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THE INFINITE DIRECT PRODUCTS OF PROBABILITY MEASURES AND STRUCTURAL SIMILARITY

VOLODYMYR KOSHMANENKO

Abstract. We show that any similar structure measure on the segment $[0,1]$ is an image-measure of the appropriate constructed infinite direct product of discrete probability measures.

1. Introduction

In this paper we continue (see [11, 9, 12]) to study a specific set of measures on the segment $[0,1]$, the so called similar structure measures, which is considerably wider than the well-known class of self-similar measures introduced by Hutchinson [6] (see also [7, 16]).

The similar structure measures have a certain similarity property on any $\varepsilon > 0$ micro-level but unlike to the self-similar measures they in general do not satisfy the transformation condition: $\mu(\cdot) = \sum_{i=1}^{n} p_i \mu(T_i^{-1}\cdot)$ for an appropriate family of similitudes $\mathbf{T} = \{T_i\}_{i=1}^n$ and some set of ratios $p_i \geq 0,\ p_1 + \cdots + p_n = 1$. In fact, each similar structure measure possesses a more general kind of invariance property with respect transformations generating by the fixed sequence of iterated function systems $T = \{\mathbf{T}_k\}_{k=1}^{\infty}$.

The main result of the paper is that every similar structure measure μ on $[0,1]$ may be considered as an image-measure $\mu = \tilde{\mu} = \pi\mu^*$ of the infinite direct product of discrete probability measures $(\Omega, \mathcal{A}, \mu^*) = \prod_{k=1}^{\infty}(\Omega_k, \mathcal{A}_k, m_k)$, where the mapping $\pi : \Omega \to [0,1]$ is defined by a fixed sequence of iterated function systems $T = \{\mathbf{T}_k\}_{k=1}^{\infty}$ uniquely associated with μ (see [1, 2, 4, 14, 15]).

We note that similar structure measures have wide applications, especially in the models describing biological populations and conflict interactions, in particular, in dynamical systems of conflict. [2, 3, 9, 10].

2. Similar structure measures

Let us describe a notion of probability similar structure measure on the segment $\Delta_0 \equiv [0,1]$ (for more details see [9, 11, 12]).

Let $T = \{\mathbf{T}_k\}_{k=1}^{\infty} = \{T_{ik}\}_{i=1}^{n}$, $\ 2 \leq n < \infty$, be a family of semilitudes (contractive similarities of the form $T_{ik}x = c_{ik}x + t_{ik},\ c_{ik}, t_{ik} < 1$) on $\mathbb{R}^1$ such that

$$T_{ik}\Delta_0 \subset \Delta_0, \quad i = 1, \dots, n, \quad k = 1, 2, \dots$$

Assume that for each k, the contractions $\mathbf{T}_k = \{T_{ik}\}_{i=1}^n$ satisfy the open set condition (see e.g. [13]), i.e., there exists a non-empty open set O such that

$$\bigcup_{i_k=1}^{n} T_{i_k k} O \subset O \quad \text{and} \quad T_{i_k k} O \bigcap T_{i'_k k} O = \emptyset, \quad i_k \neq i'_k.$$

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Therefore $\mathbf{T}_k$ is an iterated function system (for more details see [7]). So, we start with a sequence $T = \{\mathbf{T}_k\}_{k=1}^{\infty}$ of iterated function systems.

For fixed k, all possible ordered compositions of contractions

$$T_{i_1 i_2 \dots i_k} := T_{i_1 1} \circ T_{i_2 2} \circ \dots \circ T_{i_k k}$$

generate the family of subsets (closed segments from Δ_0) of rank k

$$\Delta_{i_1 \dots i_k} := T_{i_1 \dots i_k} \Delta_0, \quad i_1, \dots, i_k = 1, \dots, n.$$

By construction the obvious inclusions

$$\Delta_{i_1 i_2 \dots i_{k-1}} \supset \Delta_{i_1 i_2 \dots i_k}, \quad i_k = 1, \dots, n$$

are fulfilled, and hence

$$\Delta_{i_1 i_2 \dots i_{k-1}} \supseteq \bigcup_{i_k=1}^{n} \Delta_{i_1 i_2 \dots i_k}, \quad k = 1, 2, \dots \tag{2.1}$$

It is clear that all above segments are geometrically similar to one another. In particular, for different segments of the same rank we have

$$\Delta_{i_1 \dots i_k} = U_{i_1 \dots i_k, i'_k \dots i'_1} \Delta_{i'_1 \dots i'_k},$$

where

$$U_{i_1 \dots i_k, i'_k \dots i'_1} := T_{i_1 \dots i_k} T^{-1}_{i'_1 \dots i'_k}, \quad 1 \le i_k, \quad i'_k \le n \tag{2.2}$$

is a similarity transformation, which is well defined since each contraction $T_{i_k k}$, as well as its inverse, is bijective.

Definition 2.1. *A set $S_0 \subseteq \Delta_0$ is said to be a similar structure set, if there exists a sequence of iterated function systems, $T = \{\mathbf{T}_k\}_{k=1}^{\infty}$ such that for each $k = 1, 2, \dots,$ this set can be split into parts,*

$$S_0 = \bigcup_{i_1=1}^{n} \cdots \bigcup_{i_k=1}^{n} S_{i_1 \dots i_k}, \quad S_{i_1 \dots i_k} \subseteq \Delta_{i_1 \dots i_k}, \tag{2.3}$$

or, equivalently,

$$S_0 = \bigcup_{i_1=1}^{n} S_{i_1}, \quad S_{i_1} = \bigcup_{i_2=1}^{n} S_{i_1 i_2}, \ \cdots, \quad S_{i_1 \dots i_{k-1}} = \bigcup_{i_k=1}^{n} S_{i_1 i_2 \dots i_k}, \ \cdots, \tag{2.4}$$

where all non-empty subsets $S_{i_1 \dots i_k}$, $S_{i'_1 \dots i'_k}$ are similar to one another

$$S_{i_1 \dots i_k} = U_{i_1 \dots i_k, i'_k \dots i'_1} S_{i'_1 \dots i'_k}. \tag{2.5}$$

Directly from this definition it follows that, under the above sequence of iterated function systems T, the whole segment Δ_0 is a similar structure set with $S_{i_1 \dots i_k} = \Delta_{i_1 \dots i_k}$ if (2.1) always contains the equality sign.

We emphasize that in general

$$\Delta_0 \neq \bigcup_{i_k=1}^{n} T_{i_k k} \Delta_0,$$

and it is possible that some of the above sets $S_{i_1 \dots i_k}$ are empty.

It is also clear that

$$\operatorname{diam}(S_{i_1 \dots i_k}) \to 0, \quad k \to \infty,$$

if all $c_{ik} < c$. Besides

$$\lambda(S^{\mathrm{cl}}_{i_1 \dots i_k} \bigcap S^{\mathrm{cl}}_{i'_1 \dots i'_k}) = 0, \quad \text{if} \quad i_l \neq i'_l$$

at least for single $1 \le l \le k$, where cl stands for closure.

We note that subsets of various ranks, $S_{i_1\dots i_l},\ S_{i_1\dots i_k},\ k \neq l$ are not in general similar. In particular, no $S_{i_1\cdots i_k},\ k = 1, 2, \dots$, is in general similar to the whole set S_0. This is the important distinctive feature of a similar structure set in comparison with a self-similar one (see [6]).

Roughly speaking, a similar structure set on any ε-level ($\varepsilon > 0$) can be decomposed into a finitely many families of cells similar to each other with diameters not more than ε. However for different ε-levels the corresponding cells are not necessarily similar.

From fractal geometry (see also [16]) it is known that for each iterated function system its invariant set is self-similar. We assert that any sequence of iterated function systems generates a similar structure set.

Theorem 2.2. *Let $T = \{\mathbf{T}_k\}_{k=1}^{\infty}$ be a sequence of iterated function systems. Then the uniquely constructed from T set*

$$\Gamma := \{x \in \Delta_0 |\ x = \lim_{k\to\infty} T_{i_1 1\dots i_k k} y,\ y \in \mathbb{R}^1,\ i_1, \dots, i_k = 1, \dots, n\}$$

(the point y is arbitrary) has similar structure (see Definition 2.1)). Besides Γ is invariant in the sense that

$$\Gamma = \bigcup \mathcal{T}_{i_1\dots i_k\dots}\Gamma, \quad \mathcal{T}_{i_1\dots i_k\dots} := \lim_{k\to\infty} T_{i_1 1\dots i_k k}, \tag{2.6}$$

where the union is taken over all coordinate directions $i_1 \dots i_k \dots$

Proof. At first we recall that in the case of a single iterated function system, i.e., if $\mathbf{T}_k = \mathbf{T} = \{T_i\}_{i=1}^n$ is the same family of contractions for all k, then Γ is a usual invariant self-similar set, $\Gamma = \bigcup_{i=1}^n T_i\Gamma$ (see e.g., [7])

$$\Gamma := \{x \in \Delta_0 |\ x = \lim_{k\to\infty} T_{i_1\dots i_k} y,\ y \in \mathbb{R}^1,\ i_1, \dots, i_k = 1, \dots, n\}.$$

In other words, Γ is the set consisting of the accumulating fixed points for all possible coordinate directions $i_1 \dots i_k \dots$

$$\Gamma = \{x \in \Delta_0 |\ x = x_{i_1\dots i_k\dots} = \mathcal{T}_{i_1\dots i_k\dots}\ y,\ i_1, \dots, i_k, \dots = 1, \dots, n\},$$

where the limit point x does not depend on $y \in \mathbb{R}^1$ (instead y one take put any compact set and take an infinite intersection of its images).

Let us consider a sequence of iterated function systems $\mathbf{T}_k, k = 1, 2, \dots,$, which are, in general, different. Then we have to prove that a sequence

$$y_k := T_{i_1\dots i_k}\ y, \quad y \in \mathbb{R}^1,$$

has a unique accumulation point, i.e., converges

$$x = \lim_{k\to\infty} y_k.$$

If it is true, we may write

$$x = x_{i_1 i_2\dots i_k\dots} = \mathcal{T}_{i_1\dots i_k\dots} y,$$

since the limit point depends on only the coordinate direction $i_1 \dots i_k \dots$

Indeed, if we change y over Δ_0, then from (2.1) it follows that all $y_k \in \Delta_{i_1\dots i_k}$. So, if we fix a certain coordinate direction $i_1 \dots i_k \dots$, then there appears a sequence of associated embedded segments

$$\Delta_{i_1} \supset \Delta_{i_1 i_2} \supset \cdots \supset \Delta_{i_1\dots i_k} \supset \cdots$$

Taking into account that $\mathrm{diam}(\Delta_{i_1\dots i_k})$ goes to zero with $k \to \infty$, we conclude that there exists a unique limiting point

$$x_{i_1 i_2\dots i_k\dots} = \bigcap_{k=1}^{\infty} \Delta_{i_1\dots i_k} = \lim_{k\to\infty} y_k, \tag{2.7}$$

which does not depend of a chosen starting point y. We remark that the latter equality is valid just due to $y_k \in \Delta_{i_1 \dots i_k}$.

We observe that the mapping $\mathcal{T}_{i_1 \dots i_k \dots}$ has the image consisting of a unique point

$$\mathcal{T}_{i_1 \dots i_k \dots} : \mathbb{R}^1 \longrightarrow x_{i_1 \dots i_k \dots} \in \Delta_0,$$

in spite of that all maps $T_{i_l \dots i_k}$ are bijective. It means that the contraction ratio of $\mathcal{T}_{i_1 \dots i_k \dots}$ equals zero.

Let us define Γ as a set of all limiting points of view (2.7)

$$\Gamma := \bigcup x_{i_1 i_2 \dots i_k \dots},$$

where the union is taken over the uncountable family of all coordinate directions.

Now we decompose Γ for each $k = 1, 2, \dots$ onto subsets,

$$\Gamma = \bigcup_{i_1, \dots, i_k = 1}^{n} \Gamma_{i_1 \dots i_k},$$

where

$$\Gamma_{i_1 \dots i_k} := \Gamma \bigcap \Delta_{i_1 \dots i_k} = \bigcup_{i_1 \dots i_k \text{ is fixed}} x_{i_1 \dots i_k \dots}. \tag{2.8}$$

It proves (2.3) and (2.4) for Γ with $S_{i_1 \dots i_k} = \Gamma_{i_1 \dots i_k}$, which all are non-empty. To prove (2.5) we note that by construction for each fixed k we have

$$\Gamma_{i_1 \dots i_k} \ni x_{i_1 \dots i_k \dots} = U_{i_1 \dots i_k, i'_k \dots i'_1} x_{i'_1 \dots i'_k \dots} \in \Gamma_{i'_1 \dots i'_k}.$$

Therefore we have also

$$\Gamma_{i_1 \dots i_k} = U_{i_1 \dots i_k, i'_k \dots i'_1} \Gamma_{i'_1 \dots i'_k}. \tag{2.9}$$

Thus, Γ is a similar structure set.

Finally, (2.6) is evident since for any $y \in \mathbb{R}$

$$\mathcal{T}_{i_1 \dots i_k \dots} y = x = x_{i_1 \dots i_k \dots} \in \Gamma.$$

□

Now we are able to introduce a notion of similar structure measure.

Definition 2.3. *A Borel measure μ supported on Δ_0 is said to be a similar structure measure, if its (minimal closed) support $S_\mu = supp\mu$ is a similar structure set, i.e., admits the representations of view (2.3), (2.4)*

$$S_\mu = \bigcup_{i_1, \dots, i_k = 1}^{n} S_{i_1 \dots i_k}, \quad S_{i_1 \dots i_{k-1}} = \bigcup_{i_k = 1}^{n} S_{i_1 \dots i_k} \quad (S_{i_0} \equiv S_\mu), \tag{2.10}$$

where all subsets $S_{i_1 \dots i_k}$ for every fixed rank $k \geq 1$ are similar one to other in sense (2.5)). Besides, for each $k = 1, 2, \dots$

$$\mu(S_{i_1 \dots i_k}) = p_{i_k k} \cdot \mu(S_{i_1 \dots i_{k-1}}), \quad p_{i_k k} \geq 0, \quad \sum_{i_k = 1}^{n} p_{i_k k} = 1. \tag{2.11}$$

We remark that in (2.11) ratios $p_{i_k k}$ are independent of indices $i_1, \dots, i_{k-1}$ and $p_{i_k k} = 0$ for empty $S_{i_1 \dots i_k}$.

Thus, each similar structure measure is associated with some sequence of iterated function systems T and, by (2.11), with some stochastic matrix

$$P \equiv \{\mathbf{p}_k\}_{k=1}^{\infty} = \{p_{ik}\}_{i=1,\ k=1}^{n,\ \infty}, \tag{2.12}$$

whose columns are formed by coordinates of stochastic vectors $\mathbf{p}_k \in \mathbb{R}^n$

$$\mathbf{p}_k = (p_{1k}, \dots, p_{nk}), \quad p_{1k}, \dots, p_{nk} \geq 0, \quad p_{1k} + \dots + p_{nk} = 1, \quad k = 1, 2, \dots$$

We remark also that instead of the standard invariance property for self-similar measures,

$$\mu(B) = \sum_{i=1}^{n} p_i \mu(T_i^{-1} B), \quad p_i \geq 0, \quad p_1 + \dots + p_n = 1, \quad B \in \mathcal{B},$$

now, in the case of similar structure measures, from (2.11) it follows a more specific relation, which fulfilled separately for each k

$$\mu(B) = \sum_{i_1,\dots,i_k} \mu(B_{i_1\dots i_k}) = \sum_{i_k=1}^{n} p_{i_k k} \sum_{i_1,\dots,i_{k-1}} \mu(T_{i_k}^{-1} B \bigcap S_{i_1\dots i_{k-1}}), \quad B \in \mathcal{B}$$

where

$$B_{i_1\dots i_k} := B \bigcap S_{i_1\dots i_k}.$$

The set of probability similar structure measures on Δ_0 will be denoted by $\mathcal{M}^{\rm ss}(\Delta_0) \equiv \mathcal{M}^{\rm ss}$ (ss stands for similar structure).

3. Image-measures

Let

$$(3.1) \qquad (\Omega, \mathcal{A}, \mu^*) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, m_k)$$

be the infinite direct product (for details see, e.g., [5, 8]) of some sequence of discrete probability spaces

$$(\Omega_k, \mathcal{A}_k, m_k), \quad \Omega_k = \{\omega_{i_k}\}_{i_k=1}^{n}, \quad m_k(\omega_{i_k}) = p_{i_k k} \geq 0,$$

where Ω_k and σ-algebra $\mathcal{A}_k$ depend on k only formally (in fact they are the same objects for all k). Above numbers $p_{i_k k}$, which define the discrete measures m_k, are in general changed together with $i_k = 1, \dots, n$ and $k = 1, 2, \dots$ Thus, the measure μ^* is uniquely associated with some infinite stochastic matrix

$$P = \{\mathbf{p}_k\}_{k=1}^{\infty}, \quad \mathbf{p}_k = (p_{i_k k})_{i_k=1}^{n}, \quad p_{i_1 k} + p_{i_2 k} + \dots + p_{i_n k} = 1.$$

Its columns are denoted by $\mathbf{p}_k \in \mathbb{R}^n$, $1 < n < \infty$. The meanings of μ^* on cylindrical sets $\Omega_{i_1\dots i_k} := \omega_{i_1} \times \dots \times \omega_{i_k} \times \prod_{l=1}^{\infty} \Omega_{k+l}$ are defined by the matrix P as follows:

$$(3.2) \qquad \mu^*(\Omega_{i_1\dots i_k}) = \prod_{s=1}^{k} p_{i_s s},$$

where we take into account that $m_k(\Omega_k) = 1$.

We will correspond to μ^* its image on the segment $[0, 1]$, the so-called the image-measure, which is denoted by $\tilde{\mu}$. With this aim we need to fix a measurable mapping π from Ω onto $[0, 1]$. We introduce π using some in general non-stationary sequence of semilitudes $T = \{\mathbf{T}_k\}_{k=1}^{\infty}$ considered in the previous section.

Namely we shall define the mapping π from Ω to the invariant set Γ. Then using π we define $\tilde{\mu}$, as the image-measure of μ^*

$$(3.3) \qquad \tilde{\mu} = \pi\mu^*, \quad \tilde{\mu}(B) := \mu^*(\pi^{-1}(B)), \quad \forall B \in \mathcal{B}.$$

Let us consider at first a particular case. Assume $T = \{\mathbf{T}_k\}_{k=1}^{\infty}$ obey the following conditions.

(**a**) All contraction coefficients c_{ik} of T_{ik} are uniformly isolated from below, i.e., for all i, k

$$0 < c \leq c_{ik} < 1.$$

(**b**) For each $k = 1, 2, \ldots$ the ranges of $T_{i_k k} \equiv T_{ik}$ complete the whole segment Δ_0

$$\Delta_0 = \bigcup_{i_k=1}^{n} T_{i_k k} \Delta_0.$$

(**c**) The different sub-segments $T_{i_k k}\Delta_0$ have zero Lebesgue intersections

$$\lambda(T_{i_k k}\Delta_0 \bigcap T_{i'_k k}\Delta_0) = 0, \quad i_k \neq i'_k,$$

where λ denotes Lebesgue measure.

Condition (**c**) obviously implies that contractions $\mathbf{T}_k = \{T_{ik}\}_{i=1}^n$ for each $k = 1, 2, \ldots$ satisfy the open set condition. We may put $O = (0, 1)$ as an open set in this condition for all contractions.

Therefore, any family of iterated function systems $T = \{\mathbf{T}_k\}$ with conditions (**a**) – (**c**) defines a countable sequence of decompositions of the segment $[0, 1]$

$$\Delta_0 = [0,1] = \bigcup_{i_1=1}^{n} \Delta_{i_1}, \quad \Delta_{i_1} = \bigcup_{i_2=1}^{n} \Delta_{i_1 i_2}, \; \cdots$$

In particular, due to (**b**),

$$\Delta_{i_1 i_2 \ldots i_{k-1}} = \bigcup_{i_k=1}^{n} \Delta_{i_1 i_2 \ldots i_k}, \quad k \geq 1, \tag{3.4}$$

where recall $\Delta_{i_0} = \Delta_0$, and $\Delta_{i_1 i_2 \ldots i_k} := T_{i_1 1} \cdots T_{i_k k}\Delta_0$. Thus now the whole segment Δ_0 is the similar structure set for T. So, due to (3.4) for every point $x \in [0, 1]$ there exists a sequence of embedded segments $\Delta_{i_1 i_2 \ldots i_k}$ containing this point and such that $x = \bigcap_{k=1}^{\infty} \Delta_{i_1 i_2 \ldots i_k}$. This fact can be written in the following form:

$$x = x_{i_1 i_2 \ldots i_k \ldots} = \bigcap_{k=1}^{\infty} \Delta_{i_1 i_2 \ldots i_k}, \tag{3.5}$$

where obviously the sequence of indexes $i_1, i_2, \ldots, i_k, \ldots$ (a fixed direction) defines the point x uniquely. That is, $i_1, i_2, \ldots, i_k, \ldots$ may be considered as coordinates of x.

In the general situation there appears the one-to-one correspondence between sequences of iterated function systems $T = \{\mathbf{T}_k\}_{k=1}^{\infty}$ and mappings

$$\pi : \Omega \ni \omega^* = \{\omega_{i_1} \times \omega_{i_2} \times \cdots \times \omega_{i_k} \times \cdots\} \to x = x_{i_1 i_2 \ldots i_k \ldots} \in \Gamma, \tag{3.6}$$

where ω^* and the corresponding point x have the same coordinate direction $i_1, \ldots, i_k, \ldots$ We recall, that

$$x_{i_1 i_2 \ldots i_k \ldots} = \lim_{k \to \infty} y_k, \quad y_k = (T_{i_1 1} \circ \cdots \circ T_{i_k k})y, \quad \forall y \in \mathbb{R}^1. \tag{3.7}$$

We remark, that π is possibly not bijective, if some sets $\Gamma_{i_1 i_2 \ldots i_k}$ of a fixed rank has a common end-points. By this reason sometimes we need to replace Ω in (3.6) on $\Omega \diagdown \Omega_0$. We shall produce this replacement formally always. Although the set Ω_0 is taken non-empty only if there exists $k_0 > 1$ such that one of the following inequalities holds:

$$P_{k_0}^L = \prod_{s \geq k_0} p_{1s} > 0, \quad P_{k_0}^R = \prod_{s \geq k_0} p_{ns} > 0.$$

Namely,

$$\Omega_0 = \{\omega^* \in \Omega \mid \omega_{i_s} = \omega_1, \ \forall s \geq k_0\},$$

or

$$\Omega_0 = \{\omega^* \in \Omega \mid \omega_{i_s} = \omega_n, \ \ \forall s \geq k_0\}$$

respectively to the first or to the second case. It is easy to show that $\mu^*(\Omega_0) = 0$ in any case. We shall denote the restriction of π onto $\Omega \diagdown \Omega_0$ again by π.

Thus, under given μ^*, each sequence of iterated function systems T on $[0,1]$ fixes some mapping π and therefore uniquely defines the image-measure $\tilde{\mu} = \pi\mu^*$. Clearly, if we change T then mapping π will also changed and new image-measure appears.

In the next section we discuss the similar structure properties of image-measures.

4. The similar structure image-measures

We state that each image-measure $\tilde{\mu} = \pi\mu^*$ given by (3.3) automatically is necessary similar structure measure if the mapping π is constructed as above by some sequence of iterated function systems T. Rigorously we formulate our observation as follows.

Theorem 4.1. *Let μ^* be the infinite direct product of discrete probability measures m_k, $k = 1, 2, \dots$ (see (3.1)). And let the mapping $\pi : \Omega \to \Gamma$ is given by some in general non-stationary sequence of iterated function systems T (see (3.6), (3.7)). Then the image-measure $\tilde{\mu} = \pi\mu^*$ has the similar structure, $\tilde{\mu} \in \mathcal{M}^{\rm ss}$.*

Conversely, each similar structure measure, $\mu \in \mathcal{M}^{\rm ss}$ on $[0,1]$ which is associated with a sequence of iterated function systems T (see Definition 2.3) is the image-measure $\mu = \tilde{\mu} = \pi\mu^*$ of the infinite direct product $\mu^* = \prod_{k=1}^{\infty} m_k$ of some sequence of appropriate discrete probability measures m_k, where the mapping π is constructed by T.

Proof. A key of our arguments is based on a fact that both measures, a similar structure measure $\mu \in \mathcal{M}^{\rm ss}$ and a image-measure $\tilde{\mu} = \pi^{-1}\mu^*$ are associated with the same stochastic matrix P.

Let us consider some image-measure $\tilde{\mu} = \pi\mu^*$, where $m^* = \prod_{k=1}^{\infty} m_k$ and the mapping π is constructed by a fixed sequence of iterated function systems T. We take into account that μ^* uniquely connected with some stochastic matrix

$$P \equiv \{\mathbf{p}_k\}_{k=1}^{\infty} = \{p_{ik}\}_{i=1,\ k=1}^{n,\ \infty}, \quad p_{ik} = m_k(\omega_i).$$

We have to show that $\tilde{\mu} \in \mathcal{M}^{\rm ss}$. With this aim consider a sequence of probability measures μ_k, uniformly distributed on $\Delta_{i_1\dots i_k} = T_{i_1\dots i_k}\Delta_0 = \pi\Omega_{i_1\dots i_k}$, and defined as follows:

$$\mu_k(\Delta_{i_1\dots i_k}) := \sum_{i_1,\dots,i_k=1}^{n} C_{i_1\dots i_k}\lambda_{i_1\dots i_k}, \tag{4.1}$$

where

$$C_{i_1\dots i_k} := \frac{p_{i_1 1}\cdots p_{i_k k}}{c_{i_1 1}\cdots c_{i_k k}},$$

($c_{i_k k}$ is the contraction coefficient for $T_{i_k k}$) and

$$\lambda_{i_1\dots i_k} := \lambda|\Delta_{i_1\dots i_k}$$

denotes the restriction of Lebesgue measure on the segment $\Delta_{i_1\dots i_k}$. By (4.1) it follows that

$$\mu_1(\Delta_{i_1}) = p_{i_i 1},\ \dots\ ,\ \mu_k(\Delta_{i_1,\dots,i_k}) = p_{i_1 1}\cdots p_{i_k k}, \tag{4.2}$$

since obviously $\lambda_{i_1\dots i_k}(\Delta_{i_1\dots i_k}) = \prod_{l=1}^{k} c_{i_l l}$. From (4.1), (4.2) it also follows that the sequence of distribution functions $f_k(x) = \mu_k\{(-\infty, x)\}$ for measures μ_k uniformly converges to some left continuous non-decreasing function, $f_k(x) \to f(x), k \to \infty$. Thus, $f(x)$ is the distribution function for some probability measure, which obviously coincides with the image-measure $\tilde{\mu}$. So we have

$$\tilde{\mu} = \lim_{k\to\infty} \mu_k.$$

We shall use this fact to prove that $\tilde{\mu}$ has the similar structure.

Let us consider the geometrical structure of the support for $\tilde{\mu}$. By the above construction one can write

$$S_{\tilde{\mu}} \equiv \mathrm{supp}\tilde{\mu} = \bigcap_k S_{\mu_k}, \quad S_{\mu_k} = \mathrm{supp}\mu_k. \tag{4.3}$$

Define now the sets

$$S_{i_1\ldots i_k} := S_{\tilde{\mu}} \bigcap \Gamma_{i_1\ldots i_k}, \tag{4.4}$$

where $\Gamma_{i_1\cdots i_k}$ are "elementary" subsets of the invariant set for T (see (2.8)). Clearly that using just defined sets we have for each $k \geq 1$

$$S_{\tilde{\mu}} = \bigcup_{i_1,\ldots,i_k=1}^{n} S_{i_1\ldots i_k}.$$

Besides, from (2.9) it follows that all sets $S_{i_1\ldots i_k}$ of fixed rank are similar, i.e., (2.5) is fulfilled. Thus, we prove that the support of the measure $\tilde{\mu}$ is a similar structure set.

Further, due to (4.2) we obtain the important relations

$$\mu(S_{i_1\ldots i_k}) = \mu_k(\Delta_{i_1\ldots i_k}) = p_{i_1 1}\cdots p_{i_k k}. \tag{4.5}$$

We observe that $S_{i_1\ldots i_k}$ is non-empty, if and only if

$$\mu(S_{i_1\ldots i_k}) = p_{i_1 1}\cdots p_{i_k k} \neq 0.$$

By (4.5) the equalities (2.11) are fulfilled and therefore $\tilde{\mu} \in \mathcal{M}^{\mathrm{ss}}$.

Conversely, starting with a before given measure $\mu \in \mathcal{M}^{\mathrm{ss}}$ on $[0,1]$ we consider the sequence of discrete probability measures m_k on a some space of discrete points $\Omega = \{\omega_i\}_{i=1}^n$: $m_k(\omega_i) = p_{ik}$, where p_{ik} are matrix elements of P which is associated with μ. Using m_k we construct the infinite direct product $\mu^* = \Pi_{k=1}^{\infty} m_k$. Now the image-measure $\tilde{\mu} = \pi\mu^*$ obviously coincides with μ, where the mapping π was constructed by the sequence of iterated function systems T associated with a given starting measure. Thus for each $k \geq 1$

$$S_\mu = S_{\tilde{\mu}} = \bigcup_{i_1\ldots i_k=1}^{n} S_{i_1\ldots i_k}, \quad S_{i_1\ldots i_k} = S_\mu \bigcap \Gamma_{i_1\ldots i_k}. \tag{4.6}$$

That completes the proof. □

We remark that the subsets $S_{i_1\ldots i_k} \subseteq \Gamma_{i_1\ldots i_k}$ admits another definition

$$S_{i_1\ldots i_k} = \{x = x_{i_1\ldots i_k\ldots} \in \Gamma_{i_1\ldots i_k} \mid \lim_{l\to\infty} \frac{p_{i_l l}}{c_{i_l l}} > 0\}^{\mathrm{cl}},$$

where recall that cl stands for closure and i_l is changed along to the coordinate direction of a point $x_{i_1\ldots i_k\ldots}$.

References

1. S. Albeverio, V. Koshmanenko, M. Pratsiovytyi, G. Torbin, *$\widetilde{Q}$-representation of real numbers and fractal probability distributions*, Preprint No. 12, University of Bonn, 2002; arcXiv:math., PR/03 08 007 v1, 2003.
2. S. Albeverio, V. Koshmanenko, M. Pratsiovytyi, G. Torbin, *Spectral properties of image measures under infinite conflict interactions*, Positivity **10** (2006), 39–49.
3. S. Albeverio, V. Koshmanenko, I. Samoilenko, *The conflict interaction between two complex systems: Cyclic migration*, J. Interdisciplinary Math. **11** (2008), no. 2, 163–185.
4. S. Albeverio, G. Torbin, *Image measures of infinite product measures and generalized Bernoulli convolutions*, Transactions of the National Pedagogical University (Phys.-Math. Sci.) **5** (2004), 228–241.
5. Yu. M. Berezanskii, *Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables*, Amer. Math. Soc., Providence, RI, 1986. (Russian edition: Naukova Dumka, Kiev, 1978).
6. J. E. Hutchinson, *Fractals and selfsimilarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

7. K. J. Falconer, *Fractal Geometry*, John Wiley & Sons, Chichester, 1990.
8. S. Kakutani, *Equivalence of infinite product measures*, Ann. of Math. **49** (1948), 214–224.
9. T. Karataieva and V. Koshmanenko, *Origination of the singular continuous spectrum in the dynamical systems of conflict*, Methods Funct. Anal. Topology **15** (2009), no. 1, 15–30.
10. V. Koshmanenko, N. Kharchenko, *Spectral properties of image measures after conflict interactions*, Theory of Stochastic Processes **10(26)** (2004), no. 3–4, 73–81.
11. V. Koshmanenko, *Regeneration of the spectral type in the limiting distributions of the conflict dynamical systems*, Ukrainian Math. J. **59** (2007), 771–784.
12. V. Koshmanenko, *The full measure on the space of singular continuous measures*, Ukrainian Math. J. **61** (2009), 83–91.
13. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.
14. M. V. Pratsiovytyi, *Fractal Approach to Investigation of Singular Distributions*, National Pedagogical University, Kyiv, 1998.
15. G. M. Torbin, *Fractal properties of the distributions of random variables with independent Q-symbols*, Transactions of the National Pedagogical University (Phys.-Math. Sci.) **3** (2002), 241–252.
16. H. Triebel, *Fractals and Spectra Related to Fourier Analysis and Functional Spaces*, Birhäuser Verlag, Basel—Boston—Berlin, 1997.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine
E-mail address: kosh@imath.kiev.ua

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