POLARIZATION FORMULA FOR (p, q)-POLYNOMIALS ON A COMPLEX NORMED SPACE

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ABSTRACT. The aim of this paper to give some analogues of polarization formulas and the polarization inequality for (p, q)-polynomials between complex normed spaces. Obtained results are useful for investigation of real-differentiable mappings on complex spaces.

INTRODUCTION

Let X and Y be complex linear spaces and $X^n = X \times \cdots \times X$ be the Cartesian power of X. An *n-homogeneous polynomial* P_n from X to Y may be defined as a restriction to the diagonal of an n-linear map $B_n \colon X^n \to Y$. That is,

$$P_n(x) = B_n(x, \ldots, x).$$

It is well known that there is a unique *symmetric n-linear map* B_n which generates P_n. The map B_n can be recovered from P_n using the following *polarization formula*:

$$B_n(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_n P_n \Big(\sum_{j=1}^{n} \varepsilon_j x_j \Big).$$

The polarization formula had been known since 1931 [3] and it was rediscovered later in various forms by many authors (e.g. [6, 7]).

The polarization formula is an important tool in the theory of polynomials on normed spaces. Using it we can get the *polarization inequality*. If X and Y are normed spaces, then there is a constant $c(n, X)$ such that for all n-homogeneous polynomials P_n

$$\|P_n\| \leq \|B_n\| \leq c(n, X)\|P_n\|$$

(see e.g. [5] for details). The minimal constant satisfying the inequality in general can be estimated as $1 \leq c(n, X) \leq \frac{n^n}{n!}$. More precise estimations depends on geometrical properties of X (see [10]). For example if $X = \ell_1$, then $c(n, X)$ can not be less than $\frac{n^n}{n!}$ but if $X = \ell_2$, then we can take $c(n, X) = 1$.

The purpose of the paper to consider the case of so-called (p, q) linear maps (which are linear with respect to the first p components and anti-linear with respect to the last q components) and corresponding (p, q)-polynomials. We prove various versions of polarization formulas and a polarization inequality for this case. The polarization formulas for (p, q)-polynomials should be applicable for real-differentiable mappings on complex Banach spaces because the Taylor formula expansion of a real-differentiable mapping is a linear span of (p, q)-polynomials.

A detail information about polynomials can be found in [5, 9]. (p, q)-polynomials have been considered in [8, 9, 11].

1. Basic Concepts

We denote by S_n the group of all permutations on $\{1, \ldots, n\}$.

Definition 1.1. *The mapping* $B_{p,q}(x_1, \ldots, x_p; x_{p+1}, \ldots, x_{p+q})$, $B_{p,q} : X^{p+q} \to Y$ *is called a* (p,q)-*linear symmetric mapping if it has the following properties:*
1°. $\forall i \in \{1, \ldots, p+q\}$ $\forall x_i{}', x_i{}'' \in X$

$$B_{p,q}(x_1, \ldots, x_{i-1}, x_i{}' + x_i{}'', x_{i+1}, \ldots, x_{p+q})$$
$$= B_{p,q}(x_1, \ldots, x_{i-1}, x_i{}', x_{i+1} \ldots, x_{p+q}) + B_{p,q}(x_1, \ldots, x_{i-1}, x_i{}'', x_{i+1} \ldots, x_{p+q}).$$

2°. $\forall i \in \{1, \ldots, p\}$ $\forall \lambda \in \mathbb{C}$

$$B_{p,q}(x_1, \ldots, \lambda x_i, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}) = \lambda B_{p,q}(x_1, \ldots, x_i, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}).$$

3°. $\forall i \in \{p+1, \ldots, p+q\}$ $\forall \lambda \in \mathbb{C}$

$$B_{p,q}(x_1, \ldots, x_p; x_{p+1}, \ldots, \lambda x_i, \ldots, x_{p+q}) = \overline{\lambda} B_{p,q}(x_1, \ldots, x_p; x_{p+1}, \ldots, x_i, \ldots, x_{p+q}).$$

4°. $\forall \sigma \in S_p$

$$B_{p,q}(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)}; x_{p+1}, \ldots, x_{p+q}) = B_{p,q}(x_1, x_2, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}).$$

5°. $\forall \sigma \in S_q$

$$B_{p,q}(x_1, \ldots, x_p; x_{p+\sigma(1)}, x_{p+\sigma(2)}, \ldots, x_{p+\sigma(q)}) = B_{p,q}(x_1, \ldots, x_p; x_{p+1}, x_{p+2}, \ldots, x_{p+q}).$$

In other words, $B_{p,q}$ *is linear and symmetric with respect to* $x_1, \ldots, x_p$, *and anti-linear and symmetric with respect to* $x_{p+1}, \ldots, x_{p+q}$.

Definition 1.2. *We define a* (p,q)-*polynomial as the restriction of a* (p,q)-*linear symmetric mapping* $B_{p,q}$ *onto the diagonal*

$$P_{p,q}(x) = B_{p,q}(\underbrace{x, \ldots, x}_{p}; \underbrace{x, \ldots, x}_{q}).$$

Note that any $(p,0)$-polynomial is just a p-homogeneous polynomial and any $(p,0)$-linear symmetric mapping is just a p-linear symmetric mapping.

The aim of this work is to find a method of recovering a (p,q)-linear symmetric mapping from its restriction onto the diagonal. To do it we use the techniques of (classical) Rademacher functions (see [10]) and generalized Rademacher functions (see [1] and [2]). Note that using this approach in [4] were proved analogues of the polarization formula for nonhomogeneous polynomials and analytic mappings.

Definition 1.3. *The* i-*th Rademacher function* $r_i(t)$ *is defined on* $[0,1]$ *by* $r_i(t) = \text{sign} \sin 2^i \pi t$, $i \in \mathbb{N}$.

Rademacher functions have the following properties:
1°. $(r_i(t))^{2n} = 1$;
2°. $(r_i(t))^{2n+1} = r_i(t)$;
3°. Let $m_1, \ldots, m_n \in \mathbb{Z}$. Then

$$\int_0^1 (r_1(t))^{m_1} (r_2(t))^{m_2} \ldots (r_n(t))^{m_n} \, dt = \begin{cases} 1, & \text{if all } m_1, \ldots, m_n \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.4. (see [2]) *For a given integer* $n \geq 2$ *let* $\alpha_j = e^{2\pi i \frac{j-1}{n}}$ *be the complex* nth *roots of the unity,* $j = 1, \ldots, n$ *and* $I_j = \left(\frac{j-1}{n}, \frac{j}{n}\right)$. *The generalized Rademacher function* $S_1^{[n]} : [0,1] \to \mathbb{C}$ *is defined by setting* $S_1^{[n]}(t) = \alpha_j$ *for* $t \in I_j$ *where* $1 \leq j \leq n$. *We set* $S_1^{[n]}(t) = 1$ *for all endpoints* t.

The function $S_1^{[n]}(t)$ has the following properties:

$1°$. Since $\forall t \in [0,1] : |S_1^{[n]}(t)| = 1$,

$$\overline{S_1^{[n]}(t)} = \left(S_1^{[n]}(t)\right)^{-1};$$

$2°$.

$$\int_0^1 \left(S_1^{[n]}(t)\right)^m dt = \begin{cases} 1, & \text{if } m = 0 \mod n, \\ 0, & \text{otherwise.} \end{cases}$$

We denote

$$B_{p,q}((x_1)^{n_1}, \ldots, (x_s)^{n_s}; (x_{s+1})^{n_{s+1}}, \ldots, (x_k)^{n_k})$$
$$= B_{p,q}(\underbrace{x_1, \ldots, x_1}_{n_1}, \ldots, \underbrace{x_s, \ldots, x_s}_{n_s}; \underbrace{x_{s+1}, \ldots, x_{s+1}}_{n_{s+1}}, \ldots, \underbrace{x_k, \ldots, x_k}_{n_k}).$$

The case $n_j = 0$ for some j means that x_j does not belong to the list of arguments at the position.

Remark 1.1. Let $c_1, \ldots, c_{p+q} \in \mathbb{C}$. It is easy to see that

$$P_{p,q}(c_1 x_1 + \cdots + c_{p+q} x_{p+q}) = B_{p,q}\left((c_1 x_1 + \cdots + c_{p+q} x_{p+q})^{p+q}\right)$$

$$= \sum_{\substack{k_1 \geq 0, \ldots, k_{p+q} \geq 0 \\ k_1 + \cdots + k_{p+q} = p}} \frac{p!}{k_1! \ldots k_{p+q}!} \sum_{\substack{l_1 \geq 0, \ldots, l_{p+q} \geq 0 \\ l_1 + \cdots + l_{p+q} = q}} \frac{q!}{l_1! \ldots l_{p+q}!}$$

$$\times B_{p,q}\left((c_1 x_1)^{k_1}, \ldots, (c_{p+q} x_{p+q})^{k_{p+q}}; (c_1 x_1)^{l_1}, \ldots, (c_{p+q} x_{p+q})^{l_{p+q}}\right)$$

$$= \sum_{\substack{k_1 \geq 0, \ldots, k_{p+q} \geq 0 \\ k_1 + \cdots + k_{p+q} = p}} c_1^{k_1} c_2^{k_2} \ldots c_{p+q}^{k_{p+q}} \frac{p!}{k_1! \ldots k_{p+q}!} \sum_{\substack{l_1 \geq 0, \ldots, l_{p+q} \geq 0 \\ l_1 + \cdots + l_{p+q} = q}} \overline{c_1^{l_1} c_2^{l_2} \ldots c_{p+q}^{l_{p+q}}} \cdot \frac{q!}{l_1! \ldots l_{p+q}!}$$

$$\times B_{p,q}\left((x_1)^{k_1}, \ldots, (x_{p+q})^{k_{p+q}}; (x_1)^{l_1}, \ldots, (x_{p+q})^{l_{p+q}}\right).$$

2. Polarization formula for (p,q)-linear mappings

Theorem 2.1. Let $B_{p,q}(x_1, \ldots, x_{p+q})$ be a (p,q)-linear symmetric mapping and $P_{p,q}(x)$ be the corresponding (p,q)-polynomial. Then

$$B_{p,q}(x_1, \ldots, x_{p+q})$$

(2.1)
$$= \frac{1}{p!q!} \int_0^1 \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p} r_1(\theta) r_2(\theta) \ldots r_{p+q}(\theta)$$
$$\times P_{p,q}\left(S_1^{[2q+1]}(t)\,(r_1(\theta)x_1 + \cdots + r_p(\theta)x_p)\right.$$
$$\left. + (r_{p+1}(\theta)x_{p+1} + \cdots + r_{p+q}(\theta)x_{p+q})\right) dt\, d\theta.$$

Proof. Let us denote the right-hand side of (2.1) by A. From remark 1.1 we have

$$A = \sum_{\substack{k_1 \geq 0, \ldots, k_{p+q} \geq 0 \\ k_1 + \cdots + k_{p+q} = p}} \frac{1}{k_1! \ldots k_{p+q}!} \sum_{\substack{l_1 \geq 0, \ldots, l_{p+q} \geq 0 \\ l_1 + \cdots + l_{p+q} = q}} \frac{1}{l_1! \ldots l_{p+q}!}$$

$$\times \int_0^1 \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p} r_1(\theta) r_2(\theta) \ldots r_{p+q}(\theta)$$

$$\times \left(S_1^{[2q+1]}(t)\right)^{k_1 + \cdots + k_p} \left(\overline{S_1^{[2q+1]}(t)}\right)^{l_1 + \cdots + l_p}$$

$$\times \, (r_1(\theta))^{k_1+l_1}(r_2(\theta))^{k_2+l_2}\ldots(r_{p+q}(\theta))^{k_{p+q}+l_{p+q}}$$

$$\times \, B_{p,q}\left((x_1)^{k_1},\ldots,(x_{p+q})^{k_{p+q}};(x_1)^{l_1},\ldots,(x_{p+q})^{l_{p+q}}\right)\, dt\, d\theta$$

$$= \sum_{\substack{k_1\geq 0,\ldots,k_{p+q}\geq 0 \\ k_1+\cdots+k_{p+q}=p}} \frac{1}{k_1!\ldots k_{p+q}!} \sum_{\substack{l_1\geq 0,\ldots,l_{p+q}\geq 0 \\ l_1+\cdots+l_{p+q}=q}} \frac{1}{l_1!\ldots l_{p+q}!}$$

$$\times \, B_{p,q}\left((x_1)^{k_1},\ldots,(x_{p+q})^{k_{p+q}};(x_1)^{l_1},\ldots,(x_{p+q})^{l_{p+q}}\right)$$

$$\times \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p+k_1+\cdots+k_p-l_1-\cdots-l_p}\, dt$$

$$\times \int_0^1 (r_1(\theta))^{1+k_1+l_1}(r_2(\theta))^{1+k_2+l_2}\ldots(r_{p+q}(\theta))^{1+k_{p+q}+l_{p+q}}\, d\theta.$$

Let us consider the integral

$$\int_0^1 (r_1(\theta))^{1+k_1+l_1}(r_2(\theta))^{1+k_2+l_2}\ldots(r_{p+q}(\theta))^{1+k_{p+q}+l_{p+q}}\, d\theta.$$

If there exists a natural number i such that $k_i+l_i=0$, then there is a multiplier $(r_i(\theta))^1$ in the above expression and therefore the property $3°$ of Rademacher functions implies that the integral is equal to zero.

So for non-zero elements of the sum $k_i+l_i\geq 1,\quad i=1,\ldots,p+q.$ Thus $\sum\limits_{i=1}^{p+q}(k_i+l_i)\geq p+q$ and the equality holds only if $k_i+l_i=1,\ i=1,\ldots,p+q.$ But we know that $k_1+\cdots+k_{p+q}=p$ and $l_1+\cdots+l_{p+q}=q$, that is why $\sum\limits_{i=1}^{p+q}(k_i+l_i)=p+q.$ Hence, for non-zero elements of the sum $k_i+l_i=1,\ i=1,\ldots,p+q$ and

$$\int_0^1 (r_1(\theta))^{1+k_1+l_1}(r_2(\theta))^{1+k_2+l_2}\ldots(r_{p+q}(\theta))^{1+k_{p+q}+l_{p+q}}\, d\theta$$

$$= \int_0^1 (r_1(\theta))^2(r_2(\theta))^2\ldots(r_{p+q}(\theta))^2\, d\theta = 1.$$

We can simplify the representation of A taking into account that for non-zero elements $l_i=1-k_i,\ i=1,\ldots,p+q.$

$$A = \sum_{\substack{0\leq k_1\leq 1,\ldots,0\leq k_{p+q}\leq 1 \\ k_1+\cdots+k_{p+q}=p}} \frac{1}{k_1!\ldots k_{p+q}!} \times \frac{1}{(1-k_1)!\ldots(1-k_{p+q})!}$$

(2.2)

$$\times \, B_{p,q}\left((x_1)^{k_1},\ldots,(x_{p+q})^{k_{p+q}};(x_1)^{1-k_1},\ldots,(x_{p+q})^{1-k_{p+q}}\right)$$

$$\times \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p+k_1+\cdots+k_p-(1-k_1)-\cdots-(1-k_p)}\, dt.$$

Let us find values of k_i such that the integral is not equal to zero.

$$2q+1-p+k_1+\cdots+k_p-(1-k_1)-\cdots-(1-k_p)$$
$$= 2q+1-p+k_1+\cdots+k_p-p+k_1+\cdots+k_p$$
$$= 2q+1-2(p-k_1-\cdots-k_p) = 2q+1-2(k_{p+1}+\cdots+k_{p+q}).$$

By the property $3°$ of generalized Rademacher functions the integral in (2.2) is not equal to zero if and only if $2q+1$ is a divisor of $2q+1-2(k_{p+1}+\cdots+k_{p+q})$. But since $0\leq k_i\leq 1$,

it will be only if $2q + 1 - 2(k_{p+1} + \cdots + k_{p+q}) = 2q + 1$, that is, $k_{p+1} = \cdots = k_{p+q} = 0$. Hence $p = k_1 + \cdots + k_p + k_{p+1} + \cdots + k_{p+q} = k_1 + \cdots + k_p$, so $k_1 = \cdots = k_p = 1$.

Finally, we can rewrite A as

$$A = B_{p,q}\Big((x_1)^1, \ldots, (x_p)^1, (x_{p+1})^0, \ldots, (x_{p+q})^0; (x_1)^0, \ldots, (x_p)^0, (x_{p+1})^1, \ldots, (x_{p+q})^1\Big)$$

$$\times \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1} dt = B_{p,q}(x_1, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}).$$

$\square$

3. Polarization inequality

We can get an another form of the polarization formula from formula (2.1).

Theorem 3.1.

$$B_{p,q}(x_1, \ldots, x_{p+q})$$

(3.1)
$$= \frac{1}{2^{p+q}p!q!} \sum_{\varepsilon_1, \ldots, \varepsilon_{p+q}=\pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots \varepsilon_{p+q} \sum_{i=1}^{2q+1} \frac{1}{2q+1} \alpha_i^{2q+1-p}$$

$$\times P_{p,q}\Big(\alpha_i \varepsilon_1 x_1 + \alpha_i \varepsilon_2 x_2 + \cdots + \alpha_i \varepsilon_p x_p + \varepsilon_{p+1} x_{p+1} + \cdots + \varepsilon_{p+q} x_{p+q}\Big).$$

Proof.

$$B_{p,q}(x_1, \ldots, x_{p+q}) = \frac{1}{p!q!} \int_0^1 \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p} r_1(\theta) r_2(\theta) \ldots r_{p+q}(\theta)$$

$$\times P_{p,q}\Big(S_1^{[2q+1]}(t)\,(r_1(\theta)x_1 + \cdots + r_p(\theta)x_p)$$

$$+ (r_{p+1}(\theta)x_{p+1} + \cdots + r_{p+q}(\theta)x_{p+q})\Big)\, dt\, d\theta$$

$$= \frac{1}{p!q!} \int_0^1 r_1(\theta) r_2(\theta) r_3(\theta) \ldots r_{p+q}(\theta) \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p}$$

$$\times P_{p,q}\Big(S_1^{[2q+1]}(t) r_1(\theta)x_1 + S_1^{[2q+1]}(t) r_2(\theta)x_2 + \cdots$$

$$+ S_1^{[2q+1]}(t) r_p(\theta)x_p + r_{p+1}(\theta)x_{p+1} + \cdots + r_{p+q}(\theta)x_{p+q}\Big)\, dt\, d\theta$$

$$= \frac{1}{p!q!} \sum_{\varepsilon_1=\pm 1} \int_0^{1/2} \varepsilon_1 r_2(\theta) r_3(\theta) \ldots r_{p+q}(\theta) \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p}$$

$$\times P_{p,q}\Big(S_1^{[2q+1]}(t)\varepsilon_1 x_1 + S_1^{[2q+1]}(t) r_2(\theta)x_2 + \cdots$$

$$+ S_1^{[2q+1]}(t) r_p(\theta)x_p + r_{p+1}(\theta)x_{p+1} + \cdots + r_{p+q}(\theta)x_{p+q}\Big)\, dt\, d\theta$$

$$= \frac{1}{p!q!} \sum_{\varepsilon_1, \ldots, \varepsilon_{p+q}=\pm 1} \int_0^{1/2^{p+q}} \varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots \varepsilon_{p+q} \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p}$$

$$\times P_{p,q}\Big(S_1^{[2q+1]}(t)\varepsilon_1 x_1 + S_1^{[2q+1]}(t)\varepsilon_2 x_2 + \cdots$$

$$+ S_1^{[2q+1]}(t)\varepsilon_p x_p + \varepsilon_{p+1} x_{p+1} + \cdots + \varepsilon_{p+q} x_{p+q}\Big)\, dt\, d\theta$$

$$= \frac{1}{2^{p+q}p!q!} \sum_{\varepsilon_1, \ldots, \varepsilon_{p+q}=\pm 1} \varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots \varepsilon_{p+q} \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p}$$

$$\times P_{p,q}\Big(S_1^{[2q+1]}(t)\varepsilon_1 x_1 + S_1^{[2q+1]}(t)\varepsilon_2 x_2 + \cdots$$
$$+ S_1^{[2q+1]}(t)\varepsilon_p x_p + \varepsilon_{p+1}x_{p+1} + \cdots + \varepsilon_{p+q}x_{p+q}\Big)\, dt$$

$$= \frac{1}{2^{p+q}p!q!} \sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm 1} \varepsilon_1\varepsilon_2\varepsilon_3\ldots\varepsilon_{p+q} \sum_{i=1}^{2q+1} \frac{1}{2q+1}\alpha_i^{2q+1-p}$$
$$\times P_{p,q}\Big(\alpha_i\varepsilon_1 x_1 + \alpha_i\varepsilon_2 x_2 + \cdots + \alpha_i\varepsilon_p x_p + \varepsilon_{p+1}x_{p+1} + \cdots + \varepsilon_{p+q}x_{p+q}\Big).$$

$\square$

Let X and Y be normed spaces. Let us define norms of (p,q)-polynomials and (p,q)-linear symmetric mappings respectively by

$$\|P_{p,q}\|_1 = \sup\{\|P_{p,q}(x)\|_Y : x \in \mathcal{B}\}$$

and

$$\|B_{p,q}\|_2 = \sup\{\|B_{p,q}(x_1,\ldots,x_{p+q})\|_Y : x_1,\ldots,x_{p+q} \in \mathcal{B}\},$$

where $\mathcal{B}$ is the closed unit ball of X.

From formula (3.1) we have the following estimations:

$$\|B_{p,q}\|_2$$

$$= \sup_{\|x_1\|_X \leq 1,\ldots,\|x_{p+q}\|_X \leq 1}\left\|\frac{1}{2^{p+q}p!q!} \sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm 1} \varepsilon_1\varepsilon_2\varepsilon_3\ldots\varepsilon_{p+q} \sum_{i=1}^{2q+1} \frac{1}{2q+1}\alpha_i^{2q+1-p}\right.$$
$$\left.\times P_{p,q}\Big(\alpha_i\varepsilon_1 x_1 + \alpha_i\varepsilon_2 x_2 + \cdots + \alpha_i\varepsilon_p x_p + \varepsilon_{p+1}x_{p+1} + \cdots + \varepsilon_{p+q}x_{p+q}\Big)\right\|_Y$$

$$\leq \frac{1}{2^{p+q}p!q!} \sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm 1} \sum_{i=1}^{2q+1} \frac{1}{2q+1} \times \sup_{\|x_1\|_X \leq 1,\ldots,\|x_{p+q}\|_X \leq 1}\left\|(p+q)^{p+q}\right.$$
$$\left.\times P_{p,q}\Big(\frac{\alpha_i\varepsilon_1 x_1 + \alpha_i\varepsilon_2 x_2 + \cdots + \alpha_i\varepsilon_p x_p + \varepsilon_{p+1}x_{p+1} + \cdots + \varepsilon_{p+q}x_{p+q}}{p+q}\Big)\right\|_Y$$

$$\leq \frac{1}{2^{p+q}p!q!} \cdot 2^{p+q} \cdot \frac{2q+1}{2q+1} \cdot (p+q)^{p+q}\|P_{p,q}\|_1 = \frac{(p+q)^{p+q}}{p!q!}\|P_{p,q}\|_1.$$

Hence we have the polarization inequality:

Theorem 3.2. *Let $B_{p,q}(x_1,\ldots,x_{p+q})$ be a (p,q)-linear symmetric mapping and $P_{p,q}(x)$ be the corresponding (p,q)-polynomial between normed spaces X and Y. Then*

$$\|P_{p,q}\|_1 \leq \|B_{p,q}\|_2 \leq \frac{(p+q)^{p+q}}{p!q!}\|P_{p,q}\|_1.$$

Note that the first part of this inequality is trivial. The following example shows that the second part of the inequality is sharp.

Example 3.1. Let $X = \ell_1$.

For given $p > 0$ and $q > 0$ let $B : X^{p+q} \to \mathbb{C}$ be the following (p,q)-linear map $B(x^1,\ldots,x^p;x^{p+1},\ldots,x^{p+q}) = x_1^1 x_2^2 \ldots x_p^p \overline{x_{p+1}^{p+1}} \ldots \overline{x_{p+q}^{p+q}}$, where $x^j \in \ell_1$, $j = 1,\ldots,p+q$.

The (p,q)-symmetrization of B,

$$B_s(x^1,\ldots,x^p;x^{p+1},\ldots,x^{p+q}) = \frac{1}{p!q!}\sum_{\sigma_1 \in S_p} x_1^{\sigma_1(1)}\ldots x_p^{\sigma_1(p)} \sum_{\sigma_2 \in S_q} \overline{x_{p+1}^{p+\sigma_2(1)}\ldots x_{p+q}^{p+\sigma_2(q)}}$$

is a (p,q)-linear symmetric mapping.

$$|B_s(x^1,\dots,x^p;x^{p+1},\dots,x^{p+q})|$$

$$\leq \frac{1}{p!q!}\sum_{\sigma_1\in S_p}|x_1^{\sigma_1(1)}|\dots|x_p^{\sigma_1(p)}|\sum_{\sigma_2\in S_q}|x_{p+1}^{p+\sigma_2(1)}|\dots|x_{p+q}^{p+\sigma_2(q)}|$$

$$\leq \frac{1}{p!q!}\|x^1\|_{\ell_1}\dots\|x^{p+q}\|_{\ell_1}.$$

Hence $\|B_s\|_2 \leq \frac{1}{p!q!}$. On the other hand,

$$B_s(e^1,\dots,e^{p+q}) = \frac{1}{p!q!},$$

where

$$e^i = (\underbrace{0,\dots,0}_{i-1},1,0,\dots).$$

Therefore, $\|B_s\|_2 = \frac{1}{p!q!}$.

Let $\widehat{B}_s(x) = B_s(x,\dots,x) = x_1\dots x_p\overline{x_{p+1}}\dots\overline{x_{p+q}}$, where $x = (x_1,\dots,x_{p+q},\dots)$. Then $\widehat{B}_s$ is a (p,q)-polynomial. Since the geometric mean of positive numbers is always less than or equal to the arithmetic mean, we have

$$|\widehat{B}_s(x)| = |x_1|\dots|\overline{x_{p+q}}| \leq \frac{1}{(p+q)^{p+q}}(|x_1|+\cdots+|x_{p+q}|)^{p+q}.$$

Thus $\|\widehat{B}_s\|_1 \leq \frac{1}{(p+q)^{p+q}}$. If we take $x = \Big(\underbrace{\frac{1}{p+q},\dots,\frac{1}{p+q}}_{p+q},0,\dots\Big)$, we obtain $|\widehat{B}_s(x)| = \frac{1}{(p+q)^{p+q}}$.

This shows that $\|\widehat{B}_s\|_1 = \frac{1}{(p+q)^{p+q}}$, and hence

$$\|B_s\|_2 = \frac{(p+q)^{p+q}}{p!q!}\|\widehat{B}_s\|_1.$$

The following example shows that in the case of Hilbert space the (p,q)-polarization constant is greater than 1.

Example 3.2. Let $X = \ell_2$, $p > 1$, $q > 1$, $B_{p,q} : X^{p+q} \to \mathbb{C}$,

$$B_{p,q}(x^1,\dots,x^p;x^{p+1},\dots,x^{p+q}) = x_1^1 x_1^2\dots x_1^p\overline{x_2^{p+1}},\dots,\overline{x_2^{p+q}}.$$

Obviously, $B_{p,q}$ is a (p,q)-linear symmetric mapping. From the following estimations

$$|B_{p,q}(x^1,\dots,x^{p+q})| = |x_1^1||x_1^2|\dots|x_1^p||\overline{x_2^{p+1}}|\dots|\overline{x_2^{p+q}}|$$

$$= \sqrt{|x_1^1|^2}\sqrt{|x_1^2|^2}\dots\sqrt{|x_1^p|^2}\sqrt{|x_2^{p+1}|^2}\dots\sqrt{|x_2^{p+q}|^2}$$

$$\leq \|x^1\|_{\ell_2}\|x^2\|_{\ell_2}\dots\|x^{p+q}\|_{\ell_2}$$

we have that $\|B_{p,q}\|_2 \leq 1$. On the other hand $B_{p,q}(e^1,\dots,e^1;e^2,\dots,e^2) = 1$ and so $\|B_{p,q}\|_2 = 1$. The map $\widehat{B}_{p,q}(x) = (x_1)^p(\overline{x_2})^q$ is a (p,q)-polynomial and

$$|\widehat{B}_{p,q}(x)| = |x_1|^p|\overline{x_2}|^q = |x_1|^p|x_2|^q.$$

Hence

$$\|\widehat{B}_{p,q}\|_1 = \sup_{\|x\|_{\ell_2}\leq 1}|\widehat{B}_{p,q}(x)| = \sup_{|x_1|^2+|x_2|^2\leq 1}|x_1|^p|x_2|^q = \max_{0\leq t\leq 1} t^p\left(\sqrt{1-t^2}\right)^q.$$

It is easy to check that the function

$$f(t) = t^p\left(\sqrt{1-t^2}\right)^q$$

has a maximal value at the point

$$t = \sqrt{\frac{p}{p+q}} \in [0,1]$$

and

$$f\left(\sqrt{\frac{p}{p+q}}\right) = \left(\frac{p}{p+q}\right)^{\frac{p}{2}} \left(\sqrt{1 - \frac{p}{p+q}}\right)^q = \left(\frac{p}{p+q}\right)^{\frac{p}{2}} \left(\frac{q}{p+q}\right)^{\frac{q}{2}}.$$

Thus

$$\|\widehat{B}_{p,q}\|_1 = \frac{p^{p/2}q^{q/2}}{(p+q)^{\frac{p+q}{2}}}$$

and

$$\|B_{p,q}\|_2 = \frac{(p+q)^{\frac{p+q}{2}}}{p^{p/2}q^{q/2}} \|\widehat{B}_{p,q}\|_1.$$

Let $\mathcal{P}_{p,q}(X;Y)$ be the normed space of all continuous (p,q)-polynomials from X to Y with the norm $\|\cdot\|_1$ and $\mathcal{B}_{p,q}(X^n;Y)$ the normed space of all continuous symmetric (p,q)-linear maps from X^n to Y, $p+q=n$, with the norm $\|\cdot\|_2$.

Theorem 3.2 implies the following corollary:

Corollary 3.1. *The space $\mathcal{P}_{p,q}(X;Y)$ is isomorphic to $\mathcal{B}_{p,q}(X^n;Y)$.*

Proof. The polarization formula gives us the required linear isomorphism from $\mathcal{P}_{p,q}(X;Y)$ onto $\mathcal{B}_{p,q}(X^n;Y)$ and the polarization inequality implies its continuity.   □

Remark 3.1. In [11] the authors proved some another forms of the polarization formula using different approach

$$B_{p,q}(x_1,\ldots,x_p;x_{p+1},\ldots,x_{p+q})$$

(3.2)
$$= \frac{1}{2^m p!\,q!} \sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=0}^{1} (-1)^{p+q-(\varepsilon_1+\cdots+\varepsilon_{p+q})} \sum_{\mu_1,\ldots,\mu_m=0}^{1} (r_1^{\mu_1} r_2^{\mu_2} \ldots r_m^{\mu_m})^q$$
$$\times P_{p,q}\big((x' + \varepsilon_1 x_1 + \cdots + \varepsilon_p x_p)$$
$$+ (r_1^{\mu_1} r_2^{\mu_2} \ldots r_m^{\mu_m})(x'' + \varepsilon_{p+1} x_{p+1} + \cdots + \varepsilon_{p+q} x_{p+q})\big),$$

and

$$B_{p,q}(x_1,\ldots,x_p;x_{p+1},\ldots,x_{p+q})$$

(3.3)
$$= \frac{1}{p!q!2^{p+q}} \sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm 1} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{p+q} \sum_{\mu_1,\ldots,\mu_m=0}^{1} \frac{1}{2^m}(r_1^{\mu_1} r_2^{\mu_2} \ldots r_m^{\mu_m})^q$$
$$\times P_{p,q}\big((x' + \varepsilon_1 x_1 + \cdots + \varepsilon_p x_p)$$
$$+ (r_1^{\mu_1} r_2^{\mu_2} \ldots r_m^{\mu_m})(x'' + \varepsilon_{p+1} x_{p+1} + \cdots + \varepsilon_{p+q} x_{p+q})\big),$$

where

$$r_k = \cos\frac{\pi}{2^{k-1}} + i\sin\frac{\pi}{2^{k-1}},$$

$$m = \big[\log_2(p+q)\big] + 1$$

and x', x'' are arbitrary elements of X.

Note that from formula (3.3) we can also get the polarization inequality.

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