ON MIXING AND COMPLETELY MIXING PROPERTIES OF POSITIVE L^1-CONTRACTIONS OF FINITE REAL W*-ALGEBRAS

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ABSTRACT. We consider a non-commutative real analogue of Akcoglu and Sucheston's result about the mixing properties of positive L^1-contractions of the L^1-space associated with a measure space with probability measure. This result generalizes an analogous result obtained for the L^1-space associated with a finite (complex) W*-algebras.

1. INTRODUCTION

It is well known that there are several notions of mixing (i.e. weak mixing, strong mixing, t-mixing, mildly mixing, harmonically mixing and so on) of measure preserving transformation on probability space in ergodic theory. It is important to know how these notions are related with each other. In the last few years, a lot of papers are devoted to this subject (see. e.g., [1–3]). In [1], Akcoglu and Sucheston have studied the asymptotic properties of a positive contraction T of commutative algebra, they used weak convergence. In [3], Zaharopol and Zbaganu have introduced a commutative counterpart of the smoothing, they used the notion of smoothing instead of weak convergence. It is well known that the smoothing condition is less restrictive than the weak convergence used (see, [2] for details). In [2], it has been investigated a noncommutative extension of the result proved in [1].

In this paper, we consider a non-commutative real analogue of results in [1]. Namely, we consider the mixing properties of positive L^1-contractions of the L^1-space associated with a finite real W*-algebras. Our aim is to obtain non-commutative real analogue of the notions of mixing and completely mixing by means of the smoothing. We are going to study the mixing and completely mixing properties of positive L^1-contractions of finite real W*-algebras. Note that, the results of paper generalizes the analogous results in [2], which are proved for the L^1-space associated with a finite (complex) W*-algebras. In the paper we use the methods of the theory of real and complex W^*-algebras; the connections between the real W*-algebras and their enveloping (complex) W*-algebras, in particular the connection of real factor with their enveloping W*-algebra by means of an involutive *-antiautomorphism. Moreover, we use also the scheme of proof of the main results proved in the works [1], [3].

The paper is organized as follows. Section 2 contains some preliminary facts and definitions. In section 3 we give an non-commutative real analog of Akcoglu-Sucheston theorem (see [1]) for real L^1-spaces. We hope that this result enables to study subsequential ergodic theorems in a non-commutative real setting (see [2–4]). We note that our results are not valid when real W*-algebra is semi-finite.

2. PRELIMINARIES

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. A weakly closed *-subalgebra M with identity element $\mathbb{1}$ in $B(H)$ is called a W*-*algebra*. A real *-subalgebra $\Re \subset B(H)$ is called a *real* W*-*algebra* if it is closed in the weak operator topology and $\Re \cap i\Re = \{0\}$. A real W*-algebra $\Re$ is called a *real factor* if its center $Z(\Re)$ contains only elements of the form $\{\lambda \mathbb{1}\}$, $\lambda \in \mathbb{R}$. We say that a real W*-algebra $\Re$ is of the type I_{fin}, I_∞, II_1, II_∞, or III_λ, $(0 \leq \lambda \leq 1)$ if the enveloping W*-algebra $M = \Re + i\Re$ (i.e., the least W*-algebra containing $\Re$) is of the corresponding type with respect to the usual classification of W*-algebras. A linear mapping α of an algebra into itself with $\alpha(x^*) = \alpha(x)^*$ is called an *involutive *-antiautomorphism* if $\alpha(xy) = \alpha(y)\alpha(x)$ and $\alpha^2(x) = x$. It is known that (see [5]) the involutive *-antiautomorphism α of $\Re + i\Re$ defined by $\alpha(a + ib) = a^* + ib^*$ $(a, b \in \Re)$ generates $\Re$, and the converse if α is an involutive *-antiautomorphism of a W*-algebra M, then the set $(M, \alpha) = \{x \in M : \alpha(x) = x^*\}$ is a real W*-algebra. Therefore we shall identify from now on the real von Neumann algebra $\Re$ with the pair (M, α). In the further, for convenience instead of $\Re$ we shall often use the form (M, α).

Throughout the paper M would be a von Neumann algebra with the unit $\mathbb{1}$, and let τ be a faithful normal finite trace on M. Let α be the involutive *-antiautomorphism of M, such that $\tau \circ \alpha = \tau$. The set of all self-adjoint elements of (M, α) is denoted by $(M, \alpha)_s$; the set of all projections in (M, α) we will denote by ∇. By $(M, \alpha)_*$ we denote a pre-dual space to (M, α). It is knows that $M_* = (M, \alpha)_* + i(M, \alpha)_*$ (see [6, Remark of Proposition 6.2.1]).

The map $\| \cdot \|_1 : (M, \alpha) \to [0, \infty)$ defined by the formula $\|x\|_1 = \tau(|x|)$ is a norm. The completion of (M, α) with respect to the norm $\| \cdot \|_1$ is denoted by $L^1((M, \alpha), \tau)$. For the convenience the norm $\| \cdot \|_1$ we denote by $\| \cdot \|$. Using results of [6] it is easy to show that $L^1(M, \tau) = L^1((M, \alpha), \tau) + iL^1((M, \alpha), \tau)$, and $L^1((M, \alpha), \tau)$ is isometrically isomorphic to $(M, \alpha)_*$; moreover $L^1((M, \alpha)_s, \tau)$ is a pre-dual to $(M, \alpha)_s$.

Let $T : L^1((M, \alpha), \tau) \to L^1((M, \alpha), \tau)$ be a linear operator. We say that a linear operator T is positive if $Tx \geq 0$ whenever $x \geq 0$. A linear operator T is said to be a contraction if $\|T(x)\|_1 \leq \|x\|_1$ for all $x \in L^1((M, \alpha)_s, \tau)$.

3. MIXING AND COMPLETELY MIXING CONTRACTIONS

Let $T : L^1((M, \alpha), \tau) \to L^1((M, \alpha), \tau)$ be a linear contraction. We can extended T to $L^1(M, \tau)$ as $\overline{T}(x) = \overline{T}(x_1 + ix_2) = T(x_1)$. Let

$$\overline{\rho}(\overline{T}) = \sup \left\{ \lim_{n \to \infty} \frac{\|\overline{T}^n(u - v)\|}{\|u - v\|} : u, v \in L^1((M, \alpha)_s, \tau), u, v \geq 0, \|u\| = \|v\| \right\}.$$

We put $\overline{\rho}(T) = \overline{\rho}(\overline{T})$. If $\overline{\rho}(T) = 0$, then T is called *completely mixing*. A positive contraction T is called mixing, if for all $x \in L^1((M, \alpha)_s, \tau)$ with $\tau(x) = 0$ and $y \in (M, \alpha)$ holds

$$\lim_{n \to \infty} \tau(T^n(x)y) = 0.$$

Let T be a positive contraction of $L^1((M, \alpha), \tau)$, and let $x \in L^1((M, \alpha), \tau)$ be such that $x \geq 0$, $x \neq 0$. We say that T is *smoothing* with respect to (w.r.t.) x if for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(pT^n x) < \varepsilon$ for every $p \in \nabla$ such that $\tau(p) < \delta$ and for every $n \geq n_0$.

Theorem 1. *Let $T : L^1((M, \alpha), \tau) \to L^1((M, \alpha), \tau)$ be a positive contraction. If there is a positive element $y \in L^1((M, \alpha), \tau)$ such that T is smoothing w.r.t. y, then $\lim_{n \to \infty} \|T^n y\| = 0$ or there is a nonzero positive $z \in L^1((M, \alpha), \tau)$ such that $Tz = z$.*

Proof. It is easy to see that the limit $\lim_{n\to\infty} \|T^n y\|$ exists (which we denote by ℓ), since T is a contraction. Assume that $\ell \neq 0$. Just as in the complex case we consider the map $\lambda : (M, \alpha)_s \to \mathbb{R}$ defined as

$$\lambda(x) = L\big((\tau(xT^n y))_{n\in\mathbb{N}}\big), \quad x \in (M, \alpha)_s,$$

where L means a Banach limit. We have

$$\lambda(\mathbb{1}) = L\big((\tau(T^n y))_{n\in\mathbb{N}}\big) = \lim_{n\to\infty} \|T^n y\| = \ell \neq 0,$$

therefore $\ell \neq 0$. Besides, λ is a positive functional, since for any positive element x we have

$$\tau(xT^n y) = \tau(x^{1/2} T^n y x^{1/2}) \geq 0$$

for every $n \in \mathbb{N}$. For arbitrary $x \in (M, \alpha) = (M, \alpha)_s + (M, \alpha)_k$, we have $x = x_s + x_k$ (see [5]) and we define λ by $\lambda(x) = \lambda(x_s)$, where $(M, \alpha)_k$ is the Lie algebra of skew elements of (M, α), i.e. $(M, \alpha)_k = \{x \in M : \alpha(x) = x^* = -x\}$.

Let $T^{**} : (M, \alpha)^{**} \to (M, \alpha)^{**}$ be the second dual of T. Since

$$(T^{**}\lambda)(x) = \langle x, T^{**}\lambda \rangle = \langle T^* x, \lambda \rangle = L\big((\tau(T^n y T^* x))_{n\in\mathbb{N}}\big)$$
$$= L\big((\tau(xT^{n+1}y))_{n\in\mathbb{N}}\big) = L\big((\tau(xT^n y))_{n\in\mathbb{N}}\big) = \lambda(x)$$

the functional λ is T^{**}-invariant.

Let $\lambda = \lambda_n + \lambda_s$ be the Takesaki decomposition of real functional λ on normal and singular components defined in [6, Definition 6.2.3]. Since T is normal and $T^{**}\lambda = \lambda$, so using the idea of [7] it can be proved that $T^{**}\lambda_n = \lambda_n$. Now we will show that λ_n is nonzero. Consider a measure $\mu := \lambda|_\nabla$. It is clear that μ is an additive measure on ∇. Let us prove that it is σ-additive. To this and, it is enough to show that $\mu(p_k) \to 0$ whenever $p_{k+1} \leq p_k$ and $p_k \searrow 0$, $p_k \in \nabla$.

Let $\varepsilon > 0$. From $p_k \searrow 0$ we infer that $\tau(p_k) \to 0$ as $n \to \infty$. It follows that there exists $k_\varepsilon \in \mathbb{N}$ such that $\tau(p_k) < \varepsilon$ for all $k \geq k_\varepsilon$. Since T is smoothing w.r.t. y we obtain

$$\tau(p_k T^n y) < \varepsilon, \quad \forall k \geq k_\varepsilon,$$

for every $n \geq n_0$. From a property of the Banach limit we get

$$\lambda(p_k) = L\big((\tau(p_k T^n y))_{n\in\mathbb{N}}\big) < \varepsilon \quad \text{for every} \quad k \geq k_\varepsilon,$$

which implies $\mu(p_k) \to 0$ as $k \to \infty$. This means that the restriction of λ_n on ∇ coincides with μ. Since

$$\tau(p^\perp T^n y) > \tau(T^n y) - \varepsilon \geq \inf \|T^n y\| - \varepsilon = \ell - \varepsilon$$

as ε has been arbitrary, so $\ell - \varepsilon > 0$, and hence $\mu(p^\perp) > 0$ for all $p \in \nabla$ such that $\tau(p) < \delta$. Therefore $\mu \neq 0$, and consequently, $\lambda_n \neq 0$.

¿From this we infer that there exists a positive element $z \in L^1((M, \alpha), \tau)$ such that

$$\lambda_n(x) = \tau(zx), \quad \forall\, x \in (M, \alpha).$$

The last equality and $T^{**}\lambda_n = \lambda_n$ yield that

$$\tau(zx)) = \langle x, T^{**}\lambda_n \rangle = \langle T^* x, \lambda_n \rangle = \tau(zT^* x) = \tau(Tzx)$$

for every $x \in M$, which implies that $Tz = z$. $\qquad\square$

Theorem 1 states that if a positive contraction $T : L^1\big((M, \alpha), \tau\big) \to L^1\big((M, \alpha), \tau\big)$ is smoothing then either it's powers converge strongly to zero or it has a non-zero invariant vector in (M, α).

Corollary. Let $x \in L^1((M, \alpha), \tau)$, $x \geq 0$. Assume that $T^n x \to x^*$ weakly. Then T is smoothing w.r.t. x.

Remark. The proved Theorem 1 is a non-commutative real analog of Akcoglu and Sucheston's result [1].

Before proving the next theorem let us give the following auxiliary lemma.

Lemma. Let $x \in L^1((M, \alpha), \tau)$. If the inequality

(1) $$\tau(xy) \geq 0$$

is valid for every $y \geq 0$, $y \in (M, \alpha)$, then $x \geq 0$.

Proof. Let $\forall a \in M$ with $a \geq 0$. Since $a = y + iz$ for some $y, z \in (M, \alpha)$, by [5, Corollary 1.1.4] we have $y \geq 0$. According to (1) one gets

$$\tau(xa) = \tau(xy + ixz) = \tau(xy) \geq 0.$$

By [2, Lemma 3.4] we have $x \geq 0$.   □

¿From Lemma and Theorem 1 we find the following.

Theorem 2. *Let* $T : L^1((M, \alpha), \tau) \to L^1((M, \alpha), \tau)$ *be a positive contraction such that* $|T(x)| \leq T(|x|)$ *for every* $x \in L^1((M, \alpha), \tau)$, $x = x^*$. *Assume that there exists no nonzero* $y \in L^1((M, \alpha), \tau)$, $y \geq 0$, *such that* $Ty = y$. *If for* $z \in L^1((M, \alpha), \tau)$ *the sequence* $(T^n z)$ *converges weakly to some element of* $L^1((M, \alpha), \tau)$, *then* $\lim_{n \to \infty} \|T^n z\| = 0$. *In particular, if* T *is mixing, then* T *is completely mixing.*

Proof. As in the proof of Theorem 1 we assume that $\lim_{n \to \infty} \|T^n z\| = \ell > 0$. Just as in the complex case we consider the map $\lambda : (M, \alpha)_s \to \mathbb{R}$ defined as

$$\lambda(x) = L((\tau(x|T^n z|))_{n \in \mathbb{N}})$$

for every $x \in (M, \alpha)_s$. Using the same argument as in the proof of Theorem 1 one can show that there exists a nonzero positive element $y \in L^1((M, \alpha), \tau)$ such that

$$\lambda_n(x) = \tau(yx), \quad \forall \, x \in (M, \alpha).$$

Here λ_n is the normal part of λ.

¿From the property of T we infer

$$\tau(Tyx) = \tau(yT^*x) = L((\tau(|T^n z|T^*x))_{n \in \mathbb{N}})$$
$$= L((\tau(T|T^n z|x))_{n \in \mathbb{N}}) \geq L((\tau(|T^{n+1}z|x))_{n \in \mathbb{N}}) = \tau(yx)$$

for all $x \geq 0$. Hence, for every $x \geq 0$ we have

$$\tau((Ty - y)x) \geq 0.$$

According to Lemma we infer that $Ty \geq y$. Since T is a contraction one gets $\|y\|_1 \leq \|Ty\|_1 \leq \|y\|_1$, i.e. $\|Ty\|_1 = \|y\|_1$. Hence $Ty = |Ty| = |y| = y$, since $\tau(|Ty|) = \tau(|y|)$. But this contradicts the assumption of the theorem.   □

Remark. The proved theorem is a non-commutative real analog of [8, Ch. 8 Theorem 1.4]. Certain similar results have been obtained in [9], [10] for quantum dynamical semigroups in von Neumann algebras.

Remark. It should be noted that Theorem 1 and 2 are not valid if a von Neumann algebra is semi-finite.

Indeed, let $B(\ell_2)$ be the algebra of all linear bounded operators on Hilbert space ℓ_2. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a standard basis of ℓ_2. The matrix units of $B(\ell_2)$ can be defined by

$$e_{i,j}(\eta) = (\eta, \xi_i)\xi_j, \quad \eta \in \ell_2, \quad i, j \in \mathbb{N}.$$

Let α be a canonical involutive *-antiautomorphism of $B(\ell_2)$, i.e. $\alpha(e_{i,j}) = e_{j,i}$. A trace on $B(\ell_2)$ is defined by

$$\tau(x) = \sum_{k=1}^{\infty} (x\xi_k, \xi_k).$$

It is clear that τ is α-invariant, i.e. $\tau \circ \alpha = \tau$. By ℓ_∞^r we denote a maximal commutative real subalgebra generated by elements $\{e_{ii} : i \in \mathbb{N}\}$. Then an algebra defined by $\ell_\infty = \ell_\infty^r + i\ell_\infty^r$ is a maximal commutative (complex) subalgebra generated by elements $\{e_{ii} : i \in \mathbb{N}\}$. Let $P : \ell_\infty \to \ell_\infty$ be the natural projection on ℓ_∞^r, i.e. $P(l + il') = l$. Define a map $s : \ell_\infty \to \ell_\infty$ as follows: for every element $a \in \ell_\infty$, $a = \sum\limits_{k=1}^{\infty} a_k e_{kk}$ put

$$s(a) = \sum_{k=1}^{\infty} a_k e_{k+1,k+1}.$$

It is clear that $s(\ell_\infty^r) \subset \ell_\infty^r$. Define $T : (B(\ell_2), \alpha) \to (B(\ell_2), \alpha)$ as $T(x) = s(P(E(x)))$, $x \in (B(\ell_2), \alpha)$, where $E : B(\ell_2) \to \ell_\infty$ is the canonical conditional expectation. It is easy to see that T is positive and $\tau\big(T(x)\big) \leq \tau(x)$ for every $x \in L^1((B(\ell_2), \alpha), \tau) \cap (B(\ell_2), \alpha)$, $x \geq 0$. Hence, T is a positive L^1-contraction. But for T there is no nonzero x such that $Tx = x$. Moreover, for every $y \in L^1((B(\ell_2), \alpha), \tau)$ we have $\lim\limits_{n \to \infty} \|T^n y\| \neq 0$.

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