Methods of Functional Analysis and Topology
Vol. 16 (2010), no. 4, pp. 383–392

ON THE NUMBER OF NEGATIVE EIGENVALUES OF A MULTI-DIMENSIONAL SCHRÖDINGER OPERATOR WITH POINT INTERACTIONS

OSAMU OGURISU

Abstract. We prove that the number N of negative eigenvalues of a Schrödinger operator L with finitely many points of δ-interactions on $\mathbb{R}^d$ $(d \leq 3)$ is equal to the number of negative eigenvalues of a certain class of matrix M up to a constant. This M is expressed in terms of distances between the interaction points and the intensities. As applications, we obtain sufficient and necessary conditions for L to satisfy $N = m, n, n$ for $d = 1, 2, 3$, respectively, and some estimates of the minimum and maximum of N for fixed intensities. Here, we denote by n and m the numbers of interaction points and negative intensities, respectively.

1. Introduction and Main Theorem

We examine the number N of negative eigenvalues of a Schrödinger operator L with finitely many points of δ-interactions on $\mathbb{R}^d$ $(d \leq 3)$. We denote by n and m be the numbers of interaction points and negative intensities, respectively. The fact that $N \leq n$ is one of the classical results [1]. Regarding some recent results on $\mathbb{R}^1$, in [3] S. Albeverio and L. Nizhnik gave necessary and sufficient conditions for L to satisfy $N = n$ in the case of $m = n$. Moreover, in [2] they gave an elegant 'algorithm' for determining N. This yields the result obtained in [3] and gives necessary and sufficient conditions for L to satisfy $N = n$. In [13, 14] the author proved that $N \leq m$ and gave a necessary and sufficient condition for L to satisfy $N = m$. In [8, 9] N. I. Goloshchapova and L. L. Oridoroga proved that N is equal to the number of negative eigenvalues of a certain class of finite Jacobi matrix and gave independently a necessary and sufficient condition for L to satisfy $N = m$.

We consider the operator on $\mathbb{R}^2$ and $\mathbb{R}^3$, since in a multi-dimensional case the formulas for N are unknown. We prove that N is equal to the number of negative eigenvalues of a certain class of matrix M up to a constant, M being related to the Green function of L, and give some applications.

Let us denote by $L^{(d)}_{Y,\alpha}$ a Schrödinger operator acting in $L^2(\mathbb{R}^d)$ with point δ-interactions specified with the points of interactions $Y = \{y_1, y_2, \ldots, y_n\} \subset \mathbb{R}^d$ $(y_i \neq y_j$ if $i \neq j)$ and the intensities $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Here, $\alpha \subset \mathbb{R}$ for $d = 2, 3$ and $\alpha \subset (-\infty, +\infty] \setminus \{0\}$ for $d = 1$. It is well-known that $L^{(d)}_{Y,\alpha}$ are self-adjoint with domains characterized in Part II of [1] for each $d = 1, 2, 3$, respectively. Throughout this paper we assume that the domains of $L^{(d)}_{Y,\alpha}$ are these ones. Their spectra contain the positive semiaxis, where they are absolutely continuous, and $N(L^{(d)}_{Y,\alpha}) \leq n$. In this paper, we

2000 *Mathematics Subject Classification.* 47A10, 34L40.

Key words and phrases. Number of negative eigenvalues, point interactions, Schrödinger operators.

This work was supported by the Grant-in-Aid for Scientific Research (C) 20540204 from Japan Society for the Promotion of Science.

denote the number of negative eigenvalues of a self-adjoint operator T by $N(T)$ and that of an Hermitian matrix M by $N(M)$.

We first state our main theorem and a corollary. Let $m = |\{\alpha_i < 0; 1 \leq i \leq n\}|$, $d_{i,j} = |y_i - y_j|$ and

$$M^{(1)} = \left(-\frac{\delta_{i,j}}{\alpha_i} + \frac{d_{i,j}}{2}\right)_{i,j=1}^{n},$$
$$M^{(2)} = \left(\alpha_i\delta_{i,j} + \frac{\log d_{i,j}}{2\pi}(1-\delta_{i,j})\right)_{i,j=1}^{n},$$
$$M^{(3)} = \left(\alpha_i\delta_{i,j} - \frac{1-\delta_{i,j}}{4\pi d_{i,j}}\right)_{i,j=1}^{n},$$
$$P = \left(\delta_{i,j} - \frac{1}{n}\right)_{i,j=1}^{n}.$$

We denote the Kronecker delta by $\delta_{i,j}$ and adopt the convention that $c_{i,i}(1-\delta_{i,i}) = 0$ for any $c_{i,i}$.

Theorem 1. *The following hold.*

(i) $N(L^{(1)}_{Y,\alpha}) = N(PM^{(1)}P) + 1 + m - n.$

(ii) $N(L^{(2)}_{Y,\alpha}) = N(PM^{(2)}P) + 1.$

(iii) $N(L^{(3)}_{Y,\alpha}) = N(M^{(3)}).$

We prove this theorem in the following sections for each d. In each section, we study some properties of the discrete eigenvalues of $L^{(d)}_{Y,\alpha}$; we obtain sufficient and necessary conditions for $L^{(d)}_{Y,\alpha}$ to satisfy $N(L^{(d)}_{Y,\alpha}) = m, n, n$ for $d = 1, 2, 3$, respectively, and some bounds of $\overline{N}(L^{(d)}_{*,\alpha}) = \max\{N(L^{(d)}_{Y,\alpha}); Y\}$ and $\underline{N}(L^{(d)}_{*,\alpha}) = \min\{N(L^{(d)}_{Y,\alpha}); Y\}$. Here, $\overline{N}(L^{(d)}_{*,\alpha})$ (resp. $\underline{N}(L^{(d)}_{*,\alpha})$) stands for the maximum (resp. minimum) number of $N(L^{(d)}_{Y,\alpha})$ when n and α are fixed and the points of interactions Y are moved.

Corollary 2. *Fix α. Then the following hold.*

(i) $\overline{N}(L^{(1)}_{*,\alpha}) = m$ *and* $\underline{N}(L^{(1)}_{*,\alpha}) \leq 1$.

(ii) $\overline{N}(L^{(2)}_{*,\alpha}) = n$ *and* $\underline{N}(L^{(2)}_{*,\alpha}) = 1$.

(iii) $\overline{N}(L^{(3)}_{*,\alpha}) \geq m + 5q + \lfloor r/2 \rfloor$ *with* $q = \lfloor (n-m)/7 \rfloor$ *and* $r = n - m - 7q$, *and*

$$\underline{N}(L^{(3)}_{*,\alpha}) \leq \begin{cases} \lceil m/12 \rceil + 1, & \textit{if} \quad m = 7, 9, 10, 11 \pmod{12}, \\ \lceil m/12 \rceil, & \textit{otherwise.} \end{cases}$$

Here, $\lfloor x \rfloor$ is the largest integer not greater than x and $\lceil x \rceil$ is the smallest integer not less than x. The second half of (i) is optimal in the sense stated in Section 4. However, it is not known yet whether (iii) is also so. In the final section, we give some more applications and discussion.

We remark on the boundary conditions for $L^{(d)}_{Y,\alpha}$. Let $E_d(x)$ be a fundamental solution of Δ:

$$E_1(x) = -\frac{|x|}{2}, \quad E_2(x) = -\frac{1}{2\pi}\log|x|, \quad E_3(x) = \frac{1}{4\pi|x|}.$$

Then $L^{(d)}_{Y,\alpha}$ is defined on functions $\psi(x)$ that belong to $W^2_2(\mathbb{R}^d, X)$ and in the neighborhood of x_k have a singularity of $E_d(x)$, and we have

$$M^{(1)} = (-\delta_{i,j}/\alpha_i + \hat{E}_d(d_{i,j}))^n_{i,j=1}, \quad M^{(d)} = (\alpha_i\delta_{i,j} + \hat{E}_d(d_{i,j}))^n_{i,j=1}$$

for $d = 2, 3$ with $\hat{E}_d(0) = 0$ and $\hat{E}_d(x) = E_d(x)$ $(x \neq 0)$.

In the rest of this section, we give some notations and then mention the relation between $M^{(d)}$ and the Green function of $L^{(d)}_{Y,\alpha}$, that is, the kernel of its resolvent (see [1])

$$(1) \qquad (L^{(d)}_{Y,\alpha} - k^2)^{-1} = G^{(d)}_k + \sum_{j,j'=1}^{n} \left[\Gamma^{(d)}_{\alpha,Y}(k)\right]^{-1}_{j,j'} (\overline{G^{(d)}_k(\cdot - y_{j'}), \cdot}) G^{(d)}_k(\cdot - y_j)$$

for $k^2 \in \rho(L^{(d)}_{Y,\alpha})$ and $\Im k > 0$. Here, $G^{(d)}_k$ is the Green function of $-\Delta$ on $\mathbb{R}^d$ and $\Gamma^{(d)}_{\alpha,Y}(k)$ is an $n \times n$ matrix. In the following sections, we give explicit expressions of $\Gamma^{(d)}_{\alpha,Y}(k)$ and see that

$$M^{(3)} = \Gamma^{(3)}(0), \quad PM^{(d)}P = \lim_{\lambda \to 0+} P\Gamma^{(d)}_{\alpha,Y}(\sqrt{-1}\lambda)P \quad (d = 1, 2).$$

Our results are based on the following theorem.

Theorem 3. (Theorems 1.1.4, 2.1.3, 4.2 in Part II of [1]). *The number $-\lambda^2$, $\lambda > 0$, is an eigenvalue of $L^{(d)}_{Y,\alpha}$ if and only if $\det \Gamma^{(d)}_{\alpha,Y}(\sqrt{-1}\lambda) = 0$. In addition, the multiplicity of the eigenvalue $-\lambda^2$ is equal to the multiplicity of the eigenvalue zero of $\Gamma^{(d)}_{\alpha,Y}(\sqrt{-1}\lambda)$.*

For further exposition, we introduce the following notations. We denote by E_n the $n \times n$ identity matrix, by J_n the $n \times n$ all-one matrix and by $\mathbf{1}_n \in \mathbb{R}^n$ the all-one vector. Note that $\mathbf{1}_n$ is an eigenvector of J_n belonging to the eigenvalue n, and P is the orthogonal projection onto the orthogonal complement subspace of $\mathbf{1}_n$. This implies that $PJ_n = J_nP = 0$.

For convenience of the reader, we mention Weyl's perturbation and monotonicity theorems ([4, § III.2]); let A and B be Hermitian matrices and $\lambda_i(A)$ be the eigenvalues of A with decreasing order, $\lambda_i(A) \geq \lambda_{i+1}(A)$. Then it holds that $\max_i |\lambda_i(A) - \lambda_i(B)| \leq \|A - B\|$. If B is positive definite, then $\lambda_i(A + B) > \lambda_i(A)$ for all i.

The author would like to thank the referees for their many useful remarks and suggestions.

2. Three-dimensional case

In this section we write $L_{Y,\alpha} = L^{(3)}_{Y,\alpha}$, $N = N(L^{(3)}_{Y,\alpha})$, $\underline{N} = \underline{N}(L^{(3)}_{*,\alpha})$, $\overline{N} = \overline{N}(L^{(3)}_{*,\alpha})$ and $M = M^{(3)}$ for brevity. The matrix $\Gamma^{(3)}_{\alpha,Y}(k)$ in (1) is as follows:

$$\Gamma^{(3)}_{\alpha,Y}(k) = \left(\left(\alpha_i - \frac{\sqrt{-1}k}{4\pi} \right) \delta_{i,j} - \tilde{G}_k(d_{i,j}) \right)_{i,j=1}^{n}$$

with $\Im k > 0$, $\tilde{G}_k(x) = e^{\sqrt{-1}k|x|}/4\pi|x|$ $(x \neq 0)$ and $\tilde{G}_k(0) = 0$. See § II.1.1 of [1]. Assume that $\lambda \geq 0$ and let

$$\Gamma(\lambda) = \left(\left(\alpha_i + \frac{\lambda}{4\pi} \right) \delta_{i,j} - \frac{e^{-\lambda d_{i,j}}}{4\pi d_{i,j}} (1 - \delta_{i,j}) \right)_{i,j=1}^{n}$$

and $\mu_i(\lambda)$ $(1 \leq i \leq n)$ denote the eigenvalues of $\Gamma(\lambda)$. Note that $\Gamma(\lambda) = \Gamma^{(3)}_{\alpha,Y}(\sqrt{-1}\lambda)$ for $\lambda > 0$ and $M = \Gamma(0)$.

Proposition 4. *(i) All $\mu_i(\lambda)$ are continuous on $[0, \infty)$. (ii) All $\mu_i(\lambda)$ are monotone increasing on $(0, \infty)$. (iii) Each $\mu_i(\lambda)$ has at most one zero in $(0, \infty)$. (iv) $N(\Gamma(\lambda))$ is monotone non-increasing on $[0, \infty)$.*

Proof. [13, Proposition 1] implies (i). We can find (ii) in [1, Appendix F]. (iii) and (iv) immediately follow from (ii). □

The following is Theorem 1 (iii).

Theorem 5. *It holds that $N = N(M)$. In particular, $N \leq n$.*

Proof. Since $\lim_{\lambda\to\infty}\Gamma(\lambda)/\lambda = E_n/4\pi$, all $\mu_i(\lambda)$ are positive for sufficiently large λ. If $\mu_i(0) < 0$, then this $\mu_i(\lambda)$ has one zero. Since $N(M)$ eigenvalues $\mu_i(0)$ of M are negative, the function $\det\Gamma(\lambda)$ has $N(M)$ zeros, counting multiplicities. Thus, we obtain $N = N(M)$ by Theorem 3. □

Corollary 6. *Let $\tau = \|((1-\delta_{i,j})/4\pi d_{i,j})_{i,j=1}^n\|$.*

(i) If $m \geq 1$, then $N \geq 1$. If $m \leq n-1$, then $N \leq n-1$.

(ii) $N = n$ if and only if M is negative definite. In particular, if $N = n$, then $m = n$. If $\alpha_i < -\tau$ for all i, then $N = n$.

(iii) $N = 0$ if and only if M is non-negative definite. In particular, if $N = 0$, then $m = 0$. If $\alpha_i > \tau$ for all i, then $N = 0$.

Proof. Since $\langle e_i, Me_i\rangle = \alpha_i$ for the i-th unit vector e_i in $\mathbb{C}^n$, if $\alpha_i < 0$, then $N(M) \geq 1$ and if $\alpha_i > 0$, then $N(M) \leq n-1$. If $m \leq n-1$ and no α_i is positive, then some $\alpha_i = 0$. In this case, we can see $\langle e_i - \epsilon e_j, M(e_i - \epsilon e_j)\rangle = \epsilon/2\pi d_{i,j} + \epsilon^2\alpha_j > 0$ for $j \neq i$ and sufficiently small $\epsilon > 0$. This implies $N(M) \leq n-1$. Thus, we obtain (i).

Consider (ii). M is negative definite if and only if $N(M) = n$. Thus, the first assertion holds. The second follows from (i). The assumption of the third implies that M is negative definite. We can prove (iii) in a way similar to that above. □

Example 7. Let $n = 2$ and $c = \det M$. Then the following hold:

$$N = \begin{cases} 2, & \text{if} \quad m = 2 \text{ and } c > 0, \\ 0, & \text{if} \quad m = 0 \text{ and } c > 0, \\ 1, & \text{otherwise.} \end{cases}$$

Let $n = 3$ or 4 and assume that $\alpha_i = \alpha$ and $|y_i - y_j| = w$ $(i \neq j)$. Then the distinct eigenvalues of M are $\alpha - (n-1)/4\pi w$ and $\alpha + 1/4\pi w$, and their multiplicities are 1 and $n-1$, respectively. Thus, we have

$$N = \begin{cases} n, & \text{if} \quad \alpha < -1/4\pi w, \\ 1, & \text{if} \quad -1/4\pi w \leq \alpha < (n-1)/4\pi w, \\ 0, & \text{otherwise.} \end{cases}$$

The following theorem shows that if the distance of some pair of interactions is sufficiently small, then $L_{Y,\alpha}$ has at least one negative eigenvalue.

Theorem 8. *Assume that the distance $d_{1,2}$ of a pair $\{y_1, y_2\}$ of points of interaction with intensities $\{\alpha_1, \alpha_2\}$ is sufficiently small and that the pair lie far enough from the other points of interactions $Y' = \{y_i; i = 3, 4, \ldots, n\}$ with the intensities $\alpha' = \{\alpha_i; i = 3, 4, \ldots, n\}$. Then it holds that*

$$N(L_{Y,\alpha}) \geq N(L_{Y',\alpha'}) + 1. \tag{2}$$

Proof. Assume that $d_{1,2} < 1/4\pi\sqrt{|\alpha_1\alpha_2|}$ and let

$$M_1 = \begin{pmatrix} \alpha_1 & -\frac{1}{4\pi d_{1,2}} \\ -\frac{1}{4\pi d_{1,2}} & \alpha_2 \end{pmatrix}, \quad M_2 = \left(\alpha_i\delta_{i,j} - \frac{1-\delta_{i,j}}{4\pi d_{i,j}}\right)_{i,j=3}^n.$$

Then $N(M_1) = 1$, $N(M_2) = N(L_{Y',\alpha'})$ and $\|M - M_1 \oplus M_2\| \leq nK$ with

$$K = \max\{1/4\pi d_{i,j}; i = 1, 2, \ j = 3, 4, \ldots, n\}$$

. Since we can assume that

$$nK < \min\{|\lambda|; \lambda \text{ is a negative eigenvalue of } M_1 \oplus M_2\},$$

we obtain $N(M) \geq N(M_1) + N(M_2)$. Therefore, we obtain (2). □

Corollary 9. *$\overline{N} \geq \lfloor (n+m)/2 \rfloor$ and $\underline{N} \leq \lceil m/4 \rceil$.*

Proof. Consider the first assertion. We can assume that $\alpha_i > 0$ for $i \leq n-m$ without loss of generality. Assume that all of the distances $d_{2i-1,2i}$ of the pairs $\{y_{2i-1}, y_{2i}\}$ with $i \leq \lfloor (n-m)/2 \rfloor$ are sufficiently small and that these pairs and the other points y_i $(i > 2\lfloor (n-m)/2 \rfloor)$ lie far enough from each other. Then we can see $N(M) \geq \lfloor (n-m)/2 \rfloor + m = \lfloor (n+m)/2 \rfloor$ in a similar fashion in the proof of Theorem 8. Thus, we obtain $\overline{N} \geq \lfloor (n+m)/2 \rfloor$.

Consider the second assertion. In this proof, we denote by $M(Y')$ the corresponding matrix for $L_{Y',\alpha'}$ with a subset $Y' \subset Y$ and the corresponding intensities $\alpha' \subset \alpha$. We put $N(M(\emptyset)) = 0$.

We can assume that $\alpha_i < 0$ for $i \leq m$ without loss of generality. Divide Y into the subsets $Y_i = \{y_j; 4i-3 \leq j \leq 4i,\ 1 \leq j \leq m\}$ with $1 \leq i \leq \lceil m/4 \rceil$ and $Y_+ = \{y_i; m+1 \leq i \leq n\}$. Assume that the distances $|y_j - y_{j'}|$ are constant for any $y_j, y_{j'} \in Y_i$ $(y_j \neq y_{j'})$ and sufficiently small. Then $N(M(Y_i)) = 1$ for $1 \leq i \leq \lceil m/4 \rceil$. (cf. Example 7.) In addition, we assume that all of the subsets Y_i and the points of Y_+ lie far enough from each other. Then we can see that $N(M(Y_+)) = 0$ and $N(M(Y)) = \sum_{i=1}^{\lceil m/4 \rceil} N(M(Y_i)) = \lceil m/4 \rceil$. Thus, we obtain $\underline{N} \leq \lceil m/4 \rceil$. □

Example 10. Let $2 \leq n \leq 4$ and fix α with $m = 0$. Let $M_n = (a_{i,j})_{i,j=1}^n$ with $a_{i,i} = 0$ and $a_{i,j} = a_{j,i} < 0$. Then we can easily check that $1 \leq N(M_n) \leq \lfloor n/2 \rfloor$ for any $a_{i,j}$. This implies that $N(M) \leq \lfloor n/2 \rfloor$ and $N(M) = \lfloor n/2 \rfloor$ for some $d_{i,j}$. Thus, we have $\overline{N} = \lfloor n/2 \rfloor$ if $2 \leq n \leq 4$ and $m = 0$.

Remark 11. One of the referees suggested the author M. E. Dudkin's papers [5, 6]. He studied $N(L_{Y,\alpha})$ in cases in which point interactions lie on the vertices of regular n-gons and regular polyhedra with an equal coupling constant α. By his results, we can see that if Y is a regular n-gons $(n = 3, 4, 5, 6, 8)$ or a regular icosahedron $(n = 12)$ and negative α is sufficiently small, it holds that $N(L_{Y,\alpha}) = 1$. Therefore, we can improve the estimate of $\underline{N}$

$$\underline{N} \leq \begin{cases} \lceil m/12 \rceil + 1, & \text{if} \quad m = 7, 9, 10, 11 \ (\mathrm{mod}\ 12), \\ \lceil m/12 \rceil, & \text{otherwise}. \end{cases}$$

Similarly, if Y is a regular 7-gons and positive α is sufficiently small, then $N(L_{Y,\alpha}) = 5$. Therefore, we can improve the estimate of $\overline{N}$

$$\overline{N} \geq m + 5q + \lfloor r/2 \rfloor \quad \text{with} \quad q = \lfloor (n-m)/7 \rfloor, \quad r = n - m - 7q.$$

3. Two-dimensional case

In this section, we write $L_{Y,\alpha} = L^{(2)}_{Y,\alpha}$, $N = N(L^{(2)}_{Y,\alpha})$, $\underline{N} = \underline{N}(L^{(2)}_{*,\alpha})$, $\overline{N} = \overline{N}(L^{(2)}_{*,\alpha})$ and $M = M^{(2)}$ for brevity. The matrix $\Gamma^{(2)}_{\alpha,Y}(k)$ in (1) is as follows:

$$\Gamma^{(2)}_{\alpha,Y}(k) = \left(\frac{2\pi\alpha_i - \Psi(1) + \log(k/2\sqrt{-1})}{2\pi} \delta_{i,j} - \tilde{G}_k(d_{i,j}) \right)_{i,j=1}^n$$

with $\Im k > 0$, $\tilde{G}_k(x) = (\sqrt{-1}/4) H_0^{(1)}(k|x|)$ $(x \neq 0)$ and $\tilde{G}_k(0) = 0$. Here, $H_0^{(1)}(x)$ is the first kind of Hankel function with order 0 and the number $-\Psi(1)$ is the Euler constant. See Chapter II.4 in [1]. Assume that $\lambda > 0$ and let

$$\begin{aligned} \Gamma(\lambda) &= \Gamma^{(2)}_{\alpha,Y}(\sqrt{-1}\lambda) \\ &= \left(\frac{2\pi\alpha_i - \Psi(1) + \log(\lambda/2)}{2\pi} \delta_{i,j} - \frac{1}{2\pi} K_0(d_{i,j}\lambda)(1 - \delta_{i,j}) \right)_{i,j=1}^n \end{aligned}$$

and $\mu_i(\lambda)$ $(1 \le i \le n)$ denote the eigenvalues of $\Gamma(\lambda)$. Here, we use the fact that $H_0^{(1)}(\sqrt{-1}x) = -(2\sqrt{-1}/\pi)K_0(x)$ for $x > 0$, where $K_0(x)$ is the third kind of Bessel function with order 0.

Proposition 12. *(i) All $\mu_i(\lambda)$ are continuous on $(0,\infty)$. (ii) All $\mu_i(\lambda)$ are monotone increasing on $(0,\infty)$. (iii) Each $\mu_i(\lambda)$ has at most one zero in $(0,\infty)$. (iv) $N(\Gamma(\lambda))$ is monotone non-increasing on $(0,\infty)$.*

Proof. [10, Theorem 6.8] implies (i). We can find (ii) in [1, Appendix F]. (iii) and (iv) immediately follow from (ii). □

Since $K_0(x) \sim -\log x$ $(x \to 0+)$ and $\lim_{x\to\infty} K_0(x) = 0$, we can see that

$$\lim_{\lambda\to 0+} \frac{1}{\log\lambda}\Gamma(\lambda) = \frac{1}{2\pi}J_n, \quad \lim_{\lambda\to\infty} \frac{1}{\log\lambda}\Gamma(\lambda) = \frac{1}{2\pi}E_n. \tag{3}$$

Using (3) we can obtain the known result, $1 \le N \le n$, as in the proof of Theorem II.4.2 in [1]; the eigenvalues of J_n are n (with multiplicity 1) and 0 (with multiplicity $n-1$). Thus, at least one $\mu_i(\lambda)$ is negative for sufficiently small λ. Since all $\mu_i(\lambda)$ are positive for sufficiently large λ, $\det\Gamma(\lambda)$ has at least one zero. In addition Proposition 12 implies that $\det\Gamma(\lambda)$ has at most n zeros. Thus, we have $1 \le N \le n$ by Theorem 3.

Using (3) we obtain the following key lemma.

Lemma 13. *(i) It holds that $N = N(\Gamma(\lambda_0))$ for some λ_0. In addition, $N = N(\Gamma(\lambda))$ for any $\lambda \le \lambda_0$; (ii) $N = n$ if and only if $\Gamma(\lambda_0)$ is negative definite for some λ_0.*

Proof. Let λ_i be the zero of $\mu_i(\lambda)$ if it exists, and $\lambda_0 < \min\{\lambda_i, 1\}$. Put $k = N(\Gamma(\lambda_0))$. Then the number of negative $\mu_i(\lambda_0)$ is equal to k. Thus, each of these k negative $\mu_i(\lambda)$ has one zero, because all $\mu_i(\lambda)$ are positive for sufficiently large λ. Therefore, $\det\Gamma(\lambda)$ has k zeros, counting multiplicities. Thus, we obtain $N = k$ by Theorem 3. (ii) immediately follows from (i). □

Since $\Gamma(\lambda)$ diverges as $\lambda \to 0+$, it is difficult to determine the quantity $N(\Gamma(\lambda_0))$. Fortunately, $P\Gamma(\lambda)P$ converges as $\lambda \to 0+$.

Proposition 14. *It holds that $\lim_{\lambda\to 0+} P\Gamma(\lambda)P = PMP$.*

Proof. Since $K_0(\lambda) = 2\pi\, r(\lambda) + O(\lambda^2)$ with $r(\lambda) = (-\log(\lambda/2) + \Psi(1))/2\pi$ (cf. [7, § 7.2.5]), we have $\Gamma(\lambda) = M - r(\lambda)J_n + O(\lambda^2)$. Thus, we have $P\Gamma(\lambda)P = PMP + O(\lambda^2)$. □

The following is Theorem 1 (ii).

Theorem 15. *It holds that $N = N(PMP) + 1$.*

In the rest of this section, we write $q = N(PMP)$ for brevity.

Proof. We first prove that $N(\Gamma(\lambda)) \le q + 1$. Since $d\Gamma(\lambda)/d\lambda$ is positive definite by [1, Appendix F], $dP\Gamma(\lambda)P/d\lambda$ is non-negative definite. This implies that $N(P\Gamma(\lambda)P)$ is monotone non-increasing. Thus, $N(P\Gamma(\lambda)P) \le q$. Since $P\Gamma(\lambda)P$ is a compression of $\Gamma(\lambda)$ to $(n-1)$-dimensional subspace, we obtain $N(\Gamma(\lambda)) \le q+1$ by Cauchy's interlacing theorem [4, Corollary III.1.5].

Let us prove that $N(\Gamma(\lambda_0)) \ge q + 1$ for some λ_0. We put $I(\lambda; f) = \langle\Gamma(\lambda)f, f\rangle$ and $\xi = \mathbf{1}_n/\sqrt{n}$ and let $\varphi_1, \varphi_2, \ldots, \varphi_q$ be linearly independent eigenvectors of PMP belonging to negative eigenvalues. We examine the existence of λ_0 such that $I(\lambda_0; \psi) < 0$ for any linear combination ψ of ξ and φ_i. If $q = 0$, then this is trivial. Thus, we assume that $q \ge 1$.

We denote by $-\nu$ the largest negative eigenvalue of PMP. Let φ be a normalized linear combination of φ_i. Since there exists λ_1 such that $\|PMP - P\Gamma(\lambda)P\| < \nu/2$ for any $\lambda < \lambda_1$, we have

$$\begin{aligned} I(\lambda;\varphi) &= \langle PMP\varphi, \varphi\rangle + \langle (P\Gamma(\lambda)P - PMP)\varphi, \varphi\rangle \\ &\le -\nu + \|P\Gamma(\lambda)P - PMP\| \le -\nu/2 \end{aligned}$$

for any $\lambda \le \lambda_1$. Let $(a,b) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $\lambda \le \lambda_1$. We have

$$\begin{aligned} I(\lambda; a\varphi + b\xi) &= |a|^2 I(\lambda;\varphi) + \langle \Gamma(\lambda)a\varphi, b\xi\rangle + \langle \Gamma(\lambda)b\xi, a\varphi\rangle + |b|^2 I(\lambda;\xi) \\ &\le -(\nu/2)|a|^2 + 2|ab|\|M\| + |b|^2 I(\lambda;\xi) \\ &= -(\nu/2)(|a| - 2|b|\|M\|/\nu)^2 + |b|^2(2\|M\|^2/\nu + I(\lambda;\xi)). \end{aligned}$$

The first term is non-positive. Since $I(\lambda;\xi) \to -\infty$ as $\lambda \to 0$, the second term is negative for some $\lambda_0 \le \lambda_1$. Since both $2\|M\|^2/\nu$ and $I(\lambda;\xi)$ are independent of a, b, φ and λ_1, so is λ_0. Therefore, $N(\Gamma(\lambda_0)) \ge q + 1$. This completes the proof. $\square$

Corollary 16. *$\underline{N} = 1$ and $\overline{N} = n$.*

Proof. Let $w > 0$ and put $y_j = (jw, 0) \in \mathbb{R}^2$. Since $d_{i,j} = |i-j|w$, we have

$$M = Q + \frac{\log w}{2\pi}(J_n - E_n) \quad \text{with} \quad Q = \left(\alpha_i \delta_{i,j} + \frac{\log|i-j|}{2\pi}(1 - \delta_{i,j})\right),$$

and thus $PMP = PQP - (\log w/2\pi)P$. Therefore, if w is sufficiently large, then $q = n-1$, and thus $N = n$. Similarly, if w is sufficiently small, then $q = 0$, and thus $N = 1$. Consequently, $\underline{N} = 1$ and $\overline{N} = n$. $\square$

Example 17. Let $n = 2$. We have

$$PMP = cP \quad \text{with} \quad c = \frac{\alpha_1 + \alpha_2}{2} - \frac{\log d_{1,2}}{2\pi}.$$

Thus, if $c < 0$, then $q = 1$, else $q = 0$. Hence, if $c < 0$, then $N = 2$, else $N = 1$. Let $n = 3$ and assume that $\alpha_i = \alpha$ and $|y_i - y_j| = w$ $(i \ne j)$. Then we have

$$PMP = cP \quad \text{with} \quad c = \alpha - \frac{\log w}{2\pi}.$$

Thus, if $c < 0$, then $q = 2$, else $q = 0$. Hence, if $c < 0$, then $N = 3$, else $N = 1$.

4. One-dimensional case revisited

In this section, we write $L_{Y,\alpha} = L^{(1)}_{Y,\alpha}$, $N = N(L^{(1)}_{Y,\alpha})$, $\underline{N} = \underline{N}(L^{(1)}_{*,\alpha})$, $\overline{N} = \overline{N}(L^{(1)}_{*,\alpha})$ and $M = M^{(1)}$ for brevity. Though we already know some sufficient and necessary conditions for $L_{Y,\alpha}$ to satisfy $N = m$ and algorithms for determining N, it seems to be worthwhile to discuss $L_{Y,\alpha}$ using the method proposed in this paper. Since our discussion on $L_{Y,\alpha} = L^{(1)}_{Y,\alpha}$ is very similar to that on $L^{(2)}_{Y,\alpha}$, we give only an outline.

The matrix $\Gamma^{(1)}_{\alpha,Y}(k)$ in (1) is as follows:

$$\Gamma^{(1)}_{\alpha,Y}(k) = -\left(\frac{\delta_{i,j}}{\alpha_i} + \frac{\sqrt{-1}}{2k} e^{\sqrt{-1}k d_{i,j}}\right)_{i,j=1}^{n}$$

with $\Im k > 0$. See § II.2.1 in [1]. Assume that $\lambda > 0$ and let

$$\Gamma(\lambda) = \Gamma^{(1)}_{\alpha,Y}(\sqrt{-1}\lambda) = -\left(\frac{\delta_{i,j}}{\alpha_i} + \frac{e^{-\lambda d_{i,j}}}{2\lambda}\right)_{i,j=1}^{n}$$

and $\mu_i(\lambda)$ $(1 \le i \le n)$ denote the eigenvalues of $\Gamma(\lambda)$.

Proposition 18. *(i) All $\mu_i(\lambda)$ are continuous on $(0,\infty)$. (ii) All $\mu_i(\lambda)$ are monotone increasing on $(0,\infty)$. (iii) Each $\mu_i(\lambda)$ has at most one zero in $(0,\infty)$. (iv) $N(\Gamma(\lambda))$ is monotone non-increasing on $(0,\infty)$.*
(v) $\lim_{\lambda\to\infty}\mu_i(\lambda) = -1/\alpha_i$ for suitable numbering of $\mu_i(\lambda)$.

Proof. [10, Theorem 6.8] implies (i). We can find (ii) in [1, Appendix F]. (iii) and (iv) immediately follow from (ii). Since $\lim_{\lambda\to\infty}\Gamma(\lambda) = (-\delta_{i,j}/\alpha_i)_{i,j=1}^n$, we obtain (v) by [13, Proposition 3]. □

Lemma 19. *(i) It holds that $N = N(\Gamma(\lambda_0)) + m - n$ for some λ_0. In addition, $N = N(\Gamma(\lambda)) + m - n$ for any $\lambda \le \lambda_0$. In particular, $N \le m$. (ii) $N = m$ if and only if $\Gamma(\lambda_0)$ is negative definite for some $\lambda_0 > 0$.*

Proof. Let λ_i be the zero of $\mu_i(\lambda)$ if it exists, and $\lambda_0 < \min\{\lambda_i, 1\}$. Put $k = N(\Gamma(\lambda_0))$. Then the number of non-negative $\mu_i(\lambda_0)$ is equal to $n-k$. By Proposition 18, the number of positive $\mu_i(\lambda)$ is equal to m for sufficiently large λ. Thus, $m-(n-k)$ eigenvalues $\mu_i(\lambda)$ has one zero. Therefore, $\det\Gamma(\lambda)$ has $k+m-n$ zeros. Thus, we obtain $N = k+m-n$ by Theorem 3. (ii) immediately follows from (i). □

We remark that the author proved that "$N = m$ if and only if $M(\lambda_0)$ is *positive* definite for some $\lambda_0 > 0$" in previous papers [13, 14]. The matrix $M(\lambda)$ in these papers is equal to $-\lambda^n\Gamma(\lambda)$.

The following is Theorem 1 (i).

Theorem 20. *It holds that $N = N(PMP) + 1 + m - n$.*

Proof. Using the fact $\Gamma(\lambda) = M-(1/2\pi)J_n+O(\lambda)$, we can prove $N(\Gamma(\lambda_0)) = N(PMP)+1$ for sufficiently small λ_0 in the same fashion as that in the proof of Theorem 15. Therefore Lemma 19 implies the desired result. □

Corollary 21. *$\overline{N} = m$ and $\underline{N} \le 1$.*

Proof. We already know that $N \le m$ and that $N = m$ if all of the distances $d_{i,j}$ are sufficiently large by [13, Theorem 2]. Thus, $\overline{N} = m$. We assume $m > 0$ and examine $\underline{N}$. Assume that $d_{i,j}$ are small enough. Then $N(PMP) = N(-PA^{-1}P)$ with $A = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Since $N(-A^{-1}) = n-m$ and $-PA^{-1}P$ is a compression of $-A^{-1}$ to an $(n-1)$-dimensional subspace, we obtain $N(-PA^{-1}P) \le n-m$ by Cauchy's interlacing theorem. Therefore, Theorem 20 implies $N \le 1$. Thus, $\underline{N} \le 1$. □

We cannot write $\underline{N}$ as any expression of n and m as below. In this sense, the estimate of $\underline{N}$ is optimal; consider the case where $n = 2$ and $m = 1$ with $\alpha_1 > 0$ and $\alpha_2 < 0$. If $\alpha_1 > -\alpha_2$, then $\underline{N} = 0$, otherwise $\underline{N} = 1$. We can easily check this example by Albeverio-Nizhnik's algorithm.

Remark 22. As mentioned in Introduction, N. I. Goloshchapova and L. L. Oridoroga proved that N is equal to the number of negative eigenvalues of a certain class of finite Jacobi matrix in [8, 9]. Their arguments are based on the concept of boundary triplets and the corresponding Weyl functions developed in [11], in which different spectral properties of 1-dimensional Schrödinger operators with an infinite number of δ and δ' interactions have been investigated.

5. Discussion

In this section, we give two applications and discuss δ'-interactions and how to determine $N(M)$. Let $\Gamma^{(d)}(\lambda; Y, \alpha)$ denote the matrix $\Gamma(\lambda)$ for $L^{(d)}_{Y,\alpha}$.

The first application shows a kind of dilatation property of $L_{Y,\alpha}$.

Theorem 23. *Let $\theta > 0$. We put $Y' = \{\theta y_1, \theta y_2, \ldots, \theta y_n\}$ and*

$$\alpha' = \begin{cases} \{\alpha_1/\theta, \alpha_2/\theta, \ldots, \alpha_n/\theta\}, & \text{if} \quad d = 1, 3, \\ \{\alpha_1 + \frac{\log\theta}{2\pi}, \alpha_2 + \frac{\log\theta}{2\pi}, \ldots, \alpha_n + \frac{\log\theta}{2\pi}\}, & \text{if} \quad d = 2. \end{cases}$$

Then the following hold for all $d = 1, 2, 3$:

$$N(L^{(d)}_{Y',\alpha'}) = N(L^{(d)}_{Y,\alpha}) \quad \text{and} \quad \sigma_{\mathrm{d}}(L^{(d)}_{Y',\alpha'}) = \frac{1}{\theta}\sigma_{\mathrm{d}}(L^{(d)}_{Y,\alpha}).$$

Proof. The first assertion immediately follows from Theorem 1. The second follows from the fact that $\Gamma^{(d)}(\lambda; Y', \alpha') = \theta^{2-d}\Gamma^{(d)}(\theta\lambda; Y, \alpha)$. □

The second application is an extension of Proposition II.1.1.5 in [1].

Theorem 24. *Let $1 \le d \le 3$ and fix Y. Let $N(-\lambda^2, \alpha)$ denote the number of eigenvalues (counting multiplicities) of $L^{(d)}_{Y,\alpha}$ less than or equal to $-\lambda^2$. Assume that $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_n\}$ satisfy that $\alpha_i \le \beta_i$ for all i. Then $N(-\lambda^2, \alpha) \ge N(-\lambda^2, \beta)$.*

Proof. We can see that $N(-\lambda^2, \alpha)$ is equal to the number of zeros of $\det\Gamma(\lambda; Y, \alpha)$ on $[\lambda, \infty)$. Thus, $N(-\lambda^2, \alpha) = N(\Gamma(\lambda - 0; Y, \alpha))$. Let $\mu_i(\lambda)$ and $\nu_i(\lambda)$ be the eigenvalues of $\Gamma(\lambda; Y, \alpha)$ and $\Gamma(\lambda; Y, \beta)$ with deceasing order, respectively. Since $\Gamma(\lambda; Y, \beta) - \Gamma(\lambda; Y, \alpha) = ((\beta_i - \alpha_i)\delta_{i,j})^n_{i,j=1}$ is non-negative definite, it holds that $\mu_i(\lambda) \le \nu_i(\lambda)$. Thus, $N(\Gamma(\lambda; Y, \alpha)) \ge N(\Gamma(\lambda; Y, \beta))$. This implies the desired result. □

Let us examine $N(\tilde{L})$ of a one-dimensional Schrödineger operator $\tilde{L} = \tilde{L}_{Y,\beta}$ with finitely many points of δ'-interactions; here, we use the same definition in [1, 8, 9]. Let $\tilde{m} = |\{\beta_j < 0; 1 \le j \le n\}|$.

Theorem 25 (Goloshchapova-Oridoroga [8, 9])**.** *We have $N(\tilde{L}) = \tilde{m}$.*

We can prove this theorem using our method; let

$$\tilde{\Gamma}(\lambda) = \tilde{\Gamma}_{\beta,Y}(\sqrt{-1}\lambda) = [(\beta_j\lambda^2)^{-1}\delta_{i,j} + e^{-\lambda d_{i,j}}/2\lambda]^n_{i,j=1}.$$

We already know that $-\lambda^2$, $\lambda > 0$, is an eigenvalue of $\tilde{L}$ if and only if $\det\tilde{\Gamma}(\lambda) = 0$ [1]. Since $\tilde{\Gamma}(\lambda)$ is positive definite for sufficiently large λ, $N(\tilde{\Gamma}(\lambda)) = \tilde{m}$ for sufficiently small λ, and all eigenvalues of $\lambda^2\tilde{\Gamma}(\lambda)$ are monotone increasing (We can prove that $d(\lambda^2\tilde{\Gamma}(\lambda))/d\lambda$ is positive definite by tedious calculation.), $\det\tilde{\Gamma}(\lambda)$ always has $\tilde{m}$ zeros, counting multiplicities. This implies Theorem 25. In addition, we can see the dilatation property, too; $\sigma_{\mathrm{d}}(\tilde{L}_{Y',\beta'}) = (1/\theta)\sigma_{\mathrm{d}}(\tilde{L}_{Y,\beta})$ with $Y' = \{\theta y_1, \theta y_2, \ldots, \theta y_n\}$ and $\beta' = \{\theta\beta_1, \theta\beta_2, \ldots, \theta\beta_n\}$.

We comment on how to determine $N(M)$. We can find an algorithm for doing it in elementary linear algebra [15]; let D_k be the leading principal minors of M with order k. The number of sign changes of the sequence, $(1, D_1, D_2, \ldots, D_n)$, is equal to $N(M)$. However, this algorithm is ineffective for our purpose, because it is tedious to calculate D_k. On the other hand, S. Albeverio and L. Nizhnik gave an effective algorithm for determining $N(L^{(1)}_{Y,\alpha})$ in [2]. We must investigate more effective ones for determining $N(L^{(2)}_{Y,\alpha})$ and $N(L^{(3)}_{Y,\alpha})$.

Remark 26. One of the referees informed the author that [12] give a complete description of the number of negative eigenvalues of one dimensional Schröodinger operator with infinite number of δ'-interactions.

References

1. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, Berlin—New York, 1988.
2. S. Albeverio and L. Nizhnik, *On the number of negative eigenvalues of a one-dimensional Schrödinger operator with point interactions*, Letters Math. Phys. **65** (2003), no. 1, 27–35.
3. S. Albeverio and L. Nizhnik, *Schrödinger operators with a number of negative eigenvalues equal to the number of point interactions*, Methods Funct. Anal. Topology **9** (2003), no. 4, 273–286.
4. R. Bhatia, *Matrix Analysis*, Springer-Verlag, Berlin—New York, 1996.
5. M. E. Dudkin, *On the point spectrum of the Laplace operator with δ-potentials at the vertices of regular polyhedra*, Nonlinear Oscillations **7** (2004), no. 2, 145–153.
6. M. E. Dudkin, *Point spectrum of the Schrödinger operator with point-interactions at the vertices of regular n-gons*, Ukrainian Math. J. **56** (2004), no. 8, 1343–1352.
7. A. Erdélyi (ed.), *Higher Transcendental Functions*, vol. II, McGraw-Hill, Berlin—New York, 1953.
8. N. I. Goloshchapova and L. L. Oridoroga, *The one-dimensional Schrödinger operator with point δ- and δ'-interactions*, Math. Notes **84** (2008), no. 1, 125–129.
9. N. I. Goloshchapova and L. L. Oridoroga, *On the number of negative squares of one-dimensional Schrödinger operators with point interactions*, Integr. Equ. Oper. Theory **67** (2010), no. 1, 1–14.
10. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin—New York, 1966.
11. A. Kostenko and M. M. Malamud, *1D Schrödinger operators with local interactions on a discrete set*, J. Diff. Equat. **249** (2010), no. 2, 253–304.
12. A. Kostenko and M. M. Malamud, *Schrödinger operators with δ'-interactions and the Krein-Stieltjes string*, Doklady Mathematics **81** (2010), no. 3, 342–347.
13. O. Ogurisu, *On the number of negative eigenvalues of a Schrödinger operator with point interactions*, Letters Math. Phys. **85** (2008), no. 2–3, 129–133.
14. O. Ogurisu, *On the number of negative eigenvalues of a Schrödinger operator with δ interactions*, Methods Funct. Anal. Topology **16** (2010), 42–50.
15. I. Satake, *Linear Algebra*, Marcel Dekker, New York, 1975.

Division of Mathematical and Physical Sciences, Kanazawa University, Kanazawa, Ishikawa 920-1192, Japan
E-mail address: ogurisu@kanazawa-u.ac.jp

Received 20/10/2009; Revised 09/07/2010