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ON J-SELF-ADJOINT EXTENSIONS OF THE PHILLIPS SYMMETRIC OPERATOR

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Dedicated to the blessed memory of I. Gohberg.

Abstract. J-self-adjoint extensions of the Phillips symmetric operator S are studied. The concepts of stable and unstable C-symmetry are introduced in the extension theory framework. The main results are the following: if A is a J-self-adjoint extension of S, then either $\sigma(A) = \mathbb{R}$ or $\sigma(A) = \mathbb{C}$; if A has a real spectrum, then A has a stable C-symmetry and A is similar to a self-adjoint operator; there are no J-self-adjoint extensions of the Phillips operator with unstable C-symmetry.

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and with fundamental symmetry J (i.e., $J = J^*$ and $J^2 = I$). The space $\mathfrak{H}$ endowed with the indefinite inner product (indefinite metric) $[x,y]_J := (Jx,y)$, $\forall x,y \in \mathfrak{H}$ is called a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$.

An operator A in $\mathfrak{H}$ is called J-*self-adjoint* if A is self-adjoint with respect to the indefinite metric $[\cdot,\cdot]_J$. It is clear that A is J-self-adjoint if and only if

$$A^*J = JA. \tag{1.1}$$

During the past ten years a steady interest in the study of J-self-adjoint operators has been strongly increased by the necessity of mathematically correct and rigorous analysis of pseudo-Hermitian Hamiltonians arising in $\mathcal{PT}$-symmetric quantum mechanics (PTQM) see e.g. [10]–[19], [32, 35, 38].

In many cases, pseudo-Hermitian Hamiltonians admit the representation $A+V$, where a (fixed) self-adjoint operator A and a non-symmetric potential V satisfy certain (Krein space) symmetry properties which allow one to formalize the expression $A+V$ as a family of J-self-adjoint[1] operators A_ε acting in a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$. Here $\varepsilon \in \mathbb{C}^m$ is a complex parameter characterizing the potential V.

Let Ξ be the domain of variation of ε. One of important problems for the collection $\{A_\varepsilon\}$, which is directly inspirited by PTQM, is the description of quantitative and qualitative changes of spectra $\sigma(A_\varepsilon)$ when ε runs Ξ. Nowadays this topic has been analyzed with a wealth of technical tools (see, e.g., [7, 8, 21, 24, 39]).

In particular, if the potential V is singular, then operators A_ε turn out to be J-self-adjoint extensions of the *symmetric* operator $S = A \upharpoonright \ker V$ which *commutes* with J and spectral analysis of A_ε can be carried out by the extension theory methods [2, 3, 4, 22]. Here, the 'main ingredients' are: a holomorphic operator function characterizing S (the characteristic function $\Theta(\cdot)$ [26, 28, 37] or the Weyl function $M(\cdot)$ [16, 17, 18]) and the boundary conditions which distinguish A_ε among other J-self-adjoint extensions of S. In

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[1] Under a special choice of involution J.

such a setting, the spectral analysis of A_ε is reduced to the routine solution of algebraic equations including $\theta(\cdot)$ and boundary conditions.

In the present paper we are going to study a special case where the characteristic function of a symmetric operator S with finite deficiency indices is equal to zero ($\Theta(\mu) \equiv 0$, $\forall\mu \in \mathbb{C} \setminus \mathbb{R}$).

One of general constructions leading to symmetric operators S with the zero characteristic function is the following: let U be a bilateral shift with a wandering subspace W_0 in $\mathfrak{H}$ (see [20] for the terminology) and let V be its restriction onto $\mathfrak{H} \ominus W_0$, i.e., $V = U \upharpoonright (\mathfrak{H} \ominus W_0)$. Then the operator

$$S = i(V + I)(V - I)^{-1}, \quad \mathcal{D}(S) = \mathcal{R}(V - I) \tag{1.2}$$

is simple[2] symmetric and its deficiency induces coincide with $< \dim W_0, \dim W_0 >$.

In other words, S is the restriction of the Cayley transform of U

$$A = i(U + I)(U - I)^{-1}, \quad \mathcal{D}(A) = \mathcal{R}(U - I) \tag{1.3}$$

onto $\mathcal{D}(S) = \mathcal{R}(V - I)$.

The operator S defined by (1.2), (1.3) was used by Phillips [36] (with $\dim W_0 = 1$) as an example of the symmetric operator, which is invariant with respect to a certain set $\mathfrak{U}$ of unitary operators ($\mathfrak{U}$-invariant) but it has no $\mathfrak{U}$-invariant self-adjoint extensions. For this reason, the simple symmetric operator S determined by (1.2) and (1.3) will be referred as the *Phillips symmetric operator*.

Due to specific properties of the Phillips operator (the characteristic function is zero, there are no real points of regular type of S, etc) we obtain an evolution of $\sigma(A_\varepsilon)$ which differs from the matrix models [21, 24, 25] and models based on J-self-adjoint (symmetric) perturbations of the Schrödinger or Dirac operator [5, 15, 32, 39]. For instance, in our case, either the spectrum of an J-self-adjoint extension A_ε of S coincides with real line: $\sigma(A_\varepsilon) = \mathbb{R}$ or with complex plane: $\sigma(A_\varepsilon) = \mathbb{C}$ (Theorem 3.7).

One of the key points in PTQM is the description of a hidden symmetry C which exists for a given pseudo-Hermitian Hamiltonian A in the sector of exact $\mathcal{PT}$-symmetry [9, 10, 11]. The operator C has some rough analogy with the charge conjugation operator in the quantum field theory [10] and it is determined non-uniquely [13]. The existence of C gives rise to an inner product $(\cdot,\cdot)_C = [C\cdot,\cdot]_J$ and the dynamics generated by A is therefore governed by a unitary time evolution.

For J-self-adjoint extensions $A_\varepsilon \supset S$, where S is an *arbitrary* symmetric operator commuting with J, we introduce the concepts of stable and unstable C-symmetry (Definition 2.11). These concepts are natural in the extension theory framework. Roughly speaking, if A_ε belongs to the sector Σ_J^{st} of stable C-symmetry, then A_ε preserves the property of C-symmetry under small variation of ε.

It follows from the results of [1, 23] that for some types of singular perturbations of the Schrödinger or the Dirac operator, the sector Σ_J^{unst} of unstable C symmetry is not empty and operators A_ε with real spectra and Jordan points correspond to the case where ε belongs to the boundary of Σ_J^{st}.

In the case of the Phillips symmetric operator S, the spectral picture above can be essentially simplified. Precisely, assuming the deficiency indices $< 2, 2 >$ of S, we show that $\Sigma_J^{\mathsf{unst}} = \varnothing$ and any J-self-adjoint extension of S with real spectrum is similar to a self-adjoint operator (Theorem 3.9 and Corollary 3.10).

Throughout the paper $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of a linear operator A, respectively, while $A \upharpoonright \mathcal{D}$ stands for the restriction of

[2] An operator is called *simple* if its restriction to any nontrivial reducing subspace is not a self-adjoint operator.

A to the set $\mathcal{D}$. The set of points of regular type of a symmetric operator S is denoted by $\widehat{\rho}(S)$ (i.e., $r \in \widehat{\rho}(S) \iff \|(S - rI)u\| \geq k\|u\|, \ \forall u \in \mathcal{D}(S), \ k > 0$).

2. Preliminaries

2.1. **Elements of the Krein space theory.** Let $(\mathfrak{H}, [\cdot,\cdot]_J)$ be a Krein space with fundamental symmetry J. The corresponding orthoprojectors $P_\pm = \frac{1}{2}(I \pm J)$ determine *the fundamental decomposition* of $\mathfrak{H}$

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \quad \mathfrak{H}_- = P_-\mathfrak{H}, \quad \mathfrak{H}_+ = P_+\mathfrak{H}. \tag{2.1}$$

A subspace $\mathfrak{L}$ of $\mathfrak{H}$ is called *hypermaximal neutral* if

$$\mathfrak{L} = \mathfrak{L}^{[\perp]_J} = \{x \in \mathfrak{H} : [x,y]_J = 0, \ \forall y \in \mathfrak{L}\}.$$

A subspace $\mathfrak{L} \subset \mathfrak{H}$ is called *uniformly positive (uniformly negative)* if $[x,x]_J \geq a^2\|x\|^2$ $(-[x,x]_J \geq a^2\|x\|^2)$ $a \in \mathbb{R}$ for all $x \in \mathfrak{L}$. The subspaces $\mathfrak{H}_\pm$ in (2.1) are examples of uniformly positive and uniformly negative subspaces and they possess the property of maximality in the corresponding classes (i.e., $\mathfrak{H}_+$ ($\mathfrak{H}_-$) does not belong as a subspace to any uniformly positive (negative) subspace).

Let $\mathfrak{L}_+ (\neq \mathfrak{H}_+)$ be an arbitrary maximal uniformly positive subspace. Then its J-orthogonal complement $\mathfrak{L}_- = \mathfrak{L}_+^{[\perp]_J}$ is a maximal uniformly negative and the direct J-orthogonal sum

$$\mathfrak{H} = \mathfrak{L}_+ [\dot{+}]_J \mathfrak{L}_- \tag{2.2}$$

gives another (then (2.1)) decomposition of $\mathfrak{H}$ onto its positive $\mathfrak{L}_+$ and negative $\mathfrak{L}_-$ parts (the brackets $[\cdot]_J$ mean the orthogonality with respect to the indefinite metric).

An arbitrary decomposition of the Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$ onto its positive and negative parts (like (2.2)) is called *canonical*.

The subspaces $\mathfrak{L}_\pm$ in (2.2) can be described as

$$\mathfrak{L}_+ = (I + X)\mathfrak{H}_+, \quad \mathfrak{L}_- = (I + X^*)\mathfrak{H}_-,$$

where $X : \mathfrak{H}_+ \to \mathfrak{H}_-$ is a contraction and $X^* : \mathfrak{H}_- \to \mathfrak{H}_+$ is the adjoint of X.

The self-adjoint operator $T = XP_+ + X^*P_-$ acting in $\mathfrak{H}$ is called *an operator of transition* from the fundamental decomposition (2.1) to the canonical one (2.2). Obviously, $\mathfrak{L}_+ = (I+T)\mathfrak{H}_+$ and $\mathfrak{L}_- = (I+T)\mathfrak{H}_-$.

Operators of transition admit a simple description. Namely, a self-adjoint operator T in $\mathfrak{H}$ is an operator of transition if and only if $\|T\| < 1$ and $JT = -TJ$.

The set $\{T\}$ of all possible operators of transition is in one-to-one correspondence (via $\mathfrak{L}_\pm = (I+T)\mathfrak{H}_\pm$) with all possible canonical decompositions (2.2) of the Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$.

The projectors $P_{\mathfrak{L}_\pm} : \mathfrak{H} \to \mathfrak{L}_\pm$ onto $\mathfrak{L}_\pm$ with respect to the decomposition (2.2) are determined by the formulas

$$P_{\mathfrak{L}_-} = (I-T)^{-1}(P_- - TP_+), \quad P_{\mathfrak{L}_+} = (I-T)^{-1}(P_+ - TP_-).$$

The bounded operator

$$C = P_{\mathfrak{L}_+} - P_{\mathfrak{L}_-} = J(I-T)(I+T)^{-1} \tag{2.3}$$

also describes subspaces $\mathfrak{L}_\pm$ in (2.2)

$$\mathfrak{L}_+ = \frac{1}{2}(I+C)\mathfrak{H}, \quad \mathfrak{L}_- = \frac{1}{2}(I-C)\mathfrak{H}. \tag{2.4}$$

The set of operators C determined (2.3) is completely characterized by the conditions

$$C^2 = I, \quad JC > 0. \tag{2.5}$$

2.2. **Elements of the Von Neumann extension theory.** Let S be a closed symmetric densely defined operator in a Hilbert space $\mathfrak{H}$ with equal (finite or infinite) deficiency indices. Denote by $\mathfrak{N}_i = \mathfrak{H} \ominus \mathcal{R}(S - iI)$ and $\mathfrak{N}_{-i} = \mathfrak{H} \ominus \mathcal{R}(S + iI)$ the defect subspaces of S and consider the Hilbert space $\mathfrak{M} = \mathfrak{N}_{-i} \dot{+} \mathfrak{N}_i$ with the inner product

$$(f, g)_{\mathfrak{M}} = (f_i, g_i) + (f_{-i}, g_{-i}) \quad f = f_i + f_{-i}, \quad g = g_i + g_{-i} \quad \{f_{\pm i}, g_{\pm i}\} \subset \mathfrak{N}_{\pm i}.$$

The operator $Z(f_{-i} + f_i) = f_{-i} - f_i$ is a fundamental symmetry in the Hilbert space $\mathfrak{M}$ and its restriction onto $\mathfrak{N}_{-i}$ and $\mathfrak{N}_i$ coincide, respectively, with I and $-I$.

Let J be a fundamental symmetry in $\mathfrak{H}$. In what follows we assume that

$$SJ = JS. \tag{2.6}$$

Then the subspaces $\mathfrak{N}_{\pm i}$ reduce J and the restriction $J \upharpoonright \mathfrak{M}$ gives rise to a fundamental symmetry in the Hilbert space $\mathfrak{M}$. Moreover, according to the properties of Z mentioned above, $JZ = ZJ$. Therefore, JZ is a fundamental symmetry in $\mathfrak{M}$ and sesquilinear form

$$[f, g]_{JZ} = (JZf, g)_{\mathfrak{M}} = (Jf_{-i}, g_{-i}) - (Jf_i, g_i)$$

determines an indefinite metric on $\mathfrak{M}$.

According to von-Neumann formulas any closed intermediate extension A of S (i.e., $S \subset A \subset S^*$) is uniquely determined by the choice of a subspace $M \subset \mathfrak{M}$. Precisely,

$$\mathcal{D}(A) = \mathcal{D}(S) \dot{+} M \quad \text{and} \quad A = S^* \upharpoonright \mathcal{D}(A). \tag{2.7}$$

We use the notation A_M for J-self-adjoint extensions of S determined by (2.7).

Let A_M and $A_{\widetilde{M}}$ be arbitrary extensions of S that are defined by the subspaces M and $\widetilde{M}$, respectively. Taking (2.6) and (2.7) into account we derive

$$[A_M \psi, \phi]_J - [\psi, A_{\widetilde{M}} \phi]_J = 2i[f, g]_{JZ} \tag{2.8}$$

for all $\psi = u + f \in \mathcal{D}(A_M)$, $f \in M$, $\quad \phi = v + g \in \mathcal{D}(A_{\widetilde{M}})$, $g \in \widetilde{M}$.

It follows from (1.1) and (2.8) that an extension A_M of S is J-self-adjoint if and only if

$$M = M^{[\perp]_{JZ}} = \{f \in \mathfrak{M} \ : \ [f, g]_{JZ} = 0, \ \forall g \in M\},$$

i.e., if M is a hypermaximal neutral subspace of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_{JZ})$. Formalizing this observation we get the well-known result.

Proposition 2.1. *The correspondence $A \leftrightarrow M$ determined by (2.7) is a bijection between J-self-adjoint (self-adjoint) extensions A of S and hypermaximal neutral subspaces M of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_{JZ})$ (of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_Z)$).*

Denote by $\Sigma_J(S)$ the set of J-self-adjoint extensions of S. In general, these extensions may have complex spectra and, moreover, the existence of $A \in \Sigma_J(S)$ with empty resolvent set (i.e., $\sigma(A) = \mathbb{C}$) is also possible. To guarantee nonempty resolvent set *for any* $A \in \Sigma_J(S)$ we need to impose additional constraints. In this way we recall that a J-self-adjoint operator A is called *definitizable* if the resolvent set of A is nonempty and there exists a polynomial $p(\cdot) \not\equiv 0$ such that $p(A)$ is a nonnegative operator in the Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$.

Proposition 2.2. ([6]). *Let S have finite deficiency indices. Then if there exists a definitizable extension $A \in \Sigma_J(S)$, then an arbitrary operator from $\Sigma_J(S)$ has a nonempty resolvent set and is definitizable.*

2.3. **Boundary value spaces technique.** Proposition 2.1 provides a description of $\Sigma_J(S)$ in terms of the Krein space $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$. Another approach which allows one to avoid the use of $\mathfrak{M}$ is based on the concept of *boundary triplets* (or boundary value spaces, see [22] and the references therein).

Definition 2.3. *A triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where $\mathcal{H}$ is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings of $\mathcal{D}(S^*)$ into $\mathcal{H}$, is called a boundary triplet of S^* if the abstract Green identity*

$$(S^*\psi, \phi) - (\psi, S^*\phi) = (\Gamma_1\psi, \Gamma_0\phi)_{\mathcal{H}} - (\Gamma_0\psi, \Gamma_1\phi)_{\mathcal{H}}, \quad \psi, \phi \in \mathcal{D}(S^*)$$

is satisfied and the map $(\Gamma_0, \Gamma_1) : \mathcal{D}(S^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Denote

$$\mathfrak{N}_\mu = \mathfrak{H} \ominus \mathcal{R}(S - \mu I) = \ker(S^* - \overline{\mu} I), \quad \mu \in \widehat{\rho}(S). \tag{2.9}$$

The Weyl function $M(\cdot)$ and the characteristic function $\Theta(\cdot)$ of S associated with a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ are defined as follows [18, 27, 37]:

$$\begin{aligned} M(\mu)\Gamma_0 f_{\overline{\mu}} &= \Gamma_1 f_{\overline{\mu}}, \quad \forall f_{\overline{\mu}} \in \mathfrak{N}_{\overline{\mu}}, \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R}, \\ \Theta(\mu)(\Gamma_1 + i\Gamma_0) f_{\overline{\mu}} &= (\Gamma_1 - i\Gamma_0) f_{\overline{\mu}}, \quad \forall \mu \in \mathbb{C}_+. \end{aligned} \tag{2.10}$$

It is clear that $\Theta(\mu) = (M(\mu) - iI)(M(\mu) + iI)^{-1}$, $\forall \mu \in \mathbb{C}_+$.

The Weyl function (or, characteristic function) determines a simple symmetric operator S up to unitary equivalence.

The simplest (*canonical*) boundary triplet can immediately be constructed as a triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1)$, where

$$\Gamma_0\psi = f_{-i} + Q f_i, \quad \Gamma_1\psi = i f_{-i} - iQ f_i, \quad \psi = u + f_{-i} + f_i \in \mathcal{D}(S^*) \tag{2.11}$$

and Q is an arbitrary unitary mapping $Q : \mathfrak{N}_i \to \mathfrak{N}_{-i}$.

To underline the dependence of Γ_j on the choice of Q in (2.11), we denote by $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ the corresponding boundary triplet.

If Q commutes with J, then the boundary operators Γ_j defined by (2.11) satisfy the relations

$$\Gamma_0 J = J\Gamma_0, \quad \Gamma_1 J = J\Gamma_1. \tag{2.12}$$

By Proposition 2.1, self-adjoint extensions $A_M \supset S$ commuting with J are described by hypermaximal neutral subspaces

$$M_G = \{f_i + G f_i \mid \forall f_i \in \mathfrak{N}_i\} \tag{2.13}$$

of the Krein space $(\mathfrak{M}, [\cdot,\cdot]_Z)$ which satisfy the additional relation $JM_G = M_G$. Here $G : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ are unitary mappings. Obviously, $JM_G = M_G \iff JG = GJ$. The latter gives rise to the existence of boundary triplets $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, -G)$ defined by (2.11) with the additional properties (2.12). We prove the following simple statement:

Proposition 2.4. *Boundary triplets $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ satisfying (2.12) exist if and only if the set of self-adjoint extensions of S commuting with J is non-empty.*

For such type of boundary triplets, Proposition 2.1 can be rewritten as follows:

Proposition 2.5. *Let $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ be a boundary triplet of S^* which satisfies (2.12). Then an arbitrary $A \in \Sigma_J(S)$ with $i \notin \sigma(A)$ coincides with the restriction of S^* onto the domain*

$$\mathcal{D}(A) = \{f \in \mathcal{D}(S^*) \mid K(\Gamma_1 + i\Gamma_0)f = (\Gamma_1 - i\Gamma_0)f\}, \tag{2.14}$$

where K is a J-unitary operator in $\mathfrak{N}_{-i}$ (i.e., $J = K^*JK$).

The correspondence $A = A_K \leftrightarrow K$ determined by (2.14) is a bijection between the set of all J-self-adjoint extensions A_K of S such that $i \notin \sigma(A_K)$ and the set of J-unitary operators in $\mathfrak{N}_{-i}$. Furthermore,

$$A_K^* = A_{(K^*)^{-1}}. \tag{2.15}$$

Remark 2.6. J-Self-adjoint extensions A_M with $i \in \sigma(A_M)$ are characterized by nontrivial intersections $M \cap \mathfrak{N}_{-i}$ of the corresponding subspaces M in (2.7). In that case, the description (2.14) of $\mathcal{D}(A_M)$ is impossible (since $\ker(\Gamma_1 - i\Gamma_0) = \mathfrak{N}_{-i}$ and $\ker(\Gamma_1 + i\Gamma_0) = \mathfrak{N}_i$ by (2.11)).

2.4. **Description of $\Sigma_J(S)$. The case of deficiency indices $< 2, 2 >$.** We are going to analyze $\Sigma_J(S)$ in more detail for the case where S has deficiency indices $< 2, 2 >$. To avoid the study of self-adjoint extensions we assume $J \neq I$. Then, the following subspaces of the Hilbert space $\mathfrak{M}$:

$$\mathfrak{M}_{++} = (I + Z)(I + J)\mathfrak{M}, \qquad \mathfrak{M}_{--} = (I - Z)(I - J)\mathfrak{M},$$
$$\mathfrak{M}_{+-} = (I + Z)(I - J)\mathfrak{M}, \qquad \mathfrak{M}_{-+} = (I - Z)(I + J)\mathfrak{M}$$

are nontrivial and mutually orthogonal. Therefore, $\dim \mathfrak{M}_{\pm\pm} = 1$ (since $\dim \mathfrak{M} = 4$) and there exists an orthonormal basis $\{e_{\pm\pm}\}$ of the Hilbert space $\mathfrak{M}$ such that

$$\mathfrak{M}_{\pm\pm} =< e_{\pm\pm} >, \quad \mathfrak{N}_{-i} =< e_{++}, e_{+-} >, \quad \mathfrak{N}_i =< e_{-+}, e_{--} > .$$

In that case

$$\begin{array}{llll} Je_{++} = e_{++}, & Je_{-+} = e_{-+}, & Je_{+-} = -e_{+-}, & Je_{--} = -e_{--}; \\ Ze_{++} = e_{++}, & Ze_{-+} = -e_{-+}, & Ze_{+-} = e_{+-}, & Ze_{--} = -e_{--}. \end{array} \tag{2.16}$$

Let us consider the boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ defined by (2.11), where a unitary mapping $Q : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ acts as follows:

$$Qe_{-+} = e_{++}, \quad Qe_{--} = e_{+-}. \tag{2.17}$$

The operator Q commutes with J due to (2.16) and hence, relations (2.12) hold.

Denote by $\mathcal{K} = \|k_{ij}\|$ the matrix representation of a J-unitary operator K in $\mathfrak{N}_{-i}$ with respect to the basis $\{e_{++}, e_{+-}\}$. By (2.16), the restriction of J onto $\mathfrak{N}_{-i}$ can be identified (with respect to the basis $\{e_{++}, e_{+-}\}$) with the matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This means that $\sigma_3 = \overline{\mathcal{K}^t} \sigma_3 \mathcal{K}$ (since K is J-unitary). The simple analysis of the latter relation leads to the following description of $\mathcal{K}$:

$$\mathcal{K} = \mathcal{K}(\zeta, \phi, \omega, \xi) = e^{-i\xi} \begin{pmatrix} -(\cosh\zeta)e^{-i\phi} & (\sinh\zeta)e^{-i\omega} \\ -(\sinh\zeta)e^{i\omega} & (\cosh\zeta)e^{i\phi} \end{pmatrix}, \tag{2.18}$$

where $\zeta \in \mathbb{R}$ and $\xi, \phi, \omega \in [0, 2\pi)$. Using Proposition 2.5, we obtain the following

Proposition 2.7. *The formula*

$$S^* \restriction \mathcal{D}(A_M), \quad \mathcal{D}(A_M) = \{f \in \mathcal{D}(S^*) \mid K(\Gamma_1 + i\Gamma_0)f = (\Gamma_1 - i\Gamma_0)f\}, \tag{2.19}$$

where K is an arbitrary J-unitary operator in $\mathfrak{N}_{-i}$ and the boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ is defined by (2.11) and (2.17) establishes the one to one correspondence between J-self-adjoint extensions $A_M \in \Sigma_J(S)$ with $i \notin \sigma(A_M)$ and matrices $\mathcal{K}(\zeta, \phi, \omega, \xi)$ defined by (2.18).

Remark. It follows from Proposition 2.1 and relations (2.16) that operators $A_M \in \Sigma_J(S)$ with $i \in \sigma(A_M)$ are described by the two-parameter set of hypermaximal neutral subspaces

$$M(k_1, k_2) =< e_{++} + e^{ik_1}e_{+-};\ e_{--} + e^{ik_2}e_{-+} >, \quad k_1, k_2 \in \mathbb{R}$$

of the Krein space $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$. By virtue of (2.11) and (2.17), the subspaces $M(k_1, k_2)$ can be (formally) described by (2.14) if we put

$$\zeta = \infty, \quad \xi = 0, \quad \omega = \frac{k_2 + k_1}{2}, \quad \phi = \frac{k_2 - k_1}{2}$$

in (2.18) and consider $\cosh\infty = \sinh\infty = \infty$ as a number.

To emphasize the relationship $A_M \leftrightarrow \mathcal{K}$ established in Proposition 2.7, we will use the notation $A_{\mathcal{K}}$ instead of A_M.

Corollary 2.8. *The adjoint operator $A^*_{\mathcal{K}}$ of $A_{\mathcal{K}} \in \Sigma_J(S)$ is defined by $\mathcal{K}(-\zeta, \phi, \xi, \omega)$ i.e.,*

$$A^*_{\mathcal{K}(\zeta,\phi,\omega,\xi)} = A_{\mathcal{K}(-\zeta,\phi,\omega,\xi)}. \tag{2.20}$$

The set of self-adjoint extensions of S commuting with J is described by unitary matrices $\mathcal{K}(0, \phi, \omega, \xi)$.

Proof. The relation (2.20) follows from (2.15) and (2.18).

If a self-adjoint extension $A \supset S$ commutes with J, then A is also J-self-adjoint and $A \equiv A_{\mathcal{K}(\zeta,\phi,\omega,\xi)}$ by Proposition 2.7. Using (2.20) and taking into account (2.18), we get $\zeta = 0$ that completes the proof of Corollary 2.8. □

2.5. **The property of C-symmetry.** By analogy with [10] the definition of C-symmetry in the Krein spaces setting can be formalized as follows.

Definition 2.9. *An operator A acting in a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$ has the property of C-symmetry if there exists a bounded linear operator C in $\mathfrak{H}$ such that: (i) $C^2 = I$; (ii) $JC > 0$; (iii) $AC = CA$.*

By virtue of (2.3) and (2.5) the property of C-symmetry of A means that A can be decomposed

$$A = A_+[\dot{+}]_J A_-, \quad A_+ = A\restriction_{\mathfrak{L}_+}, \quad A_- = A\restriction_{\mathfrak{L}_-} \tag{2.21}$$

with respect to the canonical decomposition (2.2) (with subspaces $\mathfrak{L}_\pm$ determined by (2.4)).

If a J-self-adjoint operator A possesses the property of C-symmetry, then its counterparts $A_\pm$ in (2.21) turn out to be self-adjoint operators in the Hilbert spaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ with the inner products $[\cdot,\cdot]_J$ and $-[\cdot,\cdot]_J$, respectively. This simple observation leads to the following statement.

Proposition 2.10. ([1]). *A J-self-adjoint operator A has the property of C-symmetry if and only if A is similar to a self-adjoint operator in $\mathfrak{H}$. If a J-self-adjoint operator A has the property of C-symmetry then its spectrum is real and the adjoint operator C^* provides the property of C-symmetry for A^*.*

Definition 2.11. *Let $A \in \Sigma_J(S)$ have the property of C-symmetry realized by an operator C. We will say that A belongs to the sector Σ_J^{st} of stable C-symmetry if the operator C commutes with S. Otherwise ($AC = CA$ but $SC \neq CS$), the operator A belongs to the sector Σ_J^{unst} of unstable C-symmetry.*

The next statement immediately follows from Theorem 3.1 in [1].

Proposition 2.12. *Let $A_M \in \Sigma_J(S)$ be defined by (2.7). Then $A_M \in \Sigma_J^{st}$ if and only if $CM = M$, where C realizes the property of C-symmetry for S.*

3. The Phillips symmetric operator

We are going to specify general results of previous section to the case of Phillips symmetric operator S defined by (1.2) and (1.3).

3.1. **Preliminaries.** The general definition (1.2), (1.3) of S looks rather abstract and, in many cases, it is useful to work with a model realization of S in $\mathfrak{H} = l_2(\mathbb{Z}, N)$ (N is an auxiliary finite-dimensional Hilbert space). In that case

$$\begin{aligned} U(\dots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \dots) &= (\dots, x_{-3}, x_{-2}, \underline{x_{-1}}, x_0, x_1, \dots), \\ V(\dots, x_{-2}, x_{-1}, \underline{0}, x_1, x_2, \dots) &= (\dots, x_{-3}, x_{-2}, \underline{x_{-1}}, 0, x_1, \dots), \end{aligned} \tag{3.1}$$

where $x_j \in N$ and elements at the zero position are underlined.

The self-adjoint operator A takes the form

$$\begin{aligned} &Af = i(\dots, x_{-3} + x_{-2}, x_{-2} + x_{-1}, \underline{x_{-1} + x_0}, x_0 + x_1, x_1 + x_2, \dots), \\ &f \in \mathcal{D}(A) \iff f = (\dots, x_{-3} - x_{-2}, x_{-2} - x_{-1}, \underline{x_{-1} - x_0}, x_0 - x_1, x_1 - x_2, \dots), \end{aligned} \tag{3.2}$$

where $\sum_{i\in\mathbb{Z}} \|x_i\|_N^2 < \infty$ and the symmetric operator S is the restriction of A onto the set

$$u \in \mathcal{D}(S) \iff u = (\dots, x_{-3} - x_{-2}, x_{-2} - x_{-1}, \underline{x_{-1}}, -x_1, x_1 - x_2, \dots), \tag{3.3}$$

which consists of all $u \in \mathcal{D}(A)$ such that $x_0 = 0$.

Recalling (2.9) and using (3.2), (3.3), it is easily to see that (see, e.g., [29])

$$\begin{aligned} \mathfrak{N}_i &= \{f_i(x) = (\dots, 0, 0, \underline{x}, 0, 0, \dots) : \forall x \in N\}, \\ \mathfrak{N}_\mu &= \{f_\mu(x) = (\dots, \overline{r}_\mu^2 x, \overline{r}_\mu x, \underline{x}, 0, 0, \dots) : \forall x \in N\}, \quad \mu \in \mathbb{C}_+, \\ \mathfrak{N}_{-i} &= \{f_{-i}(x) = (\dots, 0, 0, \underline{0}, x, 0, \dots) : \forall x \in N\}, \\ \mathfrak{N}_{\overline{\mu}} &= \{f_{\overline{\mu}}(x) = (\dots, 0, 0, \underline{0}, x, r_\mu x, r_\mu^2 x, \dots) : \forall x \in N\}, \end{aligned} \tag{3.4}$$

where $r_\mu = \frac{\mu - i}{\mu + i}$. Direct calculation with the use of (3.3) and (3.4) gives

$$f_\mu((1 - \overline{r}_\mu)x) = u + f_i(x), \quad f_{\overline{\mu}}((1 - r_\mu)x) = v + f_{-i}(x), \quad \forall x \in N, \tag{3.5}$$

where $u, v \in \mathcal{D}(S)$. Therefore,

$$\mathfrak{N}_\mu \subset \mathcal{D}(S) \dot{+} \mathfrak{N}_i, \quad \mathfrak{N}_{\overline{\mu}} \subset \mathcal{D}(S) \dot{+} \mathfrak{N}_{-i}, \quad \forall \mu \in \mathbb{C}_+. \tag{3.6}$$

Lemma 3.1. *Let $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ be a boundary triplet of the Phillips symmetric operator S (defined by (1.2) and (1.3)). Then the corresponding characteristic function $\Theta(\cdot)$ of S is equal to zero.*

Proof. It is sufficient to verify this statement for the case where S is defined by (3.2) and (3.3). According to (3.6), an arbitrary $f_{\overline{\mu}} \in \mathfrak{N}_{\overline{\mu}}$ has the form $f_{\overline{\mu}} = u + f_{-i}$, where $u \in \mathcal{D}(S)$ and $f_{-i} \in \mathfrak{N}_{-i}$. But then $(\Gamma_1 + i\Gamma_0)f_{\overline{\mu}} = 2if_{-i}$ and $(\Gamma_1 - i\Gamma_0)f_{\overline{\mu}} = 0$ due to (2.11). Therefore, $\Theta(\mu) \equiv 0$ ($\forall \mu \in \mathbb{C}_+$) by (2.10). Lemma 3.1 is proved. □

Lemma 3.2. *Let S be defined by (3.2) and (3.3) and let J be a fundamental symmetry in $l_2(\mathbb{Z}, N)$. Then J commutes with S if and only if*

$$J(\dots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \dots) = (\dots, J_- x_{-2}, J_- x_{-1}, \underline{J_- x_0}, J_+ x_1, J_+ x_2, \dots), \tag{3.7}$$

where $J_\pm$ are fundamental symmetries in N.

Proof. Let J commute with S. It follows from (2.9) that defect subspaces $\mathfrak{N}_\mu$ are invariant with respect J. Taking (3.4) into account we conclude that the restrictions $J_- := J \restriction \mathfrak{N}_i$ and $J_+ := J \restriction \mathfrak{N}_{-i}$ determine two fundamental symmetries J_- and J_+ in N. Further, the equality $JS = SJ$ is equivalent to the relation $JV = VJ$, where V is defined by (3.1). Combining this relation with the first and third relations in (3.4) and taking the definition of $J_\pm$ into account we establish (3.7).

Conversely, if a fundamental symmetry J is defined by (3.7), then relations (3.2) and (3.3) imply that $JS = SJ$. Lemma 3.2 is proved. □

Lemma 3.3. *Let S be defined by (3.2) and (3.3), let J be a fundamental symmetry in $l_2(\mathbb{Z}, N)$ commuting with S, and let C be a bounded operator in $l_2(\mathbb{Z}, N)$ such that $C^2 = I$ and $JC > 0$. Then C commutes with S if and only if*

$$C(\dots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \dots) = (\dots, C_- x_{-2}, C_- x_{-1}, \underline{C_- x_0}, C_+ x_1, C_+ x_2, \dots), \tag{3.8}$$

where $C_\pm$ are bounded operators in N such that $C_\pm^2 = I_N$ and $J_\pm C_\pm > 0$ where $J_\pm$ are taken from the formula (3.7).

Proof. By Lemma 3.2, the operator J is defined by (3.7), where $J_\pm$ are fundamental symmetries in N.

Assume that C commutes with S. Then, using (2.6) one gets $SF = FS$, where $F = JC$ is a bounded self-adjoint operator. Hence,

$$SC^* = SFJ = FSJ = FJS = C^* S.$$

The obtained relation $C^* S = SC^*$ and $C^2 = I$ imply that the defect subspaces $\mathfrak{N}_\mu$ of S are invariant with respect C. It follows from (3.4) that the restrictions $C_- := C \upharpoonright \mathfrak{N}_i$ and $C_+ := C \upharpoonright \mathfrak{N}_{-i}$ determine bounded operators $C_\pm$ in N such that $C_\pm^2 = I_N$ and $J_\pm C_\pm > 0$. Reasoning by analogy with the proof of Lemma 3.2, we complete the proof. □

3.2. **Description of J-self-adjoint extensions.** Using (3.4) we can identify the Hilbert space $\mathfrak{M} = \mathfrak{N}_{-i} \dot{+} \mathfrak{N}_i$ with

$$N \oplus N = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in N \right\}.$$

In that case

$$Z = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad J \upharpoonright \mathfrak{M} = \begin{pmatrix} J_+ & 0 \\ 0 & J_- \end{pmatrix}. \tag{3.9}$$

Proposition 3.4. *Let S be defined by (3.2) and (3.3). Then the set $\Sigma_J(S)$ of J-self-adjoint extensions of S is non-empty if and only if*

$$\dim[(I - J_+)N] = \dim[(I - J_-)N]. \tag{3.10}$$

Proof. By Proposition 2.1, J-self-adjoint extensions of S exist if and only if the Krein space $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$ has hypermaximal neutral subspaces. This is possible only in the case where $\dim[(I + JZ)\mathfrak{M}] = \dim[(I - JZ)\mathfrak{M}]$ or, that is equivalent (see (3.9)),

$$\dim[(I + J_+)N] + \dim[(I - J_-)N] = \dim[(I - J_+)N] + \dim[(I + J_-)N].$$

This identity is equivalent to (3.10) (since $\dim[(I + J_\pm)N] + \dim[(I - J_\pm)N] = \dim N$ and $\dim N < \infty$). Proposition 3.10 is proved. □

Corollary 3.5. *Let S be defined by (3.2) and (3.3) and let J be a fundamental symmetry commuting with S in $l_2(\mathbb{Z}, N)$. Then self-adjoint extensions of S commuting with J exist if and only if the identity (3.10) holds.*

Proof. If A_M is a self-adjoint extension of S commuting with J, then $A_M \in \Sigma_J(S)$ and relation (3.10) holds due to Proposition 3.4.

Conversely, since $\dim N < \infty$, relation (3.10) is equivalent to the identity

$$\dim[(I + J_+)N] = \dim[(I + J_-)N].$$

This implies the existence of unitary mappings $G : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ such that $GJ = GJ_- = J_+G = JG$. In that case the hypermaximal neutral subspace M_G of the Krein space $(\mathfrak{M}, [\cdot,\cdot]_Z)$ (defined by (2.13)) satisfies the relation $JM_G = M_G$ and the corresponding self-adjoint extension A_M commutes with J. Corollary 3.5 is proved. □

Proposition 3.6. *Let S be the Phillips symmetric operator (defined by (1.2) and (1.3)) and let J be a fundamental symmetry commuting with S in $\mathfrak{H}$. Then boundary triplets $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ of S^* defined by (2.11) and satisfying (2.12) exist if and only if the set $\Sigma_J(S)$ is non-empty.*

Proof. It is sufficient to establish for the Phillips symmetric operator S realized by the formulas (3.2) and (3.3). In that case, by Proposition 3.4 and Corollary 3.5, $\Sigma_J(S) \neq \emptyset$ $\iff$ there exist self-adjoint extensions of S commuting with J. Using now Proposition 2.4 we complete the proof. □

Theorem 3.7. *Let S be the Phillips symmetric operator, let J be a fundamental symmetry commuting with S in $\mathfrak{H}$, and let $A_M \in \Sigma_J(S)$. Then the spectrum of A_M either coincides with $\mathbb{R}$ ($\sigma(A_M) = \mathbb{R}$) or covers the whole complex plane ($\sigma(A_M) = \mathbb{C}$) and its non-real part consists of eigenvalues of A_M.*

Proof. Since an arbitrary $A_M \in \Sigma_J(S)$ is a finite rank perturbation of the self-adjoint operator A (see (1.3)), the non-real spectrum of A_M may include complex eigenvalues.

Without loss of generality we can suppose that S is determined by the formulas (3.2) and (3.3). Assume that $\mu_0 \in \mathbb{C}_+$ is an eigenvalue of A_M. Then there exists an element $f_{\overline{\mu}_0} \in \mathfrak{N}_{\overline{\mu}_0} \cap \mathcal{D}(A_M)$ and, according to (3.5),

$$f_{\overline{\mu}_0} = f_{\overline{\mu}_0}(x) = v_0 + f_{-i}\left(\frac{x}{1 - r_{\mu_0}}\right), \quad v_0 \in \mathcal{D}(S)$$

for a certain choice of $x \in N$. Using the second relation in (3.5) for an arbitrary $\mu \in \mathbb{C}_+$, we obtain

$$f_{\overline{\mu}}\left(\frac{1 - r_\mu}{1 - r_{\mu_0}} x\right) = v + f_{-i}\left(\frac{x}{1 - r_{\mu_0}}\right), \quad v \in \mathcal{D}(S).$$

Comparing last two relations we arrive at the conclusion that the element

$$f_{\overline{\mu}}\left(\frac{1 - r_\mu}{1 - r_{\mu_0}} x\right) = v - v_0 + f_{\overline{\mu}_0}, \quad \mu \in \mathbb{C}_+,$$

belongs to $\mathfrak{N}_{\overline{\mu}} \cap \mathcal{D}(A_M)$. Hence $\mathbb{C}_+ \subset \sigma_p(A_M)$. The relation $\mathbb{C}_- \subset \sigma_p(A_M)$ is established by the same manner. Thus, $\sigma(A_M) = \mathbb{C}$ and $\mathbb{C} \setminus \mathbb{R}$ contains eigenvalues of A_M.

Assume that the spectrum of A_M is real. Since the Phillips symmetric operator has no real points of regular type (see, e.g., [26]), the spectrum of A_M coincides with $\mathbb{R}$. Theorem 3.7 is proved. □

Corollary 3.8. *Let S be the Phillips symmetric operator and let J be a fundamental symmetry commuting with S in $\mathfrak{H}$. Then the set $\Sigma_J(S)$ does not contain definitizable operators.*

Proof. By Proposition (3.6), if $\Sigma_J(S)$ is a non-empty set, then there exists a boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ of S^* which satisfies (2.12). It follows from Proposition 2.5 and Theorem 3.7 that operators $A_M \in \Sigma_J(S)$ with *real* spectrum are described by the formula (2.14) in terms of the boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$. The rest of operators $A_M \in \Sigma_J(S)$ (which can not be described by (2.14)) have empty resolvent set (due to Remark 2.6 and Theorem 3.7). By Proposition 2.2 this means that $\Sigma_J(S)$ does not contain definitizable operators. Corollary 3.8 is proved. □

3.3. J-self-adjoint extensions with C-symmetry.

Theorem 3.9. *Let S be the Phillips operator with deficiency indices $< 2, 2 >$. Then an arbitrary J-self-adjoint extension $A_M \in \Sigma_J(S)$ has the property of stable C-symmetry ($A_M \in \Sigma_J^{st}$) if and only if the spectrum of A_M is real.*

Proof. If A_M has C-symmetry, then its spectrum is real (see Proposition 2.10).

Conversely, we assume that $A_M \in \Sigma_J(S)$ has a real spectrum. In that case, by Proposition 2.7, $A_M(=A_{\mathcal{K}})$ is defined by (2.19), where $\mathcal{K}=\mathcal{K}(\zeta,\phi,\omega,\xi)$ has the form (2.18).

Without loss of generality we can assume that S is determined by the formulas (3.2) and (3.3) in the space $l_2(\mathbb{Z},N)$. Then, by virtue of Proposition 2.12 and Lemma 3.3, the operator $A_M(=A_{\mathcal{K}})$ has a stable C-symmetry if and only if $CM=M$ for at least one of operators C determined by (3.8).

It follows from (2.19) that $M=\{f=f_{-i}+f_i \mid K(\Gamma_1+i\Gamma_0)f=(\Gamma_1-i\Gamma_0)f\}$. Employing (2.11) we rewrite the latter relation as follows:

$$M=\{f_{-i}-Q^{-1}Kf_{-i} \mid \forall f_{-i}\in\mathfrak{N}_{-i}\}.$$

The obtained description of M and Lemma 3.3 imply that

$$CM=M \iff KC_+=\widehat{C}_+K \quad (\widehat{C}_+ := QC_-Q^{-1}), \tag{3.11}$$

where C_+ and $\widehat{C}_+$ act in $N=\mathfrak{N}_{-i}$ and satisfy the relations $C^2=I,\ \ JC>0 \ \ (C\in\{C_+,\widehat{C}_+\})$.

Let

$$\mathcal{C}_+=\|c_{ij}\|$$

be the matrix representation of C_+ with respect to the basis $\{e_{++},e_{+-}\}$. Then the relations $C_+^2=I,\ \ JC_+>0$ take the form

$$\mathcal{C}_+^2=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \begin{pmatrix}1&0\\0&-1\end{pmatrix}\mathcal{C}_+>0, \tag{3.12}$$

where $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ is the matrix representation of $J\restriction\mathfrak{N}_{-i}$ with respect to $\{e_{++},e_{+-}\}$ (since (2.16)). A simple analysis of (3.12) leads to the following description of $\mathcal{C}_+$:

$$\mathcal{C}_+=\mathcal{C}_{\widetilde{\chi},\widetilde{\omega}}=\begin{pmatrix}\cosh\widetilde{\chi} & (\sinh\widetilde{\chi})e^{-i\widetilde{\omega}}\\ -(\sinh\widetilde{\chi})e^{i\widetilde{\omega}} & -\cosh\widetilde{\chi}\end{pmatrix}, \quad \widetilde{\chi},\widetilde{\omega}\in\mathbb{R}. \tag{3.13}$$

Reasoning by analogy for the matrix representation $\widehat{\mathcal{C}}_+$ of $\widehat{C}_+$ we get

$$\widehat{\mathcal{C}}_+=\mathcal{C}_{\widehat{\chi},\widehat{\omega}}=\begin{pmatrix}\cosh\widehat{\chi} & (\sinh\widehat{\chi})e^{-i\widehat{\omega}}\\ -(\sinh\widehat{\chi})e^{i\widehat{\omega}} & -\cosh\widehat{\chi}\end{pmatrix}, \quad \widehat{\chi},\widehat{\omega}\in\mathbb{R}. \tag{3.14}$$

Passing to the matrix representation in (3.11) we conclude that $A_M(=A_{\mathcal{K}})$ has a stable C-symmetry if and only if

$$\mathcal{K}(\zeta,\phi,\omega,\xi)\mathcal{C}_{\widetilde{\chi},\widetilde{\omega}}=\mathcal{C}_{\widehat{\chi},\widehat{\omega}}\mathcal{K}(\zeta,\phi,\omega,\xi), \tag{3.15}$$

where $\mathcal{K}(\zeta,\phi,\omega,\xi)$ is defined by (2.18). A routine analysis of (3.15) with the use of (3.13) and (3.14) shows that (3.15) is equivalent to the system of relations

$$\begin{cases}\cosh\widehat{\chi}-\cosh\widetilde{\chi}+\tanh\zeta[e^{i(\widehat{\omega}-\omega-\phi)}\sinh\widehat{\chi}-e^{i(\omega-\widetilde{\omega}-\phi)}\sinh\widetilde{\chi}]=0,\\ \tanh\zeta[\cosh\widehat{\chi}+\cosh\widetilde{\chi}]+e^{i(\phi+\omega-\widehat{\omega})}\sinh\widehat{\chi}+e^{-i(\phi-\omega+\widetilde{\omega})}\sinh\widetilde{\chi}=0.\end{cases} \tag{3.16}$$

Let us set $\widehat{\chi}=\widetilde{\chi}=\chi$. Then the first relation in (3.16) is satisfied when

$$\omega=\frac{\widehat{\omega}+\widetilde{\omega}}{2} \tag{3.17}$$

and the second one goes over

$$\tanh\zeta+\tanh\chi\cos\left(\phi+\frac{\widetilde{\omega}-\widehat{\omega}}{2}\right)=0.$$

The latter equation can be solved with respect to χ if and only if

$$|\tanh\zeta| < \left|\cos\left(\phi + \frac{\widetilde{\omega} - \widehat{\omega}}{2}\right)\right|. \tag{3.18}$$

Since $\widetilde{\omega}, \widehat{\omega} \in \mathbb{R}$ are independent variables, conditions (3.17) and (3.18) with fixed ω and ϕ can easily be satisfied by a suitable choice of $\widetilde{\omega}$ and $\widehat{\omega}$. This means that the system (3.16) has a solution $\widetilde{\chi}, \widehat{\chi}, \widetilde{\omega}, \widehat{\omega}$ for any fixed ζ, ϕ, ω, ξ. Therefore, $A_{\mathcal{K}(\zeta,\phi,\omega,\xi)}$ has a stable C-symmetry for any choice of ζ, ϕ, ω, and ξ. Theorem 3.9 is proved. □

Corollary 3.10. *Let A be J-self-adjoint extension of the Phillips symmetric operator S with deficiency indices $< 2, 2 >$. Then A is similar to a self-adjoint operator if and only if the spectrum of A is real.*

Proof. It follows from Proposition 2.10 and Theorem 3.9 □

3.4. **Various realizations of the Phillips operator.** It follows from (1.2) and (1.3) that the Phillips symmetric operator S can be obtained as the restriction of a self-adjoint operator A with Lebesgue spectrum onto the domain

$$\mathcal{D}(S) = \{f \in \mathcal{D}(A) \mid ((A - iI)f, w) = 0, \ \forall w \in W_0\}, \tag{3.19}$$

where W_0 is a wandering subspace of the bilateral shift U. In particular, this means that the Phillips symmetric operator S naturally arises in the study of the formal expression $i\frac{d}{dx} + < \delta, \cdot > \delta(x)$ and it coincides with the operator

$$S = i\frac{d}{dx}, \quad \mathcal{D}(S) = \{u(\cdot) \in W_2^1(\mathbb{R}) \mid u(0) = 0\}$$

acting in $L_2(\mathbb{R})$. This example illustrates one of possible general approaches to the construction of the Phillips symmetric operator. Indeed, let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and let $S = S_1 \oplus S_2$, where S_1 and S_2 are simple maximal symmetric operators in the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ with deficiency indices $< m, 0 >$ and $< 0, m >$ ($m \in \mathbb{N}$), respectively. In that case S is a simple symmetric operator in $\mathfrak{H}$ with deficiency indices $< m, m >$ and its characteristic function $\Theta(\cdot)$ associated with an arbitrary boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ (see (2.11)) is equal to zero. By Lemma 3.1 this means that S is a Phillips symmetric operator.

Using (1.3) and (3.19) it is easy to calculate the defect subspaces $\mathfrak{N}_{\pm i}$ of S

$$\mathfrak{N}_i = W_0 \quad \text{and} \quad \mathfrak{N}_{-i} = UW_0.$$

According to (1.2) and (1.3), a bilateral shift U and its wandering subspace W_0 are the main ingredients for the determination of the Phillips symmetric operator S. To illustrate this point, we have presented below two mathematical constructions where U appears naturally and W_0 admits a simple description.

3.4.1. *Multiresolution approximation of* $L_2(\mathbb{R})$. We recall [33, 34] that a multiresolution approximation (MRA) of $L_2(\mathbb{R})$ is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L_2(\mathbb{R})$ such that: (i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$; (ii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$; (iii) $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R})$; (iv) $f(\cdot) \in V_j \Leftrightarrow f(2^{-j}\cdot) \in V_0$; (v) there exists a function $\varphi(\cdot) \in V_0$ such that the sequence $\{\varphi(\cdot - k), k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Let

$$Uf(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right), \quad \forall f \in L_2(\mathbb{R})$$

be the dilation operator in $\mathfrak{H} = L_2(\mathbb{R})$ and let $\{V_j\}_{j\in\mathbb{Z}}$ be a fixed multiresolution approximation of $L_2(\mathbb{R})$. Then U is a bilateral shift in $L_2(\mathbb{R})$ with a wandering subspace $W_0 = V_1 \ominus V_0$ (due to properties (i)–(iv)).

According to the general results of MRA-based wavelet theory [33, 34] the subspace W_0 is a wavelet subspace and relations (i)–(v) imply the existence of a function (wavelet)

$\psi(\cdot) \in W_0$ such that the sequence $\{\psi(\cdot - k), k \in \mathbb{Z}\}$ forms an orthonormal basis in W_0. This means that (3.19) can be rewritten as follows:

$$\mathcal{D}(S) = \{f \in \mathcal{D}(A) \mid ((A - iI)f, \psi(\cdot - k)) = 0, \ \forall k \in \mathbb{Z}\},$$

where the wavelet $\psi(\cdot)$ is directly constructed by the (scaling) function $\varphi(\cdot)$ from condition (v).

3.4.2. *Abstract wave equation.* Let us consider an operator-differential equation

$$u_{tt} = -Lu, \tag{3.20}$$

where L is a positive self-adjoint operator in a Hilbert space H. By H_L we denote the completion of domain of definition $\mathcal{D}(L)$ with respect to the norm $\|u\|^2_{H_L} := (Lu, u)_H$ and consider the Hilbert space $\mathfrak{H} = H_L \oplus H$ (the energy space). It is convenient to write elements of $\mathfrak{H}$ as column matrices $\begin{pmatrix} u \\ v \end{pmatrix}$, where $u \in H_L$ and $v \in H$. Put $u_t = v$ and rewrite (3.20) as

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = iQ\begin{pmatrix} u \\ v \end{pmatrix}, \quad Q = i\begin{pmatrix} 0 & -I \\ L & 0 \end{pmatrix}.$$

The operator Q with the domain $\mathcal{D}(Q) = \left\{\begin{pmatrix} u \\ v \end{pmatrix} \mid \{u, v\} \subset \mathcal{D}(L)\right\}$ is essentially self-adjoint in $\mathfrak{H}$. Its closure A is a generator of the group of unitary (in $\mathfrak{H}$) operators $W_A(t) = e^{iAt}$, which determines solutions of the Cauchy problem for the abstract wave equation (3.20).

The equation (3.20) is said to be *free (unperturbed) wave equation* if there exists a simple maximal symmetric operator B in H such that

$$B^2 \subset L \quad \text{and} \quad (Lu, u) = \|B^* u\|^2_H, \quad \forall u \in \mathcal{D}(L). \tag{3.21}$$

Assume that $\{u_n\}$ belong $\mathcal{D}(B^2)$ and form a Cauchy sequence in H_L. Then $\{Bu_n\}$ is the Cauchy sequence in H (due to the second relation in (3.21)) and hence $\lim_{n\to\infty} Bu_n = \gamma \in H$. In that case we will say that the sequence $\{u_n\}$ converges to the element x_γ in the space H_L. Obviously the Hilbert space H_L can be identified with the set of elements $\{\mathsf{x}_\gamma \mid \forall\gamma \in H\}$ and $(\mathsf{x}_\gamma, \mathsf{x}_\zeta)_{H_L} = (\gamma, \zeta)_H$.

In what follows, without loss of generality we assume that B has zero defect number in the lower half-plane. Then B admits the representation

$$B = T^{-1} i \frac{d}{ds} T, \quad \mathcal{D}(B) = T^{-1} \overset{0}{W}{}^1_2 (\mathbb{R}_+, \mathcal{N}), \tag{3.22}$$

where T is an isometric mapping from H onto $L_2(\mathbb{R}_+, \mathcal{N})$, $\mathcal{N}$ is an auxiliary Hilbert space of dimension equal to the nonzero defect number of B and $\mathbb{R}_+ = (0, \infty)$.

Using (3.22), we can define B in various functional spaces getting, as a result, different specific realizations of the free abstract wave equation. In particular, the classical free wave equation $u_{tt}(x, t) = \Delta u(x, t)$ in $\mathbb{R}^n$ (n is odd) can be obtained from (3.22) if we choose $\mathcal{N}$ as the Hilbert space $L_2(S^{n-1})$ of functions square-integrable on the unit sphere S^{n-1} in $\mathbb{R}^n$ and consider the isometric operator $T : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}_+, \mathcal{N})$ defined on the rapidly decreasing smooth functions $u(x) \in S(\mathbb{R}^n)$ by the formula

$$(Tu)(s, w) = (\partial_s^m Ru)(s, w), \quad m = \frac{(n-1)}{2}, \quad s \geq 0, \quad w \in S^{n-1},$$

where R is the Radon transformation. In that case the Laplace operator $L = -\Delta$ in $L_2(\mathbb{R}^n)$ satisfies condition (3.21) and equation (3.20) takes the form $u_{tt}(x, t) = \Delta u(x, t)$ (see [31] for detail).

Assume that (3.20) is the free wave equation for some choice of B. Then the corresponding generator A is the Cayley transform of a bilateral shift U and a wandering subspace W_0 can be chosen as follows [30]:

$$W_0 = \left\{ \begin{pmatrix} \mathsf{x}_h \\ -ih \end{pmatrix} \mid \forall h \in \ker(B^* + iI) \right\}. \tag{3.23}$$

Substituting (3.23) into (3.19) and taking into account that the domain $\mathcal{D}(A)$ can be described explicitly, we find S. For instance, let L be the Friedrichs extension of B^2. Then $L = B^*B$ and this operator satisfies (3.21). In this case:

$$A \begin{pmatrix} \mathsf{x}_\gamma \\ p \end{pmatrix} = -i \begin{pmatrix} \mathsf{x}_{Bp} \\ -B^*\gamma \end{pmatrix}, \quad \mathcal{D}(A) = \left\{ \begin{pmatrix} \mathsf{x}_\gamma \\ p \end{pmatrix} \mid \forall \gamma \in \mathcal{D}(B^*), \ \forall p \in \mathcal{D}(B) \right\}$$

and S is the restriction of A onto the set of elements

$$\left\{ \begin{pmatrix} \mathsf{x}_\gamma \\ p \end{pmatrix} \quad \mid \quad \gamma \in \mathcal{D}(B^*), \ p \in \mathcal{D}(B) \right\}$$

such that $((B^* + iI)(p - i\gamma), h) = 0$ for all $h \in \ker(B^* + iI)$.

Corollary 3.11. *Assume that the nonzero defect number of B is 2 and J is a fundamental symmetry in $\mathfrak{H}$ such that $SJ = JS$. Then if $A \in \Sigma_J(S)$ has a real spectrum, then $W_A(t) = e^{iAt}$ is a C_0-semigroup.*

Proof. Immediately follows from Corollary 3.10. □

4. Conclusions

In this paper we have studied the collection $\Sigma_J(S)$ of J-self-adjoint extensions of the Phillips symmetric operator S. Our attention to $\Sigma_J(S)$ was inspirited by a steady interest in the spectral analysis of new classes of J-self-adjoint operators A_ε with the aim to illustrate quantitative and qualitative changes of spectra $\sigma(A_\varepsilon)$ when parameters ε run the domain of variation Ξ. Due to specific inherent properties of the Phillips operator S (the zero characteristic function, the absence of real points of regular type, etc) we obtained a spectral picture which differs from the matrix models [21, 24, 25] and models based on J-self-adjoint (symmetric) perturbations of the Schrödinger or the Dirac operator [5, 15, 32, 39]. For instance, in our case, either the spectrum of $A \in \Sigma_J(S)$ coincides with real line: $\sigma(A) = \mathbb{R}$ or with complex plane: $\sigma(A) = \mathbb{C}$ (Theorem 3.7).

For operators $A_\varepsilon \in \Sigma_J(S)$ (where S is an arbitrary symmetric operator commuting with J) we have introduced the concepts of stable and unstable C-symmetry (Definition 2.11). These concepts are natural for sets of J-self-adjoint operators appearing in the extension theory framework. Roughly speaking, if A_ε belongs to the sector Σ_J^{st} of stable C-symmetry, then A_ε preserves the property of C-symmetry under small variation of ε.

For singular perturbations of the Schrödinger or the Dirac operator, the corresponding symmetric operator S has real points of regular type. In that case, the sector Σ_J^{unst} of unstable C symmetry is not empty and operators $A_\varepsilon \in \Sigma_J(S)$ with real spectra and Jordan points arise in the case where ε lies on the boundary of Σ_J^{st} [1, 23]. This picture is essentially simplified for the Phillips symmetric operator S since S has no real points of regular type. We have shown that the sector Σ_J^{unst} of unstable C-symmetry is the empty set and there are no J-self-adjoint extensions of A_{sym} with real spectra and Jordan points (this fact follows from Theorem 3.9 and Corollary 3.10). These results have been obtained under the assumption that S has deficiency indices $< 2, 2 >$. We believe that they remain true for the general case $< n, n >$. However, the corresponding proof requires more cumbersome analysis and the case $< n, n >$ will be considered in a forthcoming paper.

An open problem is finding an adequate physical phenomenon for which J-self-adjoint extensions of S can be served as model Hamiltonians. In this way we have just discussed certain representations of S related to abstract wave equation and multiresolution approximation.

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