. \xi_0, \xi_1, c, d \in \mathbb{C}, \quad f_H \in \mathrm{dom}(H^*) \right\}$$

and

$$(4.24) \qquad \mathrm{dom}(H_K) = \left\{ f = \xi_0 \frac{e^{-r_1}}{r_1} + \xi_0 e^{-r_1} + f_H, \ \xi_0 \in \mathbb{C}, \quad f_H \in \mathrm{dom}(H^*) \right\}.$$

Remark 4.7. The matrix Schrödinger operator with finite number of point interactions was also studied by A. Posilicano [26, Example 5.3, Example 5.4]. Particularly, the author parametrized self-adjoint extensions of the minimal symmetric operator H. A connection between our description of self-adjoint extensions and the one obtained by A. Posilicano might be established by the formulas (4.5) and (4.6) in [26, Theorem 4.5].

Remark 4.8. In [5], Yu. Arlinskii and E. Tsekanovskii described all nonnegative self-adjoint extensions $\widetilde{H}$ of H in the case $n = 1$ (see [5, Theorem 5.1]). It should be noted that the description of all nonnegative self-adjoint extensions of H close to that contained in [5] might be obtained in the framework of our scheme. It will be published elsewhere.

4.3. **Spectrum of the self-adjoint extensions of the minimal Schrödinger operator and scattering matrix.** In this subsection we describe point spectrum of the self-adjoint extensions of H and complete some results from [3] in this direction.

Theorem 4.9. *Let H be the minimal Schrödinger operator* (1.4), *let Π be the boundary triplet for H^* defined by* (4.3)–(4.4), *and $M(\cdot)$ the corresponding Weyl function defined by* (4.12). *Assume that H_Θ is a self-adjoint extension of H. Then the following assertions hold.*

(i) *Point spectrum of the self-adjoint extension H_Θ of H consists of at most nm negative eigenvalues (counting multiplicities). Moreover, $z \in \sigma_p(H_\Theta) \cap \mathbb{R}_-$ if and only if $0 \in \sigma_p(\Theta - M(z))$, i.e., the following equivalence holds:*

$$(4.25) \qquad z \in \sigma_p(H_\Theta) \cap \mathbb{R}_- \Leftrightarrow 0 \in \sigma_p(C - DM(z)).$$

The corresponding eigenfunction ψ_z has the form

$$(4.26) \qquad \psi_z = \sum_{j=1}^{m} c_j \frac{e^{i\sqrt{z}r_j}}{4\pi r_j},$$

where $(c_1, \ldots, c_m)^\top$ is eigenvector of the relation $\Theta - M(z)$ corresponding to zero eigenvalue.

(ii) *The number of negative eigenvalues of the self-adjoint extension H_Θ is equal to the number of negative eigenvalues of the relation $\Theta - M(0)$, $\kappa_-(H_\Theta) = \kappa_-(\Theta - M(0))$, i.e.,*

$$(4.27) \qquad \kappa_-(H_{C,D}) = \kappa_-(CD^* - DM(0)D^*),$$

where $M(0)$ is defined by (4.19).

Next we find sufficient conditions for the inequality $\kappa_-(H_{\alpha,X}^{(3)}) \geq m'$ (with $m' \leq m$) as well as for the equality $\kappa_-(H_{\alpha,X}^{(3)}) = m'$ to hold by applying the following Gerschgorin theorem.

Theorem 4.10. [20, Theorem 7.2.1]. *All eigenvalues of a matrix $A = (a_{ij})_{i,j=1}^{m} \in [\mathbb{C}^m]$ are contained in the union of Gerschgorin's disks*

$$(4.28) \qquad G_k = \left\{ z \in \mathbb{C} : |z - a_{kk}| \leq \sum_{k \neq j} |a_{kj}| \right\}, \quad k \in \{1, \ldots, m\}.$$

Moreover, the set consisting of m' disks that do not intersect with remaining $m - m'$ disks contains precisely m eigenvalues of the matrix A.

Proposition 4.11. *Let $H_{\alpha,X}^{(3)}$ be defined by* (4.22). *Let also $K = \{k_i\}_{i=1}^{m'}$ be a subset of $\mathbb{N}$.*

(i) *Suppose that*

$$(4.29) \qquad \alpha_{k_i} < - \sum_{j \neq k_i} \frac{1}{4\pi|x_j - x_{k_i}|} \quad for \quad k_i \in K.$$

Then $\kappa_-(H_{\alpha,X}^{(3)}) \geq m'$.

(ii) *If, in addition, $\alpha_k \geq \sum_{j \neq k} \frac{1}{4\pi|x_j - x_k|}$ for $k \notin K$, then $\kappa_-(H_{\alpha,X}^{(3)}) = m'$.*

Proof. (i) Combining Theorem 4.9(ii) with (4.19), we get

$$(4.30) \qquad \kappa_-(H_{\alpha,X}^{(3)}) = \kappa_-(B_\alpha - M(0)) = \kappa_-\left(\left(\alpha_k \delta_{jk} - \frac{1 - \delta_{jk}}{4\pi|x_j - x_k| + \delta_{jk}} \right)_{j,k=1}^{m} \right).$$

Without loss of generality we may assume that $K = \{1, \ldots, m'\}$. Denote by $B_{m'}$ the upper left $m' \times m'$ corner of the matrix $B_\alpha - M(0)$. According to the minimax principle,

$$(4.31) \qquad \kappa_-(H_{\alpha,X}^{(3)}) = \kappa_-(B_\alpha - M(0)) \geq \kappa_-(B_{m'}).$$

Conditions (4.29) yield the corresponding Gerschgorin conditions for $B_{m'}$. Applying the Gerschgorin theorem to the matrix $B_{m'}$ and using (4.29), we get $\kappa_-(B_{m'}) = m'$. Combining the latter equation with (4.31), we get $\kappa_-(H_{\alpha,X}^{(3)}) \geq m'$.

($ii $) Applying the second part of the Gerschgorin theorem to the matrix $B - M(0)$, we arrive at $\kappa_-(H^{(3)}_{\alpha,X}) = \kappa_-(B_\alpha - M(0)) = m'$. $\square$

Remark 4.12. Note that the idea of applying Gerschgorin's theorem is borrowed from [24]. This idea was also used in [14].

Consider the scattering system $\{H_\Theta, H_0\}$, where $H_\Theta = H^* \restriction \Gamma^{-1}\Theta$ with arbitrary self-adjoint relation $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$. Since H is not simple, we consider the system $\{\widehat{H}_\Theta, \widehat{H}_0\}$, $H_\Theta = \widehat{H}_\Theta \oplus H_s$. Then Theorem 2.8 and (4.12) imply

Theorem 4.13. *Scattering matrix* $\{\widehat{S}_\Theta(z)\}_{z\in\mathbb{R}_+}$ *of the scattering system* $\{\widehat{H}_\Theta, \widehat{H}_0\}$ *has the form*

$$\widehat{S}_\Theta(x) = I_{nm} + 2i\sqrt{S(x)}\Big(\Theta - I_n \otimes \Big(\tfrac{i\sqrt{x}}{4\pi}\delta_{jk} + \widetilde{G}_{\sqrt{x}}(x_j - x_k)\Big)_{j,k=1}^m\Big)^{-1}\sqrt{S(x)}, \quad x \in \mathbb{R}_+,$$

$$S(x) = I_n \otimes \Big(\tfrac{\sqrt{x}}{4\pi}\delta_{jk} + \widetilde{S}_{\sqrt{x}}(x_j - x_k)\Big)_{j,k=1}^m, \quad \widetilde{S}_{\sqrt{x}}(t) = \begin{cases} \frac{\sin(\sqrt{x}|t|)}{4\pi|t|}, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

5. Two-dimensional Schrödinger operator with point interactions

In this section, we consider in $L^2(\mathbb{R}^2, \mathbb{C}^n)$ matrix Schrödinger differential expression (1.3) (see [1, 3, 4, 16]). Minimal symmetric operator H associated with (1.3) in $L^2(\mathbb{R}^2, \mathbb{C}^n)$ is defined by (1.4). As above, the operator H is closed and the deficiency indices of H are $n_\pm(H) = nm$.

5.1. Boundary triplet and Weyl function.
In the following proposition we describe boundary triplet for the adjoint operator H^*. Let us denote

$$(5.1) \qquad r_j := |x - x_j| = \sqrt{(x^1 - x_j^1)^2 + (x^2 - x_j^2)^2}, \quad x = (x^1, x^2) \in \mathbb{R}^2.$$

Proposition 5.1. *Let H be the minimal Schrödinger operator* (1.4). *Then the following assertions hold.*
(i) *The domain of H^* is defined by*

$$(5.2) \qquad \mathrm{dom}(H^*) = \Big\{ f = \sum_{j=1}^m \big(\xi_{0j}\, e^{-r_j}\ln(r_j) + \xi_{1j}\, e^{-r_j}\big)$$

$$+ f_H \, : \, \xi_{0j}, \xi_{1j} \in \mathbb{C}^n, \, f_H \in \mathrm{dom}(H) \Big\}.$$

(ii) *The boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for H^* might be defined as follows:*

$$(5.3) \quad \mathcal{H} = \oplus_{j=1}^m \mathbb{C}^n, \quad \Gamma_0 f := \{\Gamma_{0j}f\}_{j=1}^m = -2\pi\Big\{ \lim_{x\to x_j} \frac{f(x)}{\ln|x-x_j|} \Big\}_{j=1}^m = 2\pi\{\xi_{0j}\}_{j=1}^m,$$

$$(5.4) \qquad \Gamma_1 f := \{\Gamma_{1j}f\}_{j=1}^m = \big\{ \lim_{x\to x_j}(f(x) - \ln|x-x_j|\xi_{0j}) \big\}_{j=1}^m, \quad f \in \mathrm{dom}(H^*).$$

(iii) *The operator $H_0 = H^* \restriction \ker(\Gamma_0)$ is self-adjoint with $\mathrm{dom}(H_0) = W_2^2(\mathbb{R}^2, \mathbb{C}^n)$.*

Proof. (i) It is well known (see [3, 4]) that

$$\mathrm{dom}(H^*) = \big\{ f \in L^2(\mathbb{R}^2, \mathbb{C}^n) \cap W_{2,\mathrm{loc}}^2(\mathbb{R}^2\backslash\{X\}, \mathbb{C}^n) : \Delta f \in L^2(\mathbb{R}^2, \mathbb{C}^n) \big\}.$$

Obviously, functions $f_j = \eta_j e^{-r_j}\ln(r_j)$ and $g_j = \mu_j e^{-r_j}$ ($\eta_j, \mu_j \in \mathbb{C}^n$, $j \in \{1, \ldots, m\}$) belong to $\mathrm{dom}(H^*)$. Their linear span is $2mn$-dimensional subspace in $\mathrm{dom}(H^*)$ that has trivial intersection with $\mathrm{dom}(H)$. Since $\dim(\mathrm{dom}(H^*)/\mathrm{dom}(H)) = 2mn$, the domain $\mathrm{dom}(H^*)$ takes the form (5.2).
(ii) The second Green identity is verified similarly to 3D case.

From (5.2) it follows that the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective. Namely, let $(h_0, h_1)^\top \in \mathcal{H} \oplus \mathcal{H}$, where $h_0 = \{h_{0j}\}_{j=1}^m, h_1 = \{h_{1j}\}_{j=1}^m$ are vectors from $\oplus_{j=1}^m \mathbb{C}^n$. If $f \in \mathrm{dom}(H^*)$, then, by (5.2), $f = f_H + \sum_{j=1}^m \left(\xi_{0j} e^{-r_j} \ln(r_j) + \xi_{1j} e^{-r_j} \right)$. Let us put

$$\xi_0 := \{\xi_{0j}\}_{j=1}^m, \quad \xi_1 := \{\xi_{1j}\}_{j=1}^m,$$

(5.5)
$$E_0 := \left(-e^{-|x_k - x_j|} \ln(|x_k - x_j| + \delta_{kj}) \right)_{j,k=1}^m, \quad E_1 := \left(e^{-|x_k - x_j|} \right)_{k,j=1}^m.$$

Therefore if $\xi_0 = \frac{1}{2\pi} h_0$ and $\xi_1 = (E_1 \otimes I_n)^{-1}(h_1 + \frac{1}{2\pi}(E_0 \otimes I_n)h_0)$, then $\Gamma_0 f = h_0$ and $\Gamma_1 f = h_1$. Thereby, (ii) is proved.

(iii) From (1.4) and (5.2) it follows that any function $f \in W_2^2(\mathbb{R}^2, \mathbb{C}^n)$ admits a representation $f = \sum_{j=1}^m \xi_{1j} e^{-r_j} + f_H$, where $\sum_{k=1}^m \xi_{k1} e^{-|x_k - x_j|} = f(x_j)$ which proves (iii). $\qquad\square$

Proposition 5.2. *Let H be the minimal Schrödinger operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for H^* defined by (5.3)–(5.4). Then*

(i) *the Weyl function $M(\cdot)$ corresponding to the boundary triplet Π has the form*

(5.6)
$$M(z) = \bigoplus_{s=1}^n M_s(z), \quad M_s(z) = \left(\frac{1}{2\pi}(\psi(1) - \ln(\frac{\sqrt{z}}{2i}))\delta_{jk} + \widetilde{G}_{\sqrt{z}}(x_j - x_k) \right)_{j,k=1}^m,$$

$$z \in \mathbb{C}_+,$$

where $\psi(1) = \frac{\Gamma'(1)}{\Gamma(1)}$, $\widetilde{G}_{\sqrt{z}}(x) = \begin{cases} i/4 H_0^{(1)}(\sqrt{z}|x|), & x \neq 0, \\ 0, & x = 0 \end{cases}$ and $H_0^{(1)}(\cdot)$ denotes the Hankel function of the first kind and order 0;

(ii) *the corresponding $\gamma(\cdot)$-field is*

(5.7)
$$\gamma(z)\overline{\xi} = \sum_{j=1}^m \xi_j \frac{i}{4} H_0^{(1)}(\sqrt{z} r_j), \quad \overline{\xi} = \{\xi_j\}_{j=1}^m, \quad \xi_j \in \mathbb{C}^n, \quad z \in \mathbb{C}_+.$$

Proof. Let $f_z \in \mathcal{N}_z$, $z \in \mathbb{C}_+$. Then, according to [3, chapter II.4],

(5.8)
$$f_z := \sum_{j=1}^m a_j \frac{i}{4} H_0^{(1)}(\sqrt{z} r_j), \quad a_j \in \mathbb{C}^n.$$

It is not difficult to see that, by formulas (9.01) in [25, Section 2, § 9] and (5.03), (5.07) in [25, Section 7, § 5], the function $H_0^{(1)}(z)$ has the following asymptotic expansion at 0

(5.9)
$$H_0^{(1)}(z) = 1 + \frac{2i}{\pi}(\ln(\frac{z}{2}) - \psi(1)) + o(z), \quad z \to 0.$$

Applying Γ_0 and Γ_1 to f_z and taking into account (5.9), we get

$$\Gamma_0 f_z = \{a_j\}_{j=1}^m,$$

(5.10)
$$\Gamma_1 f_z = \left\{ \left(\frac{\psi(1)}{2\pi} + \frac{i}{4} - \frac{\ln(\frac{\sqrt{z}}{2})}{2\pi} \right) a_j + \sum_{k \neq j} \frac{i}{4} H_0^{(1)}(\sqrt{z}|x_k - x_j|) a_k \right\}_{j=1}^m,$$

Further, combining (5.10) with (2.7), we get (5.6) and (5.7). $\qquad\square$

5.2. Proper extensions of the minimal Schrödinger operator H.

As in previous section, we describe proper extensions of the minimal operator H.

Theorem 5.3. *Let H be the minimal Schrödinger operator, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for H^* defined by (5.3)–(5.4), and $M(\cdot)$ the corresponding Weyl function. Assume also that ξ_0, ξ_1, E_0, E_1 are defined by (5.5) and $H_{C,D}$ is a proper extension of H.*

Then the following assertions hold.

(i) Any proper extension $H_{C,D}$ of H is described as follows:

(5.11)
$$\mathrm{dom}(H_{C,D}) = \{f \in \mathrm{dom}(H^*) : D(E_1 \otimes I_n)\xi_1 = (2\pi C + D(E_0 \otimes I_n))\xi_0\},$$
$$C, D \in [\mathcal{H}].$$

(ii) Extension $H_{C,D}$ is self-adjoint if and only if (2.6) holds.

(iii) Friedrichs extension H_F of H coincides with H_0:

(5.12)
$$\mathrm{dom}(H_F) = \mathrm{dom}(H_0) = W_2^2(\mathbb{R}^2, \mathbb{C}^n).$$

(iv) The domain $\mathrm{dom}(H_K)$ of the Krein extension H_K is

(5.13)
$$\mathrm{dom}(H_K) = \begin{cases} \mathrm{dom}(H_0), & m = 1, \\ \{f \in \mathrm{dom}(H^*) : (\Gamma_0 f, \Gamma_1 f)^\top \in M(0)\}, & m > 1, \end{cases}$$

where

$$\mathrm{dom}(M(0)_{\mathrm{op}}) = \oplus_{s=1}^n \mathrm{dom}(M_s(0)_{\mathrm{op}}),$$

(5.14)
$$\mathrm{dom}(M_s(0)_{\mathrm{op}}) = \Big\{ \xi = \{\xi_j\}_{j=1}^m \in \mathbb{C}^m : \sum_{j=1}^m \xi_j = 0 \Big\},$$

(5.15)
$$\mathrm{mul}\,(M(0)) = \oplus_{s=1}^n \mathrm{span}\{e_{\mathrm{mul}}\}, \quad e_{\mathrm{mul}} = \{e_j\}_{j=1}^m = \{1\}_{j=1}^m.$$

(v) Krein formula for canonical resolvents takes the form

(5.16)
$$R_z(H_{C,D}) = R_z(H_0) + \gamma(z)\big(C - DM(z)\big)^{-1} D\gamma(\bar{z})^*, \quad z \in \rho(H_{C,D}) \setminus \mathbb{R}_+,$$

where $\gamma(\cdot)$-field is defined by (5.7) and $R_z(H_0)$ is an integral operator with the kernel $G_{\sqrt{z}}(x, x') = i/4 H_0^{(1)}(\sqrt{z}|x - x'|) \otimes I_n$.

Proof. (i) and (ii) follow from the representation (2.4).

(iii) From the asymptotic representation (see, for instance, formula (4.03) in [25, Section 7, § 4])

(5.17)
$$H_0^{(1)}(z) \sim \sqrt{\tfrac{2}{\pi z}} e^{i(z - \pi/4)}, \quad |z| \to \infty,$$

it easily follows that $\lim_{x \downarrow -\infty}(M(x)f, f) = -\infty$, $f \in \mathcal{H} \setminus \{0\}$. Thus, by Proposition 2.7(iv), $H_F = H_0$.

(iv) In the case $m = 1$, the Weyl function has the form $M(z) = (\psi(1) - \ln(\frac{\sqrt{z}}{2i}))I_n$. The latter yields

(5.18)
$$\lim_{x \downarrow -\infty}(M(x)f, f) = -\infty, \quad \lim_{x \uparrow 0}(M(x)f, f) = +\infty, \quad f \in \mathcal{H} \setminus \{0\}.$$

By Proposition 2.7(iv), $H_F = H_K = H_0$. Furthermore, from the equality $H_K = H_F$ it follows that operator H has no other nonnegative self-adjoint extensions (see [19]).

Consider the case $m > 1$. For simplicity suppose that $n = 1$. Let $\xi = \{\xi_j\}_{j=1}^m \in \mathbb{C}^m$. Using asymptotic expansion (5.9), we get

(5.19)
$$(M(z)\xi, \xi) \sim \tfrac{1}{2\pi}(\psi(1) - \ln(\tfrac{\sqrt{z}}{2i})) \sum_{j=1}^m |\xi_j|^2$$
$$+ \sum_{k \neq j} \tfrac{1}{2\pi} \Big(\psi(1) - \ln(\tfrac{\sqrt{z}}{2i}) - \ln(|x_k - x_j|) \Big) \xi_j \overline{\xi_k}$$
$$= \tfrac{1}{2\pi}(\psi(1) - \ln(\tfrac{\sqrt{z}}{2i})) \Big(|\sum_{j=1}^m \xi_j| \Big)^2 - \tfrac{1}{2\pi} \sum_{k \neq j} \ln(|x_k - x_j|) \xi_j \overline{\xi_k}, \quad z \to 0.$$

From (5.19) it easily follows that limit $\lim_{x\uparrow 0}(M\xi,\xi)$ is finite if and only if $\sum_{j=1}^m \xi_j = 0$. Thus, the domain of the operator part $M(0)_{op}$ is described by (5.14) Finally, (5.15) takes place since $\mathrm{mul}\,(M(0))$ and $\mathrm{dom}(M(0)_{op})$ are orthogonal.

Applying Proposition 2.7(ii) completes the proof of (iv).

Combining (2.9) with the formula for the kernel of $(H_0 - z)^{-1}$ (see [3, chapter I.5]), we obtain (v). $\qquad\square$

As in the case of 3D Schrödinger operator, only certain m-parametric family $H^{(2)}_{\alpha,X}$ associated in $L^2(\mathbb{R}^2)$ with the differential expression (1.1) is described in [3, chapter II.1, Theorem 4.1].

Proposition 5.4. *Let Π be the boundary triplet for H^* defined by (5.3)–(5.4). Then the domain of $H^{(2)}_{\alpha,X}$ has the following representation:*

$$\mathrm{dom}(H^{(2)}_{\alpha,X}) = \mathrm{dom}(H^*) \restriction \ker(\Gamma_1 - B_\alpha \Gamma_0),$$
(5.20)
$$B_\alpha = \mathrm{diag}(\alpha_1,\ldots,\alpha_m), \quad \alpha_k \in \mathbb{R}, \quad k \in \{1,\ldots,m\}.$$

Note that in the case $d = 2$ it makes certain difficulty to describe nonnegative self-adjoint extensions of H since $M(0)$ appears to be the relation with nontrivial multivalued part. We may overcome this by considering the following intermediate extension of H.

$$(5.21) \qquad \widetilde{H} := H^* \restriction \mathrm{dom}(\widetilde{H}), \quad \mathrm{dom}(\widetilde{H}) = \mathrm{dom}(H_F) \cap \mathrm{dom}(H_K).$$

As above, assume that $n = 1$. It is easily seen that

(5.22)
$$\mathrm{dom}(\widetilde{H}) = \left\{ f = c \sum_{j=1}^m \widetilde{\xi}_j\, e^{-r_j} \right.$$
$$\left. + f_H \; : \; \widetilde{\xi} = \{\widetilde{\xi}_j\}_{j=1}^m = E_1^{-1}\quad e_{\mathrm{mul}},\; c \in \mathbb{C}, \quad f_H \in \mathrm{dom}(H) \right\},$$

where E_1 is defined by (5.5).

According to [29], we have

$$(5.23) \qquad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \mathcal{H}_1 = \mathrm{dom}(M(0)_{op}) \quad \text{and} \quad \mathcal{H}_2 = \mathrm{mul}\,(M(0)).$$

Let π_j, $j \in \{1,2\}$ denote the orthogonal projectors onto $\mathcal{H}_j$. Then the Weyl function $M(\cdot)$ defined by (5.6) admits the representation $M(\cdot) = (M_{kj}(\cdot))^2_{k,j=1}$ with $M_{k,j}(\cdot) = \pi_k M(\cdot) \restriction \mathcal{H}_j$, $k,j \in \{1,2\}$. One may simply verify that

$$(5.24) \qquad \widetilde{H} = H_1 := H^* \restriction \{f \in \mathrm{dom}(H^*) : \Gamma_0 f = \pi_1 \Gamma_1 f = 0\},$$

with Γ_0, Γ_1 defined by (5.3)–(5.4). From [12, Proposition 4.1] it follows that $\widetilde{H}$ is closed symmetric operator in $L^2(\mathbb{R}^2)$ with deficiency indices $n_\pm(\widetilde{H}) = \dim(\mathcal{H}_1) = m-1$. Proposition 4.1$(ii)$ in [12] also yields that $H_1^* = \widetilde{H}^* = H^* \restriction \{f \in \mathrm{dom}(H^*) : \pi_2 \Gamma_0 f = 0\}$, and boundary triplet $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for $\widetilde{H}^*$ might be defined as follows

$$(5.25) \qquad \widetilde{\mathcal{H}} = \mathcal{H}_1, \quad \widetilde{\Gamma}_0 = \Gamma_0 \restriction \mathrm{dom}(\widetilde{H}^*), \quad \widetilde{\Gamma}_1 = \pi_1 \Gamma_1 \restriction \mathrm{dom}(\widetilde{H}^*).$$

Moreover, the Weyl function $\widetilde{M}(\cdot)$ corresponding to the boundary triplet $\widetilde{\Pi}$ are given by $\widetilde{M}(\cdot) = M_{11}(\cdot)$ and the equality $\widetilde{M}(0) = M(0)_{op}$ is satisfied.

Proposition 5.5. *Let H and $M(0)_{op}$ be as above and let H' be a non-negative self-adjoint extension of H. Then*

(i) There exist pairs $C, D \in [\mathcal{H}]$ and $\widetilde{C}, \widetilde{D} \in [\widetilde{\mathcal{H}}]$ satisfying (2.6) and such that

$$(5.26) \qquad H' = H_{C,D} = H^* \restriction \ker(D\Gamma_1 - C\Gamma_0) = \widetilde{H}^* \restriction \ker(\widetilde{D}\widetilde{\Gamma}_1 - \widetilde{C}\widetilde{\Gamma}_0) =: \widetilde{H}_{\widetilde{C},\widetilde{D}}.$$

(ii) The extension $H_{C,D} = H^*_{C,D}$ is nonnegative if and only if

$$(5.27) \qquad (\widetilde{C}\widetilde{D}^* - \widetilde{D}M(0)_{\mathrm{op}}\widetilde{D}^*h, h) \geq 0, \quad h \in \mathrm{dom}(M(0)_{\mathrm{op}}) \setminus \{0\}.$$

Remark 5.6. (*i*) The uniqueness of nonnegative self-adjoint extension of 2D operator H, in the case $n = m = 1$, was established in [13] and [1].

(*ii*) In [1], V. Adamyan noted that, in the case $m > 1$ and $n = 1$, the operator H has non-unique nonnegative self-adjoint extension.

5.3. **Spectrum of the self-adjoint extensions of the minimal Schrödinger operator and scattering matrix.** Point spectrum of the self-adjoint extensions of H is described in the following theorem.

Theorem 5.7. *Let H be the operator defined by* (1.4)*, let Π be the boundary triplet for H^* defined by* (5.3)–(5.4)*, and let $M(\cdot)$ be the corresponding Weyl function. Assume also that H_Θ is a self-adjoint extension of H. Then point spectrum of the self-adjoin extension H_Θ consists of at most nm negative eigenvalues (counting multiplicities). Moreover, $z \in \sigma_p(H_\Theta) \cap \mathbb{R}_-$ if and only if $0 \in \sigma_p(\Theta - M(z))$, i.e.,*

$$(5.28) \qquad z \in \sigma_p(H_\Theta) \cap \mathbb{R}_- \Leftrightarrow 0 \in \sigma_p(C - DM(z)).$$

The corresponding eigenfunction ψ_z has the form

$$(5.29) \qquad \psi_z = \sum_{j=1}^{m} c_j \frac{i}{4} H_0^{(1)}(\sqrt{z}r_j),$$

where $(c_1, \ldots, c_m)^\top$ is eigenvector of the relation $\Theta - M(z)$ corresponding to zero eigenvalue.

As in the case of 3D Schrödinger operator, 2D Schrödinger operator H is not simple. Arguing as above, we obtain

Theorem 5.8. *Scattering matrix $\{\widehat{S}_\Theta(z)\}_{z \in \mathbb{R}_+}$ of the scattering system $\{\widehat{H}_\Theta, \widehat{H}_0\}$ has the form*

$$\widehat{S}_\Theta(x) = I_{nm} + 2i\sqrt{J(x)}$$
$$\times \left(\Theta - I_n \otimes \left(\frac{1}{2\pi}(\psi(1) - \ln(\frac{\sqrt{x}}{2i}))\delta_{jk} + \widetilde{G}_{\sqrt{z}}(x_j - x_k) \right)_{j,k=1}^{m} \right)^{-1} \sqrt{J(x)},$$
$$J(x) = I_n \otimes \left(\frac{1}{4} J_0(\sqrt{x}|x_j - x_k|) \right)_{j,k=1}^{m}, \quad x \in \mathbb{R}_+,$$

where $J_0(\cdot)$ denotes Bessel function.

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